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Global Character of a Rational Difference Equation [1]

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Abstract : In this paper, we study the global character of the solutions of higher-order rational difference equation of the form:

$$x_{N+1} = \frac{Ax_N^n + Bx_{N-1}x_{N-K}^n}{Cx_N^n + Dx_{N-1}x_{N-K}^n},$$

where n is some strictly positive integer, parameters A, B, C, D and initial conditions $x_{-K}, \ldots, x_{-1}$, x_0 are positive real numbers. We prove local stability, persistence, periodicity nature of solutions and global attractivity of equilibrium point of this equation. Some numerical examples are given to verify our theoretical results.

Keywords : stability; periodic solutions; global character.
2010 Mathematics Subject Classification : 39A10; 40A05.

1 Introduction

The theory of difference equations occupies a central position in applicable Analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. For basic theory of difference equations see [1–3]. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as

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discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics. Our aim in this paper is study the stability and global behavior of solutions of nonlinear rational difference equations of order greater than one. Our primary concern is to study the global asymptotic stability of the equilibrium solution. We also discuss the boundedness of solutions and the existence of periodic solutions. For more detail one can see [4–6].

Recently there has been a lot of interest in studying the global attractivity, boundedness character and the periodic nature of nonlinear difference equations. For some results in this area, see for example [7–11]. In the paper we study boundedness nature of solutions, the stability of the equilibrium points and the periodic character of the difference equation:

$$x_{N+1} = \frac{Ax_N^n + Bx_{N-1}x_{N-K}^n}{Cx_N^n + Dx_{N-1}x_{N-K}^n},\ N = 0, 1, \ldots, \tag{1.1}$$

where n is some strictly positive integer, parameters A, B, C, D and initial conditions $x_{-K}, \ldots, x_{-1},\ x_0$ are positive real numbers.

2 Preliminaries

A difference equation of order $(K + 1)$ is an equation of the form:

$$x_{N+1} = F(x_N, x_{N-1}, \ldots, x_{N-K}),\ N = 0, 1, \ldots, \tag{2.1}$$

where F is a continuously differentiable function which maps some set I^{K+1} into I. The set I is usually an interval of real numbers.

A solution of Equation (2.1) is a sequence $\{x_N\}_{N=-K}^{\infty}$ which satisfies Equation (2.1) for all $N \geq 0$.

Definition 2.1. A solution $\{x_N\}_{N=-K}^{\infty}$ of Equation (2.1) which is constant for all $N \geq -K$ is called an equilibrium solution of Equation (2.1). If $x_N = \bar{x}$ for all $N \geq -K$ is an equilibrium solution of Equation (2.1), then $\bar{x}$ is an equilibrium point of Equation (2.1), or equivalently a point $\bar{x} \in I$ is an equilibrium point of Equation (2.1) if

$$\bar{x} = F(\bar{x}, \bar{x}, \ldots, \bar{x}).$$

Definition 2.2. A solution $\{x_N\}_{N=-K}^{\infty}$ of difference Equation (2.1) is bounded and persists if there exist numbers m and M with $0 < m \leq M < \infty$ such that for any initial conditions $x_{-K}, \ldots, x_{-1}, x_0$ there exists a positive integer $\bar{N}$ such that $m \leq x_N \leq M$ for all $N \geq \bar{N}$.

Definition 2.3 (Stability)**.**

(i) An equilibrium point $\bar{x}$ of Equation (2.1) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\{x_N\}_{N=-K}^{\infty}$ is a solution of Equation (2.1) with $|x_{-K} - \bar{x}| + |x_{1-K} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta$, then $|x_N - \bar{x}| < \varepsilon$ for all $N \geq -K$.

(ii) An equilibrium point $\bar{x}$ of Equation (2.1) is locally asymptotically stable if it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_N\}_{N=-K}^{\infty}$ is a solution of Equation (2.1) with $|x_{-K}-\bar{x}|+|x_{1-K}-\bar{x}|+\cdots+|x_0-\bar{x}| < \gamma$, then $\lim_{N\to\infty} x_N = \bar{x}$.

(iii) An equilibrium point $\bar{x}$ of Equation (2.1) is a global attractor if for every solution $\{x_N\}_{N=-K}^{\infty}$ of Equation (2.1), we have $\lim_{N\to\infty} x_N = \bar{x}$.

(iv) An equilibrium point $\bar{x}$ of Equation (2.1) is globally asymptotically stable if it is locally stable, and $\bar{x}$ is also global attractor of Equation (2.1).

(v) An equilibrium point $\bar{x}$ of Equation (2.1) is unstable if $\bar{x}$ is not locally stable.

Definition 2.4. A solution $\{x_N\}_{N=-K}^{\infty}$ is periodic with period p if there exists an integer $p \geq 1$ such that

$$x_{N+p} = x_N \text{ for all } N \geq -K. \tag{2.2}$$

A solution is periodic with prime period p if p is the smallest positive integer for which (2.2) holds.

2.1 Linearized Stability Analysis

Suppose F is continuously differentiable in some open neighborhood of $\bar{x}$. Let $p_i = \frac{\partial F}{\partial u_i}(\bar{x},\bar{x},\ldots,\bar{x})$ for $i = 0,1,\ldots,K$ denote the partial derivatives of $F(u_0,u_1,\ldots,u_K)$ with respect to u_i evaluated at $\bar{x}$. The equation

$$z_{N+1} = p_0 z_N + p_1 z_{N-1} + \cdots + p_K z_{N-K},\; N = 0,1,\ldots \tag{2.3}$$

is called linearized equation of (2.1) about $\bar{x}$, and the equation

$$\lambda^{K+1} - p_0\lambda^K - \cdots - p_{K-1}\lambda - p_K = 0 \tag{2.4}$$

is called characteristic equation of (2.3) about $\bar{x}$.

The following result is known as the Linearized Stability Theorem, is very useful in determining the local stability character of the equilibrium point of Equation (2.1).

Theorem 2.5. *Assume that F is continuously differentiable function defined on some open neighborhood of an equilibrium point $\bar{x}$ and if all roots of Equation (2.4) have absolute value less than one, then the equilibrium point of Equation (2.1) is locally asymptotically stable.*

The following result is a sufficient condition for all roots of an equation of any order to lie inside the unit disk.

Theorem 2.6. *Assume that $p_0,p_1,\ldots,p_K$ are real numbers such that $|p_0|+|p_1|+\cdots+|p_K| < 1$. Then all roots of Equation (2.4) lie inside the open unit disk $|\lambda| < 1$.*

3 Main Results

To study the local stability character of the solutions of Equation (1.1), we let $f:(0,\infty)^3 \to (0,\infty)$ be a continuously differentiable function defined by:

$$f(x,y,z) := \frac{Ax^n + Byz^n}{Cx^n + Dyz^n}. \tag{3.1}$$

Let $\bar{x}$ be an equilibrium point of Equation (1.1), then

$$\begin{aligned}\bar{x} &= f(\bar{x},\bar{x},\bar{x}) \\ &= \frac{A\bar{x}^n + B\bar{x}^{n+1}}{C\bar{x}^n + D\bar{x}^{n+1}} \\ &= \frac{A+B\bar{x}}{C+D\bar{x}}.\end{aligned}$$

This implies that $\bar{x} = \frac{B-C\pm\sqrt{(B-C)^2+4AD}}{2D}$.

Moreover,

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{nx^{n-1}yz^n(AD-BC)}{(Cx^n+Dyz^n)^2},$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{x^n z^n(BC-AD)}{(Cx^n+Dyz^n)^2},$$

and

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{nx^n yz^{n-1}(BC-AD)}{(Cx^n+Dyz^n)^2}.$$

Furthermore, it is easy to check that

$$p_2 = \frac{\partial f}{\partial x}(\bar{x},\bar{x},\bar{x}) = \frac{n(AD-BC)}{(C+D\bar{x})^2},$$

$$p_1 = \frac{\partial f}{\partial y}(\bar{x},\bar{x},\bar{x}) = \frac{(BC-AD)}{(C+D\bar{x})^2},$$

and

$$p_0 = \frac{\partial f}{\partial z}(\bar{x},\bar{x},\bar{x}) = \frac{n(BC-AD)}{(C+D\bar{x})^2}.$$

At $\bar{x} = \frac{B-C+\sqrt{(B-C)^2+4AD}}{2D}$, one has $(C+D\bar{x})^2 = \frac{1}{4}(B+C+\sqrt{(B-C)^2+4AD})^2$.

Thus

$$p_2 = \frac{4n(AD-BC)}{(B+C+\sqrt{(B-C)^2+4AD})^2},$$

$$p_1 = \frac{4(BC-AD)}{(B+C+\sqrt{(B-C)^2+4AD})^2},$$

and

$$p_0 = \frac{4n(BC - AD)}{(B + C + \sqrt{(B - C)^2 + 4AD})^2}.$$

The linearized equation of Equation (1.1) about $\bar{x} = \frac{B-C+\sqrt{(B-C)^2+4AD}}{2D}$ is

$$z_{N+1} = p_0 z_N + p_1 z_{N-1} + p_2 z_{N-K} \tag{3.2}$$

and

$$\lambda^{K+1} - p_0\lambda^K - p_1\lambda^{K-1} - p_2 = 0 \tag{3.3}$$

is its characteristic equation.

Remark 3.1. *From above calculations it is easy to see that*

$$p_2 + np_1 = 0, \;\; np_1 - p_0 = 0, \;\; p_2 + p_0 = 0.$$

Theorem 3.2. *The equilibrium point* $\bar{x} = \frac{B-C+\sqrt{(B-C)^2+4AD}}{2D}$ *of Equation (1.1) is locally asymptotically stable if* $4(2n+1)|AD-BC| < B+C+\sqrt{(B-C)^2+4AD}$.

Proof. From Theorem 2.6 it follows that all roots of Equation (3.3) lie in an open disc $|\lambda| < 1$, if

$$|p_2| + |p_1| + |p_0| < 1.$$

This implies that

$$\frac{4n|AD - BC|}{B + C + \sqrt{(B - C)^2 + 4AD})^2} + \frac{4|BC - AD|}{B + C + \sqrt{(B - C)^2 + 4AD}} + \frac{4n|BC - AD|}{B + C + \sqrt{(B - C)^2 + 4AD}} < 1.$$

Thus one has

$$4(2n + 1)|AD - BC| < B + C + \sqrt{(B - C)^2 + 4AD}.$$

□

Theorem 3.3. *Every solution of Equation (1.1) is bounded and persists.*

Proof. Let $\{x_N\}_{N=-K}^{\infty}$ be a solution of Equation (1.1), then

$$\begin{aligned} x_{N+1} &= \frac{Ax_N^n + Bx_{N-1}x_{N-K}^n}{Cx_N^n + Dx_{N-1}x_{N-K}^n} \\ &= \frac{Ax_N^n}{Cx_N^n + Dx_{N-1}x_{N-K}^n} + \frac{Bx_{N-1}x_{N-K}^n}{Cx_N^n + Dx_{N-1}x_{N-K}^n} \\ &\le \frac{Ax_N^n}{Cx_N^n} + \frac{Bx_{N-1}x_{N-K}^n}{Dx_{N-1}x_{N-K}^n} \\ &= \frac{A}{C} + \frac{B}{D}. \end{aligned}$$

Thus $x_N \leq \frac{A}{C} + \frac{B}{D} = M$ for all $N \geq 1$.

Let there exists $m > 0$ such that $x_N \geq m$ for all $N \geq 1$. Taking $x_N = \frac{1}{y_N}$, then one has

$$\begin{aligned} y_{N+1} &= \frac{Cy_N^n + Dy_{N-1}x_{N-K}^n}{Ay_N^n + By_{N-1}x_{N-K}^n} \\ &= \frac{Cy_N^n}{Ay_N^n + By_{N-1}y_{N-K}^n} + \frac{Dy_{N-1}y_{N-K}^n}{Ay_N^n + By_{N-1}y_{N-K}^n} \\ &\leq \frac{Cy_N^n}{Ay_N^n} + \frac{Dy_{N-1}y_{N-K}^n}{By_{N-1}y_{N-K}^n} \\ &= \frac{C}{A} + \frac{D}{B}. \end{aligned}$$

Thus $x_N = \frac{1}{y_N} \geq \frac{1}{H} = \frac{AB}{AD+BC} = m$ for all $N \geq 1$. Hence, $m \leq x_N \leq M$ for all $N \geq 1$. □

Theorem 3.4. *The local asymptotic stability of equilibrium point of Equation (1.1) is independent of exponent n.*

Proof. From Remark 3.1, we have the following linear homogeneous system in p_0, p_1 and p_2:

$$\begin{pmatrix} 1 & n & 0 \\ 0 & n & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_2 \\ p_1 \\ p_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let $A = \begin{pmatrix} 1 & n & 0 \\ 0 & n & -1 \\ 1 & 0 & 1 \end{pmatrix}$, then $\det A = 0$ for all values of n. Hence, local asymptotic stability of equilibrium point of Equation (1.1) does not depend on n. □

Next we prove that Equation (1.1) has prime period two solutions.

Theorem 3.5. *Let K be even, then Equation (1.1) has prime period two solutions for all $A,\ B,\ C,\ D \in \mathbb{R}^+$.*

Proof. Suppose that $\ldots,\ p,\ q,\ p,\ q,\ldots$ be a prime period two solution of Equation (1.1). If K is even, then $x_N = x_{N-K}$ and $x_{N+1} = x_{N-1}$. From Equation (1.1) one has:

$$p = \frac{Aq^n + Bpq^n}{Cq^n + Dpq^n} = \frac{A + Bp}{C + Dp},$$

and

$$q = \frac{Ap^n + Bqp^n}{Cp^n + Dqp^n} = \frac{A + Bq}{C + Dq}.$$

Then, it is easy to see that $p + q = \frac{B-C}{D}$ and $pq = -\frac{A}{D}$. It is clear that p and q are two real distinct roots of quadratic equation given by:

$$Dt^2 - (B - C)t - A = 0,$$

for all $A,\ B,\ C,\ D \in \mathbb{R}^+$.

Second suppose that $A,\ B,\ C,\ D \in \mathbb{R}^+$. We will show that Equation (1.1) has prime period two solutions. Assume that

$$p = \frac{(B - C) + \sqrt{(B - C)^2 + 4AD}}{2D},$$

and

$$q = \frac{(B - C) - \sqrt{(B - C)^2 + 4AD}}{2D}.$$

Then, it is easy to see that p and q are distinct real numbers with $q < p$. Let K be even and $x_{-K} = p,\ x_{-K+1} = q, \ldots, x_0$. We want to show that $x_1 = q$ and $x_2 = p$. To this end, we deduce from the original difference Equation (1.1) that

$$x_1 = \frac{Ax_0^n + Bx_{-1}x_{-K}^n}{Cx_0^n + Dx_{-1}x_{-K}^n} = \frac{A + Bq}{C + Dq}. \tag{3.4}$$

Furthermore,

$$x_1 - q = \frac{A + Bq}{C + Dq} - \frac{(B - C) - \sqrt{(B - C)^2 + 4AD}}{2D}. \tag{3.5}$$

Taking $q = \frac{(B-C)-\sqrt{(B-C)^2+4AD}}{2D}$ in right hand side of Equation (3.5), one has $x_1 - q = 0$, i.e., $x_1 = q$. Similarly one can show that $x_2 = p$. Hence, by mathematical induction one can easily prove that $x_N = q$ and $x_{N+1} = p$ for all $N \geq -K$. Hence, if K is even and for all $A,\ B,\ C,\ D \in \mathbb{R}^+$, we have a prime period two solution of (1.1) and the proof is completed. □

Theorem 3.6. *Let K be odd, then Equation (1.1) has no prime period two solutions for all $A,\ B,\ C,\ D \in \mathbb{R}^+$.*

Proof. Assume that there exists a distinct solution

$$\ldots,\ p,\ q,\ p,\ q, \ldots$$

of prime order two of Equation (1.1). If K is odd, then $x_{N+1} = x_{N-K}$. It follows from Equation (1.1) that

$$p = \frac{Aq^n + Bqp^n}{Cq^n + Dqp^n} = \frac{Aq^{n-1} + Bp^n}{Cq^{n-1} + Dp^n},$$

and

$$q = \frac{Ap^n + Bpq^n}{Cp^n + Dpq^n} = \frac{Ap^{n-1} + Bq^n}{Cp^{n-1} + Dq^n}.$$

Hence, one has

$$Cpq^{n-1} + Dp^{n+1} = Aq^{n-1} + Bp^n, \tag{3.6}$$

and

$$Cqp^{n-1} + Dq^{n+1} = Ap^{n-1} + Bq^n. \tag{3.7}$$

From (3.6) and (3.7), one has $C(p-q) = 0$. Hence, $p = q$, which is a contradiction. □

Lemma 3.7. *For any values of quotients $\frac{A}{C}$ and $\frac{B}{D}$ the function $f(x,y,z) = \frac{Ax^n+Byz^n}{Cx^n+Dyz^n}$ is monotonic in each of its three arguments.*

Theorem 3.8. *The equilibrium point $\bar{x}$ is a global attractor of Equation (1.1) if one of the following statements holds:*

(i) $AD \geq BC$ *and* $2nC(\frac{B}{D})^{2n} - 2nA(\frac{B}{D})^{2n-1} > -(B+C)(\frac{A}{C})^{2n}$.

(ii) $AD \leq BC$ *and* $(2n+1)D(\frac{A}{C})^{2n} - 2nB(\frac{A}{C})^{2n-1} > -A(\frac{B}{D})^{2n-2}$.

Proof. Let $\{x_N\}_{N=-K}^{\infty}$ be a solution of Equation (1.1). In case of (i), when $AD \geq BC$, the function $f(x,y,z) = \frac{Ax^n+Byz^n}{Cx^n+Dyz^n}$ is non-decreasing in x and non-increasing in y and z. Thus we have $\frac{B}{D} \leq x_N \leq \frac{A}{C}$ for all $N \geq 1$. Let $\{x_N\}_{N=0}^{\infty}$ be a solution of Equation (1.1) with $L = \lim_{N\to\infty} \inf x_N$ and $U = \lim_{N\to\infty} \sup x_N$. We want to show that $L = U$.

According to Lemma 3.7, $L \geq f(L,U,U)$ and $U \leq f(U,L,L)$, which implies that

$$AL^{2n} + BL^nU^{n+1} - CL^{2n+1} \leq DL^{n+1}U^{n+1}, \tag{3.8}$$

and

$$AU^{2n} + BU^nL^{n+1} - CL^{2n+1} \geq DL^{n+1}U^{n+1}. \tag{3.9}$$

Combining (3.8) and (3.9), one has

$$AL^{2n} + BL^nU^{n+1} - CL^{2n+1} \leq AU^{2n} + BU^nL^{n+1} - CL^{2n+1}. \tag{3.10}$$

Inequality (3.10) can be written as:

$$\begin{aligned}(L-U)[C(L^{2n} + L^{2n-1}U + \cdots + U^{2n}) + BL^nU^n \\ - A(L^n+U^n)(L^{n-1} + L^{n-2}U + \cdots + U_{n-1})] \geq 0.\end{aligned} \tag{3.11}$$

Now $L - U > 0$, if

$$C(L^{2n}+L^{2n-1}U+\cdots+U^{2n})+BL^nU^n-A(L^n+U^n)(L^{n-1}+L^{n-2}U+\cdots+U_{n-1}) \geq 0. \tag{3.12}$$

Inequality (3.12) can be written as:

$$\begin{aligned}C(L^{2n}+L^{2n-1}U + \cdots + L^{n+1}U^{n-1} + L^{n-1}U^{n+1} + \cdots + U^{2n}) \\ + (B+C)L^nU^n - A(L^n+U^n)(L^{n-1} + L^{n-2}U + \cdots + U^{n-1}) \geq 0.\end{aligned}$$

To prove Inequality (3.12), let us consider

$$\Omega = C(L^{2n} + L^{2n-1}U + \cdots + L^{n+1}U^{n-1} + L^{n-1}U^{n+1} + \cdots + U^{2n}) - A(L^n + U^n)(L^{n-1} + L^{n-2}U + \cdots + U^{n-1}).$$

Then, one has

$$\begin{aligned}\Omega &\geq 2nC\left(\frac{B}{D}\right)^{2n} - 2nA\left(\frac{B}{D}\right)^{2n-1}\\ &\geq -(B+C)\left(\frac{A}{C}\right)^{2n}\\ &\geq -(B+C)L^nU^n.\end{aligned}$$

Now assume that $AD \leq BC$, then the function $f(x,y,z) = \frac{Ax^n+Byz^n}{Cx^n+Dyz^n}$ is non-increasing in x and non-decreasing in y and z. In this case, one has $\frac{A}{C} \leq x_N \leq \frac{B}{D}$ for all $N \geq 1$. Furthermore, in this case $L \geq f(U,L,L)$ and $U \leq f(L,U,U)$. It is easy to see that instead of inequalities (3.8) and (3.9), one has following two inequalities:

$$AL^{n-1}U^n + BL^{2n} - DL^{2n+1} \leq CL^nU^n, \tag{3.13}$$

and

$$AL^nU^{n-1} + BU^{2n} - DU^{2n+1} \geq CL^nU^n. \tag{3.14}$$

Combining (3.13) and (3.14), one has

$$AL^{n-1}U^n + BL^{2n} - DL^{2n+1} \leq AL^nU^{n-1} + BU^{2n} - DU^{2n+1}. \tag{3.15}$$

Inequality (3.15) can be written as:

$$(L-U)[AL^{n-1}U^{n-1} - B(L^n+U^n)(L^{n-1}+\cdots+U^{n-1}) + D(L^{2n}+\cdots+U^{2n})] \geq 0. \tag{3.16}$$

In Inequality (3.16) $L-U \geq 0$, if

$$AL^{n-1}U^{n-1} - B(L^n+U^n)(L^{n-1}+\cdots+U^{n-1}) + D(L^{2n}+\cdots+U^{2n}) \geq 0. \tag{3.17}$$

To prove Inequality (3.17), we let

$$\Upsilon = D(L^{2n} + \cdots + U^{2n}) - B(L^n + U^n)(L^{n-1} + \cdots + U^{n-1}).$$

Then, one has

$$\begin{aligned}\Upsilon &\geq D(2n+1)\left(\frac{A}{C}\right)^{2n} - 2nB\left(\frac{A}{C}\right)^{2n-1}\\ &\geq -A\left(\frac{B}{D}\right)^{2n-2}\\ &\geq -AL^{n-1}U^{n-1}.\end{aligned}$$

Hence, $L = U$ and the proof is therefore completed. □

4 Examples

In order to verify our theoretical results and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the nonlinear difference Equation (1.1). In this section we consider some numerical examples and discuss local stability points with their plots.

Example 4.1. *Consider the Equation (1.1) with* $A = 5,\ B = 34.8,\ C = 1,\ D = 400,\ n = 1,\ K = 2$. *Then,*

$$x_{N+1} = \frac{5x_N + 34.8x_{N-1}x_{N-2}}{x_N + 400x_{N-1}x_{N-2}}.$$

Figure 1 shows stability of equilibrium point

$$\bar{x} = \frac{B - C + \sqrt{(B-C)^2 + 4AD}}{2D} = 0.16177013428707315$$

of the Equation (1.1) with initial conditions $x_0 = 0.11,\ x_{-1} = 0.17,\ x_{-2} = 0.16$.

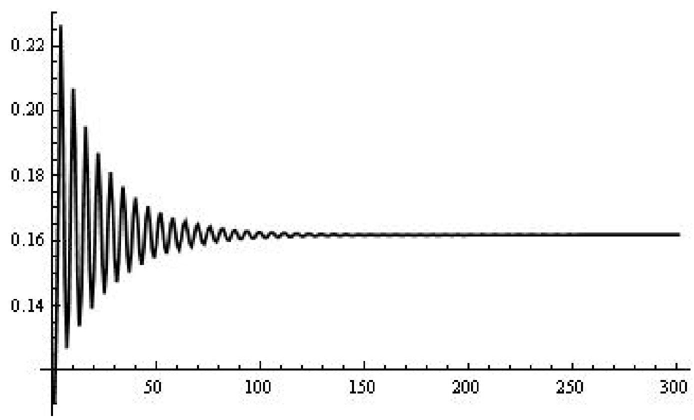

Figure 1: Plot of $x_{N+1} = \frac{5x_N+34.8x_{N-1}x_{N-2}}{x_N+400x_{N-1}x_{N-2}}$.

Example 4.2. *Consider the Equation (1.1) with* $A = 2,\ B = 7,\ C = 3,\ D = 1,\ n = 2,\ K = 2$. *Then,*

$$x_{N+1} = \frac{2x_N^2 + 7x_{N-1}x_{N-2}^2}{3x_N^2 + x_{N-1}x_{N-2}^2}.$$

Figure 2 shows stability of equilibrium point $\bar{x} = \frac{B-C+\sqrt{(B-C)^2+4AD}}{2D} = 4.44949$ *of the Equation (1.1) with initial conditions* $x_0 = 0.001,\ x_{-1} = 0.07,\ x_{-2} = 0.09$.

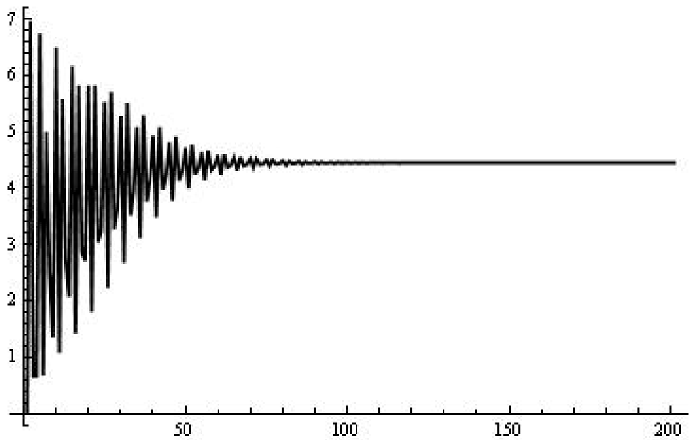

Figure 2: Plot of $x_{N+1} = \frac{2x_N^2 + 7x_{N-1}x_{N-2}^2}{3x_N^2 + x_{N-1}x_{N-2}^2}$.

Example 4.3. *Consider the Equation (1.1) with* $A = 1,\ B = 1,\ C = 3,\ D = 1,\ n = 3,\ K = 2$. *Then,*

$$x_{N+1} = \frac{x_N^3 + x_{N-1}x_{N-2}^3}{3x_N^3 + x_{N-1}x_{N-2}^3}.$$

Figure 3 shows stability of equilibrium point $\bar{x} = \frac{B-C+\sqrt{(B-C)^2+4AD}}{2D} = 0.414214$ *of the Equation (1.1) with initial conditions* $x_0 = 0.11,\ x_{-1} = 0.127,\ x_{-2} = 0.46.$

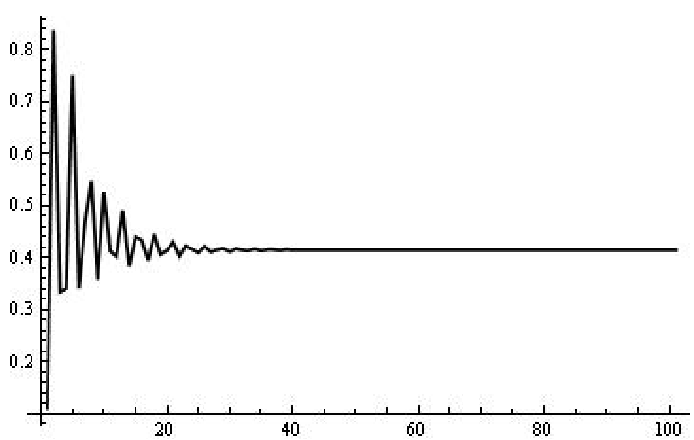

Figure 3: Plot of $x_{N+1} = \frac{x_N^3 + x_{N-1}x_{N-2}^3}{3x_N^3 + x_{N-1}x_{N-2}^3}$.

Example 4.4. *Consider the Equation (1.1) with* $A = 2,\ B = 8,\ C = 2,\ D = 4.5,\ n = 7,\ K = 2$. *Then,*

$$x_{N+1} = \frac{2x_N^7 + 8x_{N-1}x_{N-2}^7}{2x_N^7 + 4.5x_{N-1}x_{N-2}^7}.$$

Figure 4 shows stability of equilibrium point $\bar{x} = \frac{B-C+\sqrt{(B-C)^2+4AD}}{2D} = 1.60948$ *of the Equation (1.1) with initial conditions* $x_0 = 1,\ x_{-1} = 0.57,\ x_{-2} = 0.69$.

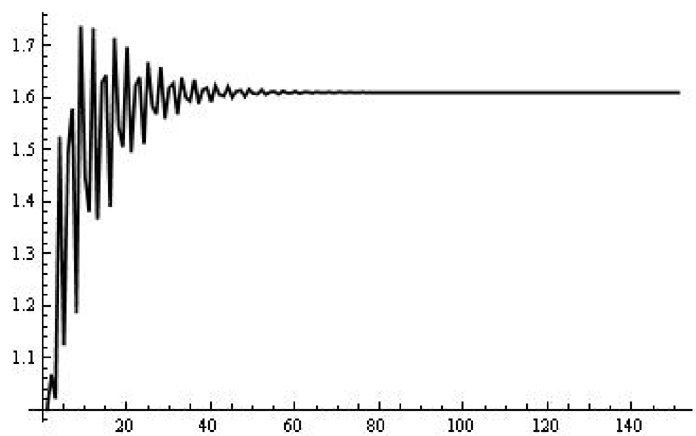

Figure 4: Plot of $x_{N+1} = \frac{2x_N^7+8x_{N-1}x_{N-2}^7}{2x_N^7+4.5x_{N-1}x_{N-2}^7}$.

Example 4.5. *Consider the Equation (1.1) with* $A = 2,\ B = 8,\ C = 2,\ D = 4.5,\ n = 30,\ K = 3$. *Then,*

$$x_{N+1} = \frac{x_N^{30} + 2.3x_{N-1}x_{N-3}^{30}}{2x_N^{30} + 4.4x_{N-1}x_{N-3}^{30}}.$$

Figure 5 shows stability of equilibrium point $\bar{x} = \frac{B-C+\sqrt{(B-C)^2+4AD}}{2D} = 0.51204$ *of the Equation (1.1) with initial conditions* $x_0 = 0.5,\ x_{-1} = 0.7,\ x_{-2} = 0.9,\ x_{-3} = 0.3$.

Example 4.6. *Consider the Equation (1.1) with* $A = 0.5,\ B = 0.008,\ C = 6.8,\ D = 0.9,\ n = 50,\ K = 5$. *Then,*

$$x_{N+1} = \frac{0.5x_N^{50} + 0.008x_{N-1}x_{N-5}^{50}}{6.8x_N^{50} + 0.9x_{N-1}x_{N-5}^{50}}.$$

Figure 6 shows stability of equilibrium point $\bar{x} = \frac{B-C+\sqrt{(B-C)^2+4AD}}{2D} = 0.0729116$ *of the Equation (1.1) with initial conditions* $x_0 = 0.35,\ x_{-1} = 0.67,\ x_{-2} = 0.19,\ x_{-3} = 0.13,\ x_{-4} = 0.62,\ x_{-5} = 0.93$.

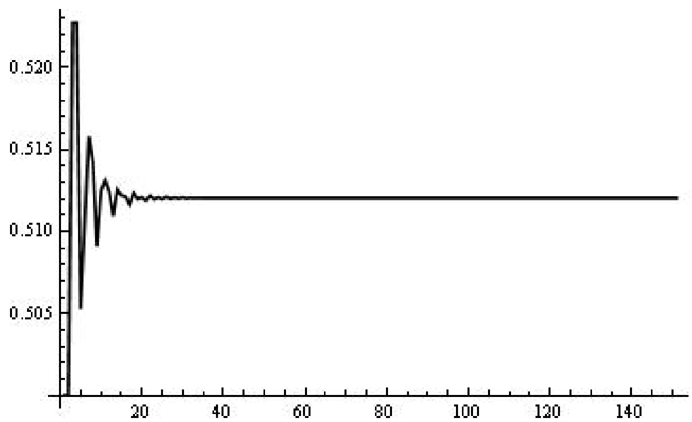

Figure 5: Plot of $x_{N+1} = \frac{x_N^{30}+2.3x_{N-1}x_{N-3}^{30}}{2x_N^{30}+4.4x_{N-1}x_{N-3}^{30}}$.

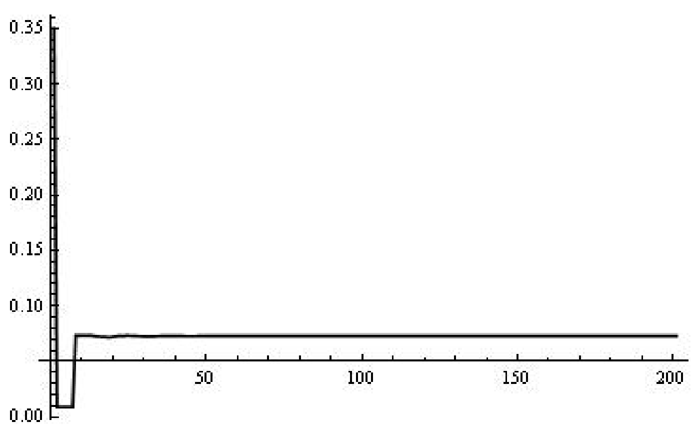

Figure 6: Plot of $x_{N+1} = \frac{0.5x_N^{50}+0.008x_{N-1}x_{N-5}^{50}}{6.8x_N^{50}+0.9x_{N-1}x_{N-5}^{50}}$.

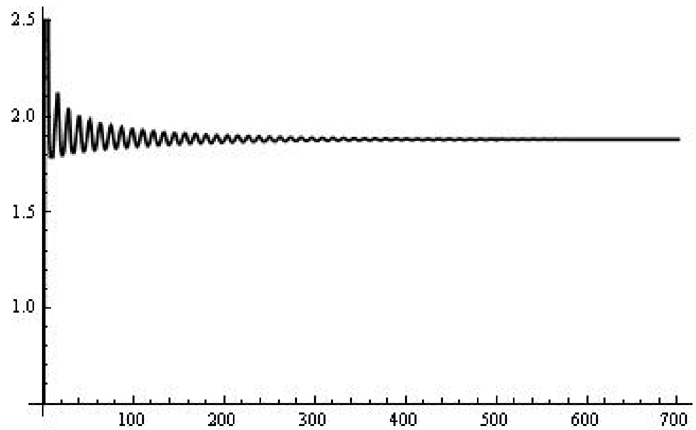

Figure 7: Plot of $x_{N+1}=\frac{5x_N^{13}+12.5x_{N-1}x_{N-4}^{13}}{2x_N^{13}+7x_{N-1}x_{N-4}^{13}}$.

Example 4.7. *Consider the Equation (1.1) with* $A=5,\ B=12.5,\ C=2,\ D=7,\ n=13,\ K=4$. *Then,*

$$x_{N+1}=\frac{5x_N^{13}+12.5x_{N-1}x_{N-4}^{13}}{2x_N^{13}+7x_{N-1}x_{N-4}^{13}}.$$

Figure 7 shows stability of equilibrium point $\bar{x}=\frac{B-C+\sqrt{(B-C)^2+4AD}}{2D}=1.87995$ *of the Equation (1.1) with initial conditions* $x_0=0.5,\ x_{-1}=0.7,\ x_{-2}=0.9,\ x_{-3}=0.3,\ x_{-4}=0.2$.

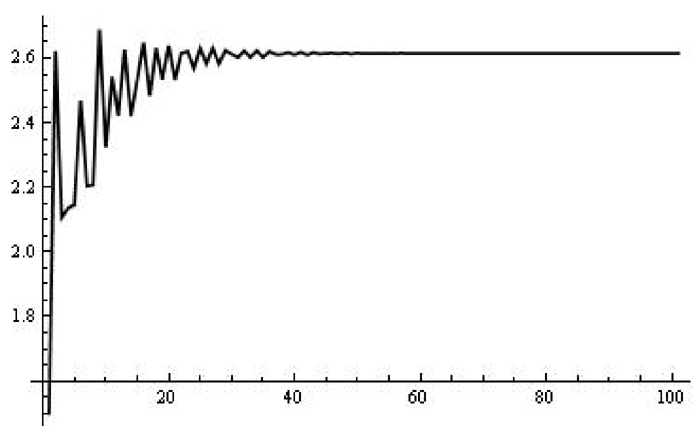

Figure 8: Plot of $\frac{4x_N^{8}+9x_{N-1}x_{N-6}^{8}}{1.9x_N^{8}+3.3x_{N-1}x_{N-6}^{8}}$.

Example 4.8. *Consider the Equation (1.1) with* $A = 4,\ B = 9,\ C = 1.9,\ D = 3.3,\ n = 8,\ K = 6$. *Then,*

$$x_{N+1} = \frac{4x_N^8 + 9x_{N-1}x_{N-6}^8}{1.9x_N^8 + 3.3x_{N-1}x_{N-6}^8}.$$

Figure 8 shows stability of equilibrium point $\bar{x} = \frac{B-C+\sqrt{(B-C)^2+4AD}}{2D} = 2.61504$ *of the Equation (1.1) with initial conditions* $x_0 = 1.5,\ x_{-1} = 1.7,\ x_{-2} = 1.9,\ x_{-3} = 1.3,\ x_{-4} = 1.2,\ x_{-5} = 1.1,\ x_{-6} = 1.6$.

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