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# A Note on Brauer Commutative Monoid

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**Abstract :** The notion of separable semialgebras over a commutative semirings has been defined in [1]. Here we give some new characterisation about central separable semialgebras and introduce the Brauer commutative monoid.

**Keywords :** semirings; semimodules; semialgebras; ideals; k-ideals.

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## 1 Introduction

This paper is concerned with generalizing some results in central separable algebras over commutative rings. The notion of central separable semialgebras over commutative semirings has been defined in [1]. Using some new generalizations in module theory, in this paper we will try to develop this structure theory for central separable semialgebras further and introduce the Brauer commutative monoid.

By a semiring $R$ we always mean a commutative semiring $R$ with 1. Throughout the paper we assume all semimodules are unitary semimodules.

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## 2 Preliminaries

Semirings, additively cancellative semirings, commutative semirings, semimodules, additively cancellative semimodules, ideals, k-ideals (subtractive ideals), homomorphisms, steady homomorphisms are as defined in [2]. Henceforth cancellative semirings (semimodules) mean additively cancellative semirings (semimodules).

*Convention*: $CS$ denotes cancellative and semisubtractive semialgebras as defined in [1].

**Definition 2.1.** A sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ of $R$-semimodules and $R$- homomorphisms is short exact if $f$ is one-one, $g$ is onto, $g$ is a steady $R$-homomorphism and $Im\ f = Ker g$.

**Definition 2.2.** A short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ of $R$-semimodules and $R$-homomorphisms splits if there exists a splitting map $g' \colon N \to M$ such that $gg' = Id_N$.

**Proposition 2.3** ([3])**.** *For any $R$-semimodule $K$ and an exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ the induced sequence $0 \to Hom(K, L) \xrightarrow{f^*} Hom(K, M) \xrightarrow{g^*} Hom(K, N)$ is exact.*

For any $R$-semimodule $M$, consider the subset $I_R(M)$ of $R$ consisting of the elements of the form $\sum_{i=1}^{n} f_i(m_i)$ where $f_i \in Hom(M, R)$ and $m_i \in M$. $I_R(M)$ is a two sided ideal in $R$ and is called the *trace ideal* of $M$. A left $R$-semimodule $M$ is an $R$- generator if and only if the trace ideal $I_R(M) = R$.

**Proposition 2.4** ([4])**.** *Let $R$ be a commutative and lattice ordered semiring and $M$ a finitely generated, projective and cancellative $R$-semimodule. Then $I_R(M) \oplus annih_R(M) = R$.*

An $R$-semimodule $M$ is an *$R$-progenerator* if $M$ is a finitely generated, projective and generator over $R$.

**Proposition 2.5** ([4])**.** *Let $R$ be a commutative and lattice ordered semiring and $M$ be a cancellative $R$-semimodule. $M$ is an $R$-progenerator if and only if $M$ is a finitely generated, projective and faithful.*

**The Dual Basis Lemma [1]** Let $M$ be an $R-$semimodule. Then $M$ is projective if and only if there exists $\{m_i\}_{i \in I} \subset M$ and $\{f_i\}_{i \in I} \subset Hom_R(M, R)$ ($I$ some indexing set) such that

a) for every $m \in M$, $f_i(m) = 0$ for all but finitely many $i \in I$, and

b) for every $m \in M$, $\sum_{i \in I} f_i(m) m_i = m$.

The collection $\{m_i, f_i\}$ is called dual a basis for $M$.

**Lemma 2.6** ([5]). *Let $R$ be any cancellative semiring and $M$ be any cancellative $R$-semimodule. Then $\theta_R$ is onto if and only if $M$ is a finitely generated and projective. Moreover if $\theta_R$ is onto then it is one-one.*

**Lemma 2.7** ([5]). *Let $R$ be any cancellative semiring, $M$ be any cancellative $R$-semimodule and $S = Hom_R(M, M)$ be a cancellative semiring. Then $\theta_S$ is onto if and only if $M$ is a generator. Moreover if $\theta_S$ is onto then it is one-one.*

**Corollary 2.8** ([1]). *Let $R$ be a commutative semiring and let $M$ and $N$ be $R$-semimodules. Then $M \otimes_R N$ is an $R$-progenerator if both $M$ and $N$ are.*

**Proposition 2.9** ([1]). (Hom-Tensor Relation) *Let $R$ be a commutative semiring and let $A$ and $B$ be $R$-semialgebras. Let $M$ be a finitely generated and projective $A$-semimodule and $N$ be a finitely generated and projective $B$-semimodule. Then $Hom_A(M, M) \otimes Hom_B\ (N, N) \cong Hom_{A \otimes B}(M \otimes N, M \otimes N)$ where $\otimes = \otimes_R$.*

In [4], we have introduced $R^e = \{[a, b]/a, b \in R\}$, the ring of differences of any cancellative semiring $R$ w.r.t. the following well defined operations

$$[a, b] + [c, d] = [a + c, b + d] \text{ and}$$

$$[a, b][c, d] = [ac + bd, ad + bc].$$

Let $R$ be any cancellative semiring. If $I$ is a [left, right] ideal of $R$, then $I^e = \{[a, b]/a, b \in I\}$ is a [left, right] ideal of $R^e$. Conversely, if $J$ is a [left, right] ideal of $R^e$, then $J^c = \{a \in R/[a, 0] \in J\}$ is a [left, right] ideal of $R$.

**Proposition 2.10** ([4]).

*(a) Let $R$ be a cancellative semiring. Then for any $k$- ideal $I$ of $R, I = (I^e)^c$.*

*(b) Let $R$ be a cancellative semiring and $I$ be a proper $k$-ideal of $R$ then $I^e$ is a proper ideal of $R^e$.*

*(c) If $I, I'$ are any two $k$-ideals of a cancellative semiring $R$ and $I \subset I'$, then $I^e \subset I'^e$.*

*(d) Let $R$ be a cancellative semiring. Then for any two ideals $J$ and $J'$ of $R^e, J \subset J' \Rightarrow J^c \subset J'^c$.*

(e) *Let $R$ be a cancellative semiring, $I$ and $I'$ be any two ideals in $R$, then $(I+I')^e = I^e + I'^e$.*

**Proposition 2.11** ([4]). *Let $R$ be a cancellative, semisubtractive semiring. Then for any ideal $J$ of $R^e, J = (J^c)^e$.*

**Proposition 2.12** ([4]). *Let $R$ be a cancellative, semisubtractive and lattice ordered semiring. If $J$ and $J'$ are ideals of a ring of difference $R^e$, then $(J+J')^c = J^c + J'^c$.*

## 3 Central Separable Semialgebras

Eventhough some of the statements in this section are more or less known from [1], but we organize and prove them in some how different way for our purpose for a semirings which are not necessarily zerosumfree.

For any $R$-semialgebra $A$, we shall let $A^0$ denote the *opposite semialgebra* of $A$, whose underlying additive semigroup is $A$, multiplication is $a^0b^0 = (ba)^0$ and the $R$ - semimodule structure coincides with $A$ (to avoid confusion, for any element $a \in A$, while considering an element in $A^0$ we shall denote it by $a^0$). The *enveloping semialgebra* is defined by $A \otimes A^0$. For convenience we will write $A^E$ for the enveloping semialgebra $A \otimes A^0$ of $A$.

**Remark 3.1.** *$A \otimes A^0$ is a cancellative $R$-semimodule.*

The semialgebra $A$ has a structure as a left $A^E-$ semimodule induced by $(a \otimes b^0)x = axb$. If $A$ is a cancellative $R$- semialgebra then a map $\mu$ from semialgebra $A^E$ onto $A$ given by $\mu(\sum_i a_i \otimes b_i^0) = \sum_i a_i b_i$, is a left $A^E$ - semimodule homomorphism, which in case $A$ is commutative is a semiring homomorphism.

**Definition 3.2.** A cancellative $R$-semialgebra $A$ is said to be *$R$-separable* if $\mu$ splits as an $A^E$-homomorphism or equivalently $A$ is a retract of $A^E$ (i.e. there exists an $A^E$- homomorphism $\rho : A \rightarrow A^E$ such that $\mu\rho = Id_A$).

**Proposition 3.3** ([1]). *Let $A$ be a cancellative $R$-semialgebra. Then $A$ is $R-$ separable if and only if there exists an element $e$ in $A^E$ satisfying $\mu(e) = 1$ and $(1 \otimes a^0)e = (a \otimes 1^0)e$ for any $a$ in $A$.*

The element $e$ in $A^E$ in the above Proposition is called *separability idempotent* of $A$ and is indeed an idempotent.

An $R$-semialgebra $A$ is called *central* if $A$ is faithful as an $R$-semimodule and $R \cdot 1$ coinsides with the center of $A$. We call $A$ is central separable $R$-semialgebra if $A$ is both central and separable.

For any cancellative $R$-semialgebra $A$, we have seen that $A$ is naturally a left $A^E$-semimodule. This structure induces an $R$-semialgebra homomorphism $\phi \colon A^E \to Hom_R(A, A)$ by associating to any element $\alpha$ in $A^E$, the element $\phi(\alpha) \in Hom_R(A, A)$ which is scalar multiplication in $A$ by $\alpha$. If $\alpha = \sum_i a_i \otimes b_i^0$, then $\phi(\alpha)(a) = \alpha \cdot a = \sum_i a_i a b_i$

**Corollary 3.4** ([1])**.** *Let $A$ be a cancellative $R$- semialgebra. Then $Hom_{A^E}$ $(A,\ A) \cong C(A)$, the center of $A$ under the correspondence $f \to f(1)$.*

**Lemma 3.5.** *Let $R$ be a cancellative semiring and let $A$ be a cancellative, $R$-semialgebra. If $A$ is central separable then $R = Im\phi(e) \subseteq A$.*

*Proof.* Let $e$ be a separability idempotent for $A$ (in $A^E$). Consider the homomorphism $\phi(e)$ in $Hom_R(A, A)$ where $\phi$ is the map defined just above.

Now, $a(\phi(e)(b)) = (a \otimes 1^0)(e \cdot b) = (1 \otimes a^0)(e \cdot b) = (\phi(e)(b))a$, for any $a \in A$, implies that $\phi(e) \cdot (b) \in C(A) = R$, shows that $Im\phi(e) = R$. Hence the proof. □

**Proposition 3.6** ([1])**.** *Let $R$ be a cancellative semiring and let $A$ and $B$ be cancellative, $R$-semialgebras. If $A$ and $B$ are central separable over $R$, then $A \otimes B$ is central separable $R$-semialgebra.*

**Lemma 3.7** ([1])**.** *Let $R$ be a cancellative semiring and let $A$ be a $CS$ $R$-semialgebra. If $A$ is $R$-central, then $C(A^e) = (C(A))^e$.*

**Lemma 3.8** ([1])**.** *Let $R$ be a cancellative semiring and let $A$ be a $CS$ $R$-semialgebra. If $A$ is central separable over $R$, then $A^e$ is central separable over $R^e$.*

**Lemma 3.9.** *Let $R$ be a cancellative semiring and let $A$ be a $CS$ $R$-semialgebra. If $A$ is an $A^E$-progenerator and $R$-central, then $A$ is central separable over $R$.*

*Proof.* If $A$ is $A^E$-progenerator, then $A$ is a finitely generated $A^E$-semimodule which is projective as an $A^E$-semimodule. Since $A$ is a $CS$ $R$-semialgebra, therefore $A$ is an $A^E$-projective, that implies $A$ is $R$-separable. Hence $A$ is central separable over $R$. □

**Lemma 3.10** ([1])**.** *Let $R$ be a cancellative semiring and let $A$ be cancellative $R$-semialgebra. If $A$ is an $A^E$-progenerator and $A$ is $R$-central then $A$ is an $R$-progenerator and the map $\phi \colon A^E \to Hom_R(A,\ A)$ is an isomorphism.*

**Lemma 3.11.** *Let $R$ be a cancellative semiring and let $A$ be cancellative $R$-semialgebra. If $A$ is an $R$-progenerator and the map $\phi\colon A^E \to Hom_R(A, A)$ is an isomorphism, then $A$ is an $A^E$-progenerator and $A$ is $R$-central.*

*Proof.* If $A$ is an $R$-progenerator, then $A$ is an $A^E$-progenerator. Now $A^E \cong Hom_R(A, A)$, implies that $A$ is $R-$ central. □

**Lemma 3.12.** *For any maximal k-ideal $m$ of a central separable $R-$ semialgebra $A$, there exists an k-ideal $J^c$ of $R$ with $J^c A = m$.*

*Proof.* Let $m$ be any maximal k-ideal of $A$. Then $m^e$ is a maximal ideal in $A^e$ for if here exists an ideal $J$ of $A^e$ such that $m^e \subset J \subset A^e$, then $m \subset J^c \subset A$, this then imply $m$ is not a maximal k-ideal in $A$. Therefore there exists an ideal $J$ of $R^e$ such that $JA^e = m^e$, implies that $J^c A = m$. □

The above Propositions and Lemmas are needed to prove the extremely important Theorem.

**Theorem 3.13.** *Let $R$ be a cancellative semiring and let $A$ be a CS $R$-semialgebra. Then the following conditions are equivalent*

1. *$A$ is central separable over $R$.*
2. *$A$ is an $A^E$-progenerator and $R$-central.*
3. *$A$ is an $R$-progenerator and the map $\phi$ from $A^E$ to $Hom_R(A,\ A)$ is an isomorphism.*

*Proof.* 3. $\Rightarrow$ 1. It is obvious that $A$ is $A^E$-projective and finitely generated over $A^E$. It remains to prove that $A$ is an $A^E$-generator, that is, to prove

$$A^* \otimes_{Hom_{A^E}(A,\ A)=R} A \cong Hom_{A^E}(A, A^E) \otimes A \cong A^E$$

under the map $f \otimes a = f(a)$. But by Corollary 1.4

$$A^* \cong Hom_{A^E}(A, A^E) \cong (0 : J)$$

under the map $f \mapsto f(1)$. Therefore we have to show that $(0 : J) \otimes A \cong A^E$ under $b \otimes a = (a \otimes 1)b = (1 \otimes a)b$. But this is equal to $A^E(0 : J) = A^E$.

Suppose that $A^E(0 : J) \neq A^E$

$$\begin{aligned}
&\Rightarrow [A^E(0 : J)]^e \neq (A^E)^e \\
&\Rightarrow (A^E)^e(0 : J)^e \neq (A^E)^e \\
&\Rightarrow (A^E)^e = (A^e)^E = m^e \\
&\Rightarrow A^E = m, \quad \textit{a contradiction.}
\end{aligned}$$

Shows that $A^E(0 : J) = A^E$. Hence 3. $\Rightarrow$ 1. □

**Remark 3.14.** *When $A$ is a central separable, $CS$ $R$-semialgebra where $R$ is a cancellative semiring, we have seen that $A$ is an $R$-progenerator and that $Hom_R(A,\ A)$, being isomorphic to $A \otimes A^0$, is a central separable, $CS$ $R$-semialgebra.*

**Proposition 3.15.** *Let $R$ be a lattice ordered semiring and let $E$ be any cancellative $R$-progenerator. Then $A = Hom_R(E, E)$ is a central separable $R$-semialgebra.*

*Proof.* Note that if $E$ is cancellative, then $A = Hom_R(E, E)$ is cancellative. Now we have to show that $A = Hom_R(E, E)$ is a central separable $R$-semialgebra.

Let $\{x_i, f_i\}$ be a dual basis of $E$. Then we have $\sum f_i(x)x_i = x$ for any $x$ in $E$ and $f_i \in Hom_R(E, R)$. Let $g_j\colon E \to R$, $y_j$ in $E$ satisfy $\sum g_j(y_j) = 1$ as $E$ is a generator over $R$. Define $E_{ij}, F_{ji}$ in $A$ by $E_{ij}(x) = g_j(x)x_i,\ \ F_{ji}(x) = f_i(x)y_j$.

Let $e = \sum_{i,j} E_{ij} \otimes F^0_{ji}$ in $A \otimes A^0$. By easy computation one can verify that $e$ is the separability idempotent for $A$ and $\mu(e) = 1$ and $(f \otimes 1^0)e = (1 \otimes f^0)e$. Now for any $a \in A$, $ea \in eA$, implies that $(ea)b = (1 \otimes b^0)ea = (b \otimes 1^0)ea = b(ea)$, for any $b \in A$. Hence $eA$ is contained in $C(A)$. Conversely, suppose that $x \in C(A)$. Therefore $ex = \mu(e)x = x$, so $x \in eA$.. Shows that $C(A) = eA = A^A = R$. Hence $A$ is $R-$central. Faithfulness of semimodule follows by Proposition 2.4 and 2.5.

**Alternative Proof:** $E^* \otimes E \cong Hom_R(E, E) = A$ and $A \cong E^* \otimes E$ is finitely generated and projective. Moreover it is clear that $A$ is $R$-faithful since $E$ is $R$-faithful. Therefore $A$ is an $R$-progenerator by Proposition 2.4 and 2.5. Also by corollary to Morita Theorem it follows that $A^E$ is then isomorphic to $Hom_R(E, E)$, implies that $R = Hom_{A^E}(A, A) = C(A)$. □

# 4 The Brauer Commutative Monoid

For any commutative lattice ordered semiring $R$, consider a collection $\mathscr{C}(R)$ of central separable, cancellative $R$-semialgebras such that every central separable, cancellative $R$-semialgebra is isomorphic to exactly one member of $\mathscr{C}(R)$.

We observe that any element $A$ in $\mathscr{C}(R)$ is a finitely generated cancellative $R$-semimodule. Thus up to $R$-semimodule isomorphism $\mathscr{C}(R)$ is only a set of finitely generated cancellative $R$-semimodules. The semialgebra structure of $A$ is determined by the mapping from $A \otimes A^0$ to $A$ and the collection of all such maps is also a set. Hence for each isomorphism class of finitely generated cancellative $R$-semimodules, $\mathscr{C}(R)$ is only a set of semialgebra structures which can be given to a representative of that class, so $\mathscr{C}(R)$ is a set.

We can put a commutative, associative binary operation on $\mathscr{C}(R)$ by identifying $A \otimes B$ with the element of $\mathscr{C}(R)$ to which it is isomorphic, where $A$ and $B$ are any two element of $\mathscr{C}(R)$.

Since $\mathscr{C}(R)$ contains an element isomorphic to $R$, $\mathscr{C}(R)$ possesses an identity for these operations therefore forms a *commutative monoid* under $\otimes$.

For any cancellative $R$-progenerator $E$, we have by Proposition 3.13 that $Hom_R(E, E)$ is a central separable, cancellative $R$-semialgebra.

Let $\mathscr{C}^0(R)$ be the subset of $\mathscr{C}(R)$ consisting of those central separable, cancellative $R$-semialgebra $A$ such that $A \cong Hom_R(E, E)$ for some cancellative $R$-progenerator $E$. If $E_1$ and $E_2$ are cancellative $R$-progenerators, so is $E_1 \otimes E_2$ by Corl. 2.8 and by hom-tensor relation 2.9 we then have,

$$Hom_R(E_1 \otimes E_2, E_1 \otimes E_2) \cong Hom_R(E_1, E_1) \otimes Hom_R(E_2, E_2).$$

So $\mathscr{C}^0(R)$ is closed with respect to the tensor product. Furthermore if $R$ is an $R$-progenerator with $R \cong Hom_R(R, R)$, $\mathscr{C}^0(R)$ contains the identity of $\mathscr{C}(R)$. $\mathscr{C}^0(R)$ is a submonoid of $\mathscr{C}(R)$.

We introduce a relation $\sim$ in $\mathscr{C}(R)$ by specifying that two elements $A$ and $B$ of $\mathscr{C}(R)$ are in a relation (written as $A \sim B$) if and only if there exists $X_1$ and $X_2$ in $\mathscr{C}^0(R)$ such that $A \otimes X_1 \cong B \otimes X_2$ an cancellative $R$-semialgebras that is if and only if there exist cancellative $R$-progenerators $E_1$ and $E_2$ such that

$$A \otimes_R Hom_R(E_1, E_1) \cong B \otimes_R Hom_R(E_2, E_2).$$

Obviously, $\sim$ is an equivalent relation $\mathscr{C}(R)$. Thus $\mathscr{C}(R)$ partitions into disjoint equivalence classes with respect to the equivalence relation $\sim$.

**Definition 4.1.** Let $\mathscr{B}(R)$ denote the equivalence classes of $\mathscr{C}(R)$ under the relation $\sim$ and let $[A]$ denote the class containing $A$. Define a binary operation in $\mathscr{B}(R)$ by

$$[A][B] = [A \otimes B].$$

If $A' \in [A]$ and $B' \in [B]$, then by definition there exist $Y$, $Y'$ and $Z$, $Z'$ in $\mathscr{C}^0(R)$ such that $A \otimes Y \cong A' \otimes Y'$ and $B \otimes Z \cong B' \otimes Z'$. Thus we obtain,

$$(A \otimes B) \otimes (Y \otimes Z) \cong (A' \otimes B') \otimes (Y' \otimes Z'),$$

where $Y \otimes Z, Y' \otimes Z'$ are in $\mathscr{C}^0(R)$. This shows that the operation is well defined.
Now,

$$\begin{aligned}[A][B] &= [A \otimes B] \\ &= [B \otimes A], \quad \text{since } R \text{ is commutative} \\ &= [B][A].\end{aligned}$$

Hence commutativity. Associativity is obvious.
Now for any $[A] \in B(R)$,

$$\begin{aligned}[A][R] &= [A \otimes R] \\ &= [R \otimes A] \\ &= [A].\end{aligned}$$

$B(R)$ is a commutative monoid with identity element $[R]$ with respect to the binary operation as defined above and is called the *Brauer Commutative Monoid*.

For any commutative and lattice ordered semiring $R$, if we consider a collection $\mathscr{C}(R)$ of $CS$ $R$-semialgebras which are central separable with above assumed conditions, then by Theorem 3.12, we obtain $A^E = A \otimes A^0 \cong Hom_R(A, \ A)$ in $\mathscr{C}^0(R)$ where $A$ is an $R$-progenerator which is $R-$cancellative and

$$\begin{aligned}[A][A^0] &= [A \otimes A^0] \\ &= [Hom_R(A, A)] \\ &= [R] \\ \Rightarrow [A^0] &= [A^{-1}].\end{aligned}$$

So, $B(R)$ forms an abelian group under the composition of equivalence classes, and is called the *Brauer group* of $R$.

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