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Orbits for Products of Maps

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Abstract : We study the behaviour of the dynamical zeta function and the orbit Dirichlet series for products of maps. The behaviour under products of the radius of convergence for the zeta function, and the abscissa of convergence for the orbit Dirichlet series, are discussed. The orbit Dirichlet series of the cartesian cube of a map with one orbit of each length is shown to have a natural boundary.

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1 Introduction

A fundamental topological invariant of a dynamical system – here thought of as a continuous map $T : X \to X$ on a compact metric space – is its orbit-counting data. Analytic properties of functions capturing this data have been widely exploited in dynamics. For example, the dynamical zeta function ζ_T associated to T is related directly to the so-called transfer operator of T, and has been used to study the distribution of periodic points. Recently the authors [1] studied functorial properties of orbit-counting functions from a purely combinatorial point of view, relating disjoint unions, Cartesian products, and iterates of maps to corresponding operations on the orbit-counting functions. Here we focus on some

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analytic questions in the same spirit, a simple example being this: What is the relationship between the analytic properties of the dynamical zeta functions ζ_{T_1}, ζ_{T_2} and $\zeta_{T_1\times T_2}$? Similar questions arise for the orbit Dirichlet series introduced in [2], where analytic properties of the orbit Dirichlet series are directly related to the usual orbit-growth function π_T.

In order to highlight the underlying combinatorial questions, we take a cavalier attitude to maps in the following sense. For any sequence $a=(a_n)_{n\geqslant 1}$ of non-negative integers, there is (manifestly) a map on $\mathbb{N}$ with a_n closed orbits of length n for each $n\geqslant 1$; via a compactification there is a continuous map on a compact metric space with the same property; finally, by a beautiful theorem of Windsor [3], there is a C^∞ diffeomorphism of the two-torus with the same property. Thus all our remarks below may be seen as being about abstract combinatorial maps or about (unspecified) smooth examples. In the combinatorial setting, the paradigmatic examples are those for which the sequence a is arithmetically simple, the prototype being a map with exactly one orbit of each length, with the natural analytic tool being the orbit Dirichlet series. In the smooth setting, the paradigmatic example might be an Axiom A diffeomorphism of the torus, with the natural analytical tool being the dynamical zeta function. Thus two examples of the arguments below are the following.

- If T has one orbit of each length, then the orbit Dirichlet series d_T is the Riemann zeta function, and a calculation shows that $\mathsf{d}_{T\times T}(s)=\frac{\zeta(s)^2\zeta(s-1)}{\zeta(2s)}$, with abscissa of convergence at 2 and a meromorphic extension to the plane; more surprising is the fact that for the Cartesian cube we find that $\mathsf{d}_{T\times T\times T}(s)$ has abscissa of convergence at 3, a meromorphic extension to $\Re(s)>1$, and a natural boundary at $\Re(s)=1$. This is a striking instance of a naturally-occurring Dirichlet series with a natural boundary.
- If T_1 and T_2 are maps with rational dynamical zeta functions, what relates the discs of convergence of ζ_{T_1} and ζ_{T_2} to that of $\zeta_{T_1\times T_2}$?

2 Products and Iterates

Let T (or $T_1,T_2,\dots$) be a map (or list of maps). A closed orbit τ of length $|\tau|$ is a set of the form $\{x,Tx,\dots,T^{|\tau|}x=x\}$ with cardinality $|\tau|$; write $\mathsf{O}_T(n)$ for the number of closed orbits of length n under T. We always assume that $\mathsf{O}_T(n)<\infty$ for all $n\geqslant 1$.

The number of points of period n (that is, the number of points fixed by the nth iterate T^n) is

$$\mathsf{F}_T(n)=\sum_{d|n} d\mathsf{O}_T(d).$$

The dynamical zeta function associated to T is the function

$$\zeta_T(z)=\exp\sum_{n=1}^{\infty}\mathsf{F}_T(n)\frac{z^n}{n},$$

with radius of convergence $\varrho(\zeta_T) = 1/\limsup_{n\to\infty} \mathsf{F}_T(n)^{1/n}$ (which may be zero), and the orbit Dirichlet series associated to T is

$$\mathsf{d}_T(s) = \sum_{n=1}^{\infty} \frac{\mathsf{O}_T(n)}{n^s},$$

convergent on a (possibly empty) half-plane $\Re(s) > \sigma(\mathsf{d}_T)$, where $\sigma(\mathsf{d}_T)$ is the abscissa of convergence. Analytic properties of ζ_T and d_T may be used in several ways, the most immediate being that asymptotics for the orbit-counting function

$$\pi_T(N) = |\{\tau \mid |\tau| \leqslant N\}|$$

may be found via Tauberian theorems if we have enough information about the analytic properties of ζ_T or d_T. The usual Möbius relation between the sequences $(\mathsf{O}_T(n))$ and $(\mathsf{F}_T(n))$ means that

$$\mathsf{d}_T(s) = \frac{1}{\zeta(s+1)} \sum_{n=1}^{\infty} \frac{\mathsf{F}_T(n)}{n^{s+1}} \tag{2.1}$$

formally and, viewed via the Euler transform, the same formal relation means that there is an Euler product expansion

$$\zeta_T(z) = \prod_{\tau} (1 - z^{|\tau|})^{-1} = \prod_{n=1}^{\infty} (1-z)^{-\mathsf{O}_T(n)}.$$

Clearly $\mathsf{F}_{T_1\times T_2}(n) = \mathsf{F}_{T_1}(n)\mathsf{F}_{T_2}(n)$ for all $n \geqslant 1$, and as pointed out in [1, Lem. 1], it follows that

$$\mathsf{O}_{T_1\times T_2}(n) = \sum_{\operatorname{lcm}(d_1,d_2)=n} \gcd(d_1,d_2)\mathsf{O}_{T_1}(d_1)\mathsf{O}_{T_2}(d_2) \tag{2.2}$$

(this may be seen using (2.1), or by pure thought). The arithmetic properties of the relation (2.2) are rather subtle.

Turning now to iterates of a single map (rather than products of pairs of maps), write $\mathcal{D}(n)$ for the set of prime divisors of $n \in \mathbb{N}$, and for a prime decomposition $n = \boldsymbol{p}^{\boldsymbol{a}} = p_1^{a_1} \cdots p_r^{a_r}$ and a subset $J \subset \mathcal{D}(n)$, write $\boldsymbol{p}_J^{\boldsymbol{a}_J}$ for the restricted product $\prod_{p_j \in J} p_j^{a_j}$. The basic formula for orbit-counting under iteration is found in [1, Th. 4]: if $m = \boldsymbol{p}^{\boldsymbol{a}}$ and $J = J(n) = \mathcal{D}(m) \setminus \mathcal{D}(n)$, then

$$\mathsf{O}_{T^m}(n) = \sum_{d \mid \boldsymbol{p}_J^{\boldsymbol{a}_J}} \tfrac{m}{d} \mathsf{O}_T(\tfrac{mn}{d}). \tag{2.3}$$

In this expression J depends on n, so it involves a splitting into cases depending on the set of primes dividing n. The corresponding formula for fixed points is once again trivial: $\mathsf{F}_{T^k}(n) = \mathsf{F}_T(kn)$ for all $n, k \geqslant 1$.

Example 2.1. *The quadratic map $T : x \mapsto 1 - cx^2$ on the interval $[-1, 1]$ at the Feigenbaum parameter value $c = 1.401155\cdots$ has exactly one orbit of length 2^k for each $k \geqslant 0$ and no other closed orbits, so (as pointed out by Ruelle [4])*

$$\zeta_T(z) = \prod_{n=0}^{\infty}\left(1 - z^{2^n}\right)^{-1} = \prod_{n=0}^{\infty}\left(1 + z^{2^n}\right)^{n+1},$$

satisfying the functional equation $\zeta_T(z^2) = (1 - z)\zeta_T(z)$. More enlightening from an analytic point of view is to note that

$$\mathsf{d}_T(s) = \frac{1}{1 - 2^{-s}}, \tag{2.4}$$

with $\sigma(\mathsf{d}_T) = 0$. It is clear that $\pi_T(N) = \frac{\log N}{\log 2} + \mathrm{O}(1)$; this toy case may also be found by applying Perron's theorem [5] or Agmon's Tauberian theorem [6] to (2.4). Even in this simple case some care is needed as there are infinitely many poles on the critical line $\Re(s) = 0$, and the corresponding residue sums are only conditionally convergent. A calculation using (2.3) (see [1] for the details) shows that

$$\mathsf{d}_{T^k}(s) = |k|_2^{-1} - 1 + |k|_2^{-1}\mathsf{d}_T(s),$$

so in this case $\sigma(\mathsf{d}_{T^k}) = \sigma(\mathsf{d}_T)$ for all $k \geqslant 1$. Similarly,

$$\mathsf{d}_{T\times T}(s) = \frac{3}{1 - 2^{-(s-1)}} - \frac{2}{1 - 2^{-s}},$$

so $\sigma(\mathsf{d}_{T\times T}) = \sigma(\mathsf{d}_T) + \sigma(\mathsf{d}_T) + 1$ in this case.

In pursuit of the behaviour of the abscissa of convergence for products, Ramanujan's formula [7] for the Dirichlet series with coefficients $\sigma_a(n)\sigma_b(n)$ may be used together with (2.2) to give the following (the detailed calculation is in the first author's thesis [8]).

Example 2.2. *Let T_1 be map with n^a orbits of length n and let T_2 be a map with n^b orbits of length n for some integers $a, b \geqslant 0$, so that $\mathsf{d}_{T_1}(s) = \zeta(s - a)$ and $\mathsf{d}_{T_2}(s) = \zeta(s - b)$. Then*

$$\begin{aligned}
d_{T_\times T_2}(s) &= \sum_{n=1}^{\infty}\frac{\mathsf{O}_n(T_1\times T_2)}{n^s}\\
&= \sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{d|n}\frac{1}{n}\mu\left(\frac{n}{d}\right)\sigma_{a+1}(d)\sigma_{b+1}(d)\\
&= \sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{d|n}\frac{d}{n}\mu\left(\frac{n}{d}\right)\frac{1}{d}\sigma_{a+1}(d)\sigma_{b+1}(d)\\
&= \sum_{n=1}^{\infty}\frac{\mu(n)}{n^{s+1}}\cdot\sum_{n=1}^{\infty}\frac{\sigma_{a+1}(n)\sigma_{b+1}(n)}{n^{s+1}}\\
&= \frac{\zeta(s-a)\zeta(s-b)\zeta(s-a-b-1)}{\zeta(2s-a-b)}
\end{aligned}$$

by Ramanujan's formula [7], *where we write as usual* $\sigma_k(n)=\sum_{d|n} d^k$. *Thus*

$$\sigma(\mathsf{d}_{T_1\times T_2})=\sigma(\mathsf{d}_{T_1})+\sigma(\mathsf{d}_{T_2})$$

in this case. Perron's theorem [5] *applies to show that*

$$\begin{aligned}\pi_{T_1\times T_2}(N) &\sim \operatorname{Res}\left(\mathsf{d}_{T_1\times T_2}(s)N^s/s\right)_{s=a+b+2}\\ &= \tfrac{\zeta(a+2)\zeta(b+2)}{2\zeta(a+b+4)+(a+b)\zeta(a+b+4)}N^{a+b+2}.\end{aligned}$$

Example 2.3. *Let* T_1 *be the full shift on* a *symbols, and* T_2 *the full shift on* b *symbols, so that* $\zeta_{T_1}(z)=1/(1-az)$ *and* $\zeta_{T_2}(z)=1/(1-bz)$. *Clearly in this case* $\zeta_{T_1\times T_2}(z)=1/(1-abz)$, *so* $\varrho(\zeta_{T_1\times T_2})=\varrho(\zeta_{T_1})\varrho(\zeta_{T_2})$.

Our first result is that the phenomena in Example 2.3 holds for rational zeta functions. Recall that a linear recurrence sequence is said to be non-degenerate if among the non-trivial ratios of zeros of the characteristic polynomial no unit roots are found (see [9, Sect. 1.1.9]), and we say that a rational zeta function ζ_T is non-degenerate if the linear recurrence sequence satisfied by the sequence $(\mathsf{F}_T(n))$ is non-degenerate.

Theorem 2.4. *If* ζ_{T_1} *and* ζ_{T_2} *are non-degenerate rational functions, then*

$$\varrho(\zeta_{T_1^k})=\varrho(\zeta_{T_1})^k$$

and

$$\varrho(\zeta_{T_1\times T_2})=\varrho(\zeta_{T_1})\varrho(\zeta_{T_2}).$$

Proof. The first assertion is immediate: if ζ_{T_1} is rational, then by [10] there are algebraic numbers $\beta_1,\dots,\beta_r$ and $\alpha_1,\dots,\alpha_s$ with

$$\mathsf{F}_{T_1}(n)=\sum_{i=1}^{r}\beta_i^n-\sum_{i=1}^{s}\alpha_i^n, \tag{2.5}$$

giving the statement at once.

The second statement is more delicate. If

$$\zeta_{T_j}(s)=\prod_{i=1}^{r^{(j)}}\left(1-\alpha_i^{(j)}z\right)\prod_{i=1}^{s^{(j)}}\left(1-\beta_i^{(j)}z\right)^{-1}$$

for $j=1,2$ then

$$\zeta_{T_1\times T_2}(z)=\frac{\prod_{i=1}^{r^{(1)}}\prod_{j=1}^{s^{(2)}}\left(1-\alpha_i^{(1)}\beta_j^{(2)}z\right)\prod_{i=1}^{r^{(2)}}\prod_{j=1}^{s^{(1)}}\left(1-\alpha_i^{(2)}\beta_j^{(1)}z\right)}{\prod_{i=1}^{r^{(1)}}\prod_{j=1}^{r^{(2)}}\left(1-\alpha_i^{(1)}\alpha_j^{(2)}z\right)\prod_{i=1}^{s^{(2)}}\prod_{j=1}^{s^{(1)}}\left(1-\beta_i^{(1)}\beta_j^{(2)}z\right)}.$$

Thus $\varrho(\zeta_{T_1\times T_2})$ is the reciprocal of

$$\max\{\alpha_i^{(1)}\alpha_j^{(2)},\beta_k^{(1)}\beta_\ell^{(2)}\mid 1\leqslant i\leqslant r^{(1)},1\leqslant j\leqslant r^{(2)},1\leqslant k\leqslant s^{(1)},1\leqslant \ell\leqslant s^{(2)}\},\quad (2.6)$$

and we claim that the reciprocal of (2.6) is equal to

$$\max\{\beta_i^{(1)}\beta_j^{(2)}\mid 1\leqslant i\leqslant s^{(1)},1\leqslant j\leqslant s^{(2)}\}^{-1}.$$

That is, we claim that the exponential growth due to the poles of the zeta function dominates the growth due to the zeros. In simple cases like Example 2.3 this is obvious, but in general account needs to be taken of possible cancellation among terms of equal modulus in (2.5). □

Lemma 2.5. *If ζ_T is a non-degenerate zeta function with* (2.5)*, then*

$$\max\{|\beta_i|\mid 1\leqslant i\leqslant r\}\geqslant\max\{|\alpha_i|\mid 1\leqslant i\leqslant s\}.$$

Proof. If $\max\{|\alpha_i|,|\beta_j|\}<1$ then $\mathsf{F}_T(n)\to 0$ as $n\to\infty$, so $\mathsf{F}_T(n)=0$ for all large n, and therefore the function is degenerate (see [9, Th. 2.1]). It follows that $\max\{|\alpha_i|,|\beta_j|\}\geqslant 1$. If $\max\{|\alpha_i|\}=1$, then $\max\{|\beta_j|\}$ cannot be less than 1 since $\mathsf{F}_T(n)\geqslant 0$ for all $n\geqslant 1$ and we are done. Assume therefore that $\max\{|\alpha_i|\}>1$, and for the purposes of a contradiction assume that

$$1\leqslant\max\{|\beta_j|\}<\max\{|\alpha_i|\}.$$

Choose $\epsilon>0$ so that

$$\max\{|\beta_j|\}<\left(\max\{|\alpha_i|\}\right)^{1-\epsilon}.\quad (2.7)$$

By [10, Prop. 1] the numbers α_i and β_j are algebraic numbers (indeed, are reciprocals of algebraic integers), so that the estimates of Evertse [11] or van der Poorten and Schlickewei [12] may be applied to see that there is an $N(T,\epsilon)$ with

$$\left|\sum_{i=1}^{r}\alpha_i^n\right|\geqslant s\left(\max\{|\alpha_i|\}\right)^{n(1-\epsilon)}$$

for $n\geqslant N(T,\epsilon)$. Then, by (2.7),

$$\begin{aligned}\left|\sum_{i=1}^{r}\alpha_i^n\right| &> s\max\{|\beta_j|\}^n\quad\text{(for all large } n)\\ &\geqslant\sum_{j=1}^{s}|\beta_j|^n\quad\text{(for all large } n),\end{aligned}$$

which would make $\mathsf{F}_T(n)$ negative for large n, an impossibility.

This completes the proof, since Lemma 2.5 shows that $\varrho(\zeta_{T_j})=\max\{|\beta_i^{(j)}|\}$ for $j=1,2$ and that $\varrho(\zeta_{T_1\times T_2})$ is the product. □

The next two examples show that the relationships found in Theorem 2.4 do not hold in general.

Example 2.6. *Let $\mathcal{P} = \{p_1, p_2, \dots\}$ be the set of primes written in order, and let $\mathcal{P}_1 = \{p_2, p_4, \dots\}$, $\mathcal{P}_2 = \{p_1, p_3, \dots\}$ be the primes of even and of odd index respectively. Let T_j be a map with*

$$\mathsf{O}_{T_j}(n) = \begin{cases} A_j^n & \text{if } n \in \mathcal{P}_j; \\ 0 & \text{if not,} \end{cases}$$

for $j = 1, 2$, where $A_1 = 2$ and $A_2 = 3$. Then $\mathsf{F}_{T_j}(n) = \sum_{p \in \mathcal{P}_j; |n|_p < 1} pA_j^p$, and so

$$\mathsf{F}_{p_{2k+j}}(T_j)^{1/p_{2k+j}} \to A_j \tag{2.8}$$

as $k \to \infty$ for each $j = 1, 2$. On the other hand, a simple induction argument shows that

$$\left(a_1 A_j^{a_1} + a_2 A_j^{a_2} + \cdots + a_r A_j^{a_r}\right)^{1/a_1 a_2 \cdots a_r} \leqslant A_j$$

for distinct $a_1, \dots, a_r \geqslant 1$, so A_j is in fact the upper limit in (2.8), and $\varrho(\zeta_{T_j}) = 1/A_j$ for $j = 1, 2$. Turning to the product, let $n = n_1 n_2$, where

$$n_j = \prod_{i=1}^{u(j)} q_{i,j}^{a_{i,j}}$$

with $q_{i,j} \in \mathcal{P}_j$ and $a_{i,j} > 0$. Then

$$\mathsf{F}_{T_1 \times T_2}(n) = \left(\sum_{i=1}^{u(1)} s_{i,1} 2^{s_{i,1}}\right) \left(\sum_{i=1}^{u(1)} s_{i,2} 3^{s_{i,2}}\right),$$

and straightforward estimates show that

$$\limsup_{n \to \infty} \mathsf{F}_{T_1 \times T_2}(n)^{1/n} < 6.$$

Thus, for this example, $\varrho(\zeta_{T_1 \times T_2}) < \varrho(\zeta_{T_1}) \varrho(\zeta_{T_2})$.

Example 2.7. *The map T_1 from Example 2.6 has 2^n orbits of length n if $n \in \mathcal{P}_1$, and none otherwise, and we have seen in (2.8) that $\varrho(\zeta_{T_1}) = \frac{1}{2}$. On the other hand, $\mathsf{F}_{T_1^2}(n) = \mathsf{F}_{T_1}(2n) = \sum_{p \in \mathcal{P}_1; |n|_p < 1} p2^p$, so $\varrho(\zeta_{T_1^2}) = \frac{1}{2}$ also.*

Example 2.1 has $\sigma(\mathsf{d}_{T_1 \times T_2}) - (\sigma(\mathsf{d}_{T_1}) + \sigma(\mathsf{d}_{T_2})) = 1$; some simple estimates show that this discrepancy cannot be any larger.

Proposition 2.8. *For any maps T_1 and T_2 for which $\sigma(\mathsf{d}_{T_1})$ and $\sigma(\mathsf{d}_{T_2})$ exist, we have*

$$\sigma(\mathsf{d}_{T_1 \times T_2}) \leqslant \sigma(\mathsf{d}_{T_1}) + \sigma(\mathsf{d}_{T_2}) + 1.$$

Proof. Let $\sigma_j = \sigma(T_j)$ for $j = 1, 2$. Then, for any $\epsilon > 0$, $\mathsf{d}_{T_j}(\sigma_j + \epsilon) < \infty$ and so

$$\sum_{n=1}^{\infty} \frac{\mathsf{F}_{T_j}(n)}{n^{1+\sigma_j+\epsilon}} < \infty$$

for $j = 1, 2$ by (2.1). Thus

$$\sum_{n=1}^{\infty} \frac{\mathsf{F}_{T_1\times T_2}(n)}{n^{2+\sigma_1+\sigma_2+2\epsilon}} < \infty,$$

and therefore $\mathsf{d}_{T_1\times T_2}(1 + \sigma_1 + \sigma_2 + 2\epsilon) < \infty$ by (2.1) again. □

3 Higher Products

Even in the simplest of situations, higher products have quite subtle combinatorial and analytic properties, and for simplicity we will shortly restrict attention to the case of a map with a single orbit of each length. Similar methods will apply to maps for which the sequence $(\mathsf{O}_T(n))$ is multiplicative.

Proposition 3.1. *Let T be a map with $\mathsf{O}_T(n) = n^a$ for all $n \geqslant 1$ for some integer $a \geqslant 0$. Then $\mathsf{O}_{T\times\cdots\times T}(n)$ is equal to*

$$\prod_{p|n} \frac{1}{(p^{a+1}-1)^{m-1}} \left(\sum_{r=0}^{m-1} (-1)^r \binom{m}{m-r} p^{((m-r)(a+1)-1)\operatorname{ord}_p(n)} \sum_{j=0}^{(m-r)(a+1)-1} p^j \right)$$

where there are m terms in the Cartesian product.

Proof. We have $\mathsf{F}_T(n) = \sigma_{a+1}(d)$ and fixed points for iterates simply multiply for Cartesian products so, for a prime p and $k \geqslant 1$, by (2.1),

$$\begin{aligned}\mathsf{O}_{T\times\cdots\times T}(p^k) &= \frac{1}{p^k} \sum_{d|p^k} \mu(p^k/d)(\sigma_{a+1}(d))^m \\ &= \frac{1}{p^k} \left(\left(\tfrac{p^{(a+1)(k+1)}-1}{p^{a+1}-1} \right)^m - \left(\tfrac{p^{(a+1)k}-1}{p^{a+1}-1} \right)^m \right).\end{aligned}$$

Clearly $n \mapsto \mathsf{O}_{T\times\cdots\times T}(n)$ is multiplicative, so this proves the proposition. □

Proposition 3.1 allows the orbit Dirichlet series for higher powers to be computed (in the case $\mathsf{d}_T(s) = \zeta(s)$, with trivial changes for $\mathsf{O}_T(s)$ polynomial). To this end, assume that $\mathsf{d}_T(s) = \zeta(s - a)$, let $f(n) = \mathsf{O}_{T\times\cdots\times T}(n)$, write

$$\mathsf{d}_{T\times\cdots\times T}(s) = \prod_{p\in\mathcal{P}} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right) = \prod_{p\in\mathcal{P}} E_p(s),$$

and define θ by

$$f(p^k) = \frac{1}{(p^{a+1}-1)^{m-1}} \theta(p^k).$$

Then

$$E_p(s) = 1 + \frac{1}{(p^{a+1}-1)^{m-1}} \sum_{b=0}^{m-1} A_b p^{(b+1)a+b-s} \frac{1}{1-p^{(b+1)a+b-s}},$$

where $A_b = (-1)^r \binom{m}{m-r} \sum_{j=0}^{(b+1)a+b} p^j$, and $m-r-1=b$, so by rearranging

$$E_p(s) = \frac{M_p(s)}{(1-p^{a-s})(1-p^{2a+1-s})(1-p^{3a+2-s})\cdots(1-p^{ma+(m-1)-s})}$$

with $M_p(s) \neq 0$. Thus $\mathsf{d}_{T\times\cdots\times T}(s)$ is given by

$$\zeta(s-a)\zeta(s-(2a+1))\zeta(s-(3a+2))\cdots\zeta(s-(ma+m-1)) \prod_{p\in\mathcal{P}} M_p(s),$$

where $M_p(s)$ is (in principle) explicitly computable, and so

$$\sigma(\mathsf{d}_{T\times\cdots\times T}) = ma+m.$$

Example 3.2. *By Perron's theorem* [5] *we deduce that if* $\mathsf{d}_T(s) = \zeta(s)$*, then*

$$\pi_{T\times\cdots\times T}(N) \sim C_m \zeta(m)\zeta(m-1)\cdots\zeta(2)\frac{N^m}{m}$$

where $C_m = \prod_p M_p(m)$ *is an explicit constant. Thus, for example,* $\pi_T(N) \sim N$ *and* $\pi_{T\times T}(N) \sim \frac{\pi^2}{12}N^2$*, while*

$$\pi_{T\times T\times T}(N) \sim C_3 \frac{\pi^2\zeta(3)}{18} N^3,$$

where

$$C_3 = \prod_p (1+p^{-5}+2p^{-2}+2p^{-3}) = 2.835979\ldots.$$

Example 3.3. *Example* 2.2 *with* $a=b=0$ *gives*

$$\mathsf{d}_{T\times T}(s) = \frac{\zeta(s)^2\zeta(s-1)}{\zeta(2s)},$$

and the calculation above gives

$$\mathsf{d}_{T\times T\times T}(s) = \zeta(s)\zeta(s-1)\zeta(s-2)\prod_{p\in\mathcal{P}}\left(1+(2p+2)p^{-s}+p^{1-2s}\right). \tag{3.1}$$

Remark 3.4. *The Euler product* $\prod_p(1+p^{1-2s}+2p^{1-s}+2p^{-s})$ *resembles an L-function, but this is deceptive. Under the Hecke correspondence, the modular form with Fourier series* $c(0)+\sum_{n=1}^{\infty} c(n)e^{2\pi i n\tau}$ *has associated Dirichlet series*

$$\sum_{n=1}^{\infty}\frac{c(n)}{n^s} = \prod_p \left(1-c(p)p^{-s}+p^{2k-1}p^{-2s}\right)^{-1}.$$

However, there is no real connection because of the Weil bounds $|r_2| = |r_2| = \sqrt{p}$ *where* $1-c(p)x+p^{2k-1}x^2 = (1-r_1x)(1-r_2x)$*, which clearly do not hold here.*

4 Natural Boundaries

Estermann's theorem [13] gives a large class of Euler products of the form

$$\prod_p h(p^{-s})$$

with natural boundaries. The example below is closer to the work of Grunewald, du Sautoy and Woodward [14, 15] on zeta functions for subgroup growth, where products of 'ghost' polynomials are used to exhibit natural boundaries for Euler products of the form $\prod_p h(p^{-s}, p)$. Natural boundaries also arise for dynamical zeta functions in several natural dynamical settings, including certain random maps [16] and automorphisms of certain solenoids [17].

We exhibit a natural boundary for a specific case, but the appearance of a natural boundary for triple (and higher) products of systems with polynomial orbit growth is a widespread phenomena.

Theorem 4.1. *If $\mathsf{d}_T(s) = \zeta(s)$, then $\mathsf{d}_{T\times T\times T}(s)$ has abscissa of convergence at 3, a meromorphic extension to $\Re(s) > 1$, and a natural boundary at $\Re(s) = 1$.*

Proof. By (3.1) we know that

$$\mathsf{d}_{T\times T\times T}(s) = \zeta(s)\zeta(s-1)\zeta(s-2)\prod_{p\in\mathbb{P}} f(p^{-s}, p)$$

where $f(p^{-s}, p) = (1 + (2p+2)p^{-s} + p^{1-2s})$. The term $\prod_p f(p^{-s}, p)$ converges for $\Re(s) > 2$, so the abscissa of convergence is determined by the term $\zeta(s-2)$.

We first show that there is a meromorphic extension to the half-plane $\Re(s) > 1$, and here we follow the methods of [15]. Using the lexicographic ordering on $\mathbb{N}^2$ to eliminate terms in ascending powers of x and then y, there is a unique decomposition

$$f(x, y) = \prod_{(m,n)\in\mathbb{N}^2} (1 - x^m y^n)^{c(m,n)}$$

with $c(m, n) \in \mathbb{Z}$, where $f(x, y) = 1 + 2x + 2xy + yx^2$. This may be constructed using factors of the shape $(1 - x^a y^b)^e$ to eliminate a term $-ex^a y^b$ ($e > 0$) and factors of the shape $(1 - x^{2a} y^{2b})^e (1 - x^a y^b)^{-e}$ to eliminate a term $ex^a y^b$ ($e > 0$), obtaining an approximation valid to larger and larger powers of x by induction. Thus, for example, we find that

$$f(x, y) = (1 - x^2)^3(1 - x)^{-2}(1 - x^2y^2)^2(1 - xy)^{-2}(1 - x^2y)^3(1 - x^2y^2) + O(x^3).$$

By construction, if

$$f(x, y) = \prod_{m\leqslant M} (1 - x^m y^n)^{c(m,n)} + \sum_{m>M} e(m, n)x^m y^n$$

then if $c(m,n)$ and $e(m,n)$ are non-zero we must have $n \leqslant m$. Thus for each M and $\Re(s) > \max\{(n+1)/m \mid e(m,n) \neq 0\}$ the product

$$f_M(s) = \prod_p \left(1 + \frac{\sum_{m>M} e(m,n)p^{n-ms}}{\prod_p (1-p^{n-ms})^{c(m,n)}}\right)$$

converges absolutely, allowing $\prod_p f(p^{-s}, p)$ to be defined there by

$$\prod_{(m,n)\in\mathbb{N}^2, m\leqslant M} \zeta(ms-n)^{-c(m,n)} f_M(s).$$

Letting $M \to \infty$ gives a meromorphic extension to $\Re(s) > 1$.

To show that $\Re(s) = 1$ is a natural boundary, we show that each point s with $\Re(s) = 1$ is a limit of a sequence (s_n) of zeros of $\prod_p f(p^{-s}, p)$ with $\Re(s_n) > 1$. Solving the quadratic $f(x,p) = 0$ for x gives the solutions

$$\alpha_p^{\pm} = -\left(1 + \tfrac{1}{p}\right) \pm \sqrt{\left(1 + \tfrac{1}{p}\right)^2 - \tfrac{1}{p}},$$

and the zero $\alpha_p^+ = p^{-s}$ has solutions

$$s_{n,p} = \frac{-\log|\alpha_p^+|}{\log p} + \frac{\pi \mathrm{i} + 2k\pi\mathrm{i}}{\log p}$$

for $k \in \mathbb{Z}$. Notice that $\frac{-3+\sqrt{7}}{2} < \alpha_p^+ < 0$ for all p, $\alpha_p^+ \to 0$ as $p \to \infty$, and by the binomial theorem $\alpha_p^+ \sim -\frac{1}{2p}$ for large primes p. It follows that $\Re(s_{n,p}) > 1$ and $\Re(s_{n,p}) \to 1$ as $p \to \infty$. Thus given any s with $\Re(s) = 1$ we may choose a sequence (s_{n_k,p_k}) with the properties that

1. $\Re(s_{n_k,p_k}) > 1$;
2. $s_{n_k,p_k} \to s$ as $k \to \infty$;
3. $\prod_p f(p^{-s_{n_k,p_k}}, p) = 0$ for all $k \geqslant 1$.

This shows that $\Re(s) = 1$ is a natural boundary. □

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