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On Intra-Regular Ordered $\star$-Semigroups

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Abstract : Nordahl and Scheiblich [1] considered a unary operation $\star$ on semigroups and introduced the concept of regularity on $\star$-semigroups. In this paper we impose this operation on ordered semigroups under the assumption of order preserving, i.e. if $a \geq b$ then $a^\star \geq b^\star$. Then we can characterize intra-regular ordered $\star$-semigroups. Indeed since $\star$ can be considered to be the identity mapping particularly, the results in this paper can be considered to be the extensions of some properties in ordered semigroups [2-5].

Keywords : ordered $\star$-semigroup; intra-regular ordered $\star$-semigroup; left (right) ideal; filter.
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1 Introduction

Szász [6] has shown that the ideals of a semigroup S are prime if and only if S is intra-regular and any two ideals are comparable. He also proved that an ideal of a semigroup S is prime if and only if it is both weakly prime and semiprime; and that in commutative semigroups the prime and weakly prime ideals coincide. Ordered semigroups in which the ideals are prime, weakly prime have been considered by Kehayopulu [2, 3]. Above results, which Szász presented in semigroups, are also true in case of ordered semigroups [4]. Furthermore a characterization for intra-regular ordered semigroups was done [4].

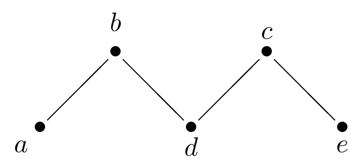
In this paper we will present analogous results on ordered $\star$-semigroups. It will be seen that the ideals requires virtually no changes from that in ordered semigroups. However in order to guarantee ideals being able to be ideals after operated by $\star$, it is necessary to assume the operator $\star$ preserves ordering. Section 2 will characterize ordered $\star$-semigroups in which all ideals are (weakly) prime. The final section is devoted to construct the concept of filters and creates a characterization on intra-regular ordered $\star$-semigroups in terms of the least filter.

An ordered semigroup S is a partial ordering set at the same semigroup such that for any $a, b, x \in S$, $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. An ordered semigroup S with a unary operation $\star : S \longrightarrow S$ is called an ordered $\star$-semigroup if it satisfies $(x^\star)^\star = x$ and $(xy)^\star = y^\star x^\star$ for any $x, y \in S$. Such a unary operation $\star$ is called an involution [1]. If for any a, b with $a \geq b$, we have $a^\star \geq b^\star$, then $\star$ is called an order preserving involution.

Example 1.1. *Let $S = \{a, b, c, d, e\}$ be an ordered semigroup. The multiplication "$\cdot$", the order "$\leq$" and the corresponding Hasse diagram are given below [4]. Define the involution $\star$ by $a^\star = e$ (hence $e^\star = a$), $b^\star = c$ and $d^\star = d$. It is easy to check that S is an ordered $\star$-semigroup with order preserving involution $\star$.*

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (d, b), (d, c), (d, d), (e, c), (e, e)\}$$

$\cdot$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c



2 Characterization of Ordered $\star$-Semigroups in which all Ideals are (Weakly) Prime

Many of the deepest properties of ordered $\star$-semigroups depend on ideals. We shall introduce the basic concepts and derive some crucial important properties. Then they will permit us to characterize ordered $\star$-semigroups.

Let S be an ordered $\star$-semigroup. For $H \subseteq S$, we denote $(H] := \{t \in S \mid t \leq h$ for some $h \in H\}$. If $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$ for convenience (cf. [5]). A non-empty subset L (resp. R) of S is called a left (resp. right) ideal of S if (1) $SL \subseteq L$ (resp. $RS \subseteq R$), and (2) $a \in L$ (resp. R), $S \ni b \leq a$ implies $b \in L$ (resp. R). I is called an ideal of S if it is both a left and a right ideal of S (cf. [5]). We denote by $L(a)$, $R(a)$ and $I(a)$ the left ideal, right ideal and the ideal of S, respectively, generated by a. Clearly $L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup SaS]$ (cf. [2, 3]).

Let S be an ordered $\star$-semigroup with order preserving involution $\star$. We will see that $L^\star$ is a right ideal for any left ideal L of S, and $R^\star$ is a left ideal for any right ideal R of S.

Proposition 2.1 (cf. [5, Lemma 1]). *Let S be an ordered $\star$-semigroup.*

1. $A \subseteq (A]$ *for any* $A \subseteq S$.
2. $(A] \subseteq (B]$ *for any* A, B *with* $A \subseteq B \subseteq S$.
3. $(A](B] \subseteq (AB]$ *for any* $A, B \subseteq S$.
4. $((A]] = (A]$ *for any* $A \subseteq S$.
5. $(T] = T$ *for any ideal* T *of* S.
6. $(AB]$ *and* $A \cap B$ *are ideals for any ideals* A, B *of* S.
7. $(SaS]$ *is an ideal for any* $a \in S$.

Proposition 2.2. *Let* S *be an ordered* $\star$*-semigroup with order preserving involution* $\star$.

1. $(bSa]^\star = (a^\star Sb^\star]$ *for any* $a, b \in S$.
2. $(SaS]^\star = (Sa^\star S]$ *for any* $a \in S$.
3. $I^\star$ *is an ideal for any ideal* I *of* S.

Proof. 1) Let $y \in (bSa]^\star$. Since $y^\star \in (bSa]$, $y^\star \leq bua$ for some $u \in S$. This implies $y \leq (bua)^\star = a^\star u^\star b^\star \in a^\star Sb^\star$ because $\star$ is an order preserving involution. Therefore $y \in (a^\star Sb^\star]$ and we get that $(bSa]^\star \subseteq (a^\star Sb^\star]$. On the other hand if $y \in (a^\star Sb^\star]$, then $y \leq a^\star ub^\star$ for some $u \in S$. Hence $y^\star \leq bu^\star a \in bSa$ because $a^\star ub^\star = (bu^\star a)^\star$. This implies $y^\star \in (bSa]$ and $y \in (bSa]^\star$. Therefore $(a^\star Sb^\star] \subseteq (bSa]^\star$. Consequently $(bSa]^\star = (a^\star Sb^\star]$.

2) The proof is handled similarly.

3) Let I be an ideal of S. Since $SI \subseteq I$, we have $(SI)^\star \subseteq (I)^\star$. Therefore $I^\star S^\star \subseteq I^\star$. Since $\star$ is an involution on S, $(x^\star)^\star = x$ for every $x \in S$, whence $S^\star = S$. Thus $I^\star S \subseteq I^\star$. Similarly since $IS \subseteq I$ we have $SI^\star \subseteq I^\star$. Let $a \in I^\star$ and $b \leq a$, then $b^\star \leq a^\star$. Observe that $a^\star \in I$ and I is an ideal. Thus $b^\star \in I$, whence $b \in I^\star$ and we conclude that $I^\star$ is an ideal of S. □

Definition 2.3. Let S be an ordered $\star$-semigroup and $T \subseteq S$. T is called *prime* if $AB \subseteq T$, then $A^\star \subseteq T$ or $B^\star \subseteq T$.

Equivalent definition: if $ab \in T$, then $a^\star \in T$ or $b^\star \in T$.

Definition 2.4. Let S be an ordered $\star$-semigroup and $T \subseteq S$. T is called *weakly prime* if for any ideals A, B of S such that $AB \subseteq T$ we have $A^\star \subseteq T$ or $B^\star \subseteq T$.

Definition 2.5. Let S be an ordered $\star$-semigroup. A subset T of S is called *semiprime* if $AA \subseteq T$, then $A^\star \subseteq T$.

Equivalent definition: if $aa \in T$, then $a^\star \in T$.

Theorem 2.6. *Let* S *be an ordered* $\star$*-semigroup with order preserving involution* $\star$. *An ideal of* S *is prime if and only if it is both weakly prime and semiprime. Furthermore, if* S *is commutative, then the prime and weakly prime ideals coincide.*

Proof. Suppose that I is a prime ideal of S. It is trivial that I is both weakly prime and semiprime.

Conversely, suppose that T is an ideal of S which is weakly prime and semiprime. Let $ab \in T$, we need to show that $a^\star \in T$ or $b^\star \in T$. First note that by Proposition 2.1 $(bSa](bSa] \subseteq (SabS] \subseteq (STS] \subseteq (T] = T$. Then T is semiprime implies that $(bSa]^\star \subseteq T$. Therefore $(Sa^\star S](Sb^\star S] \subseteq (Sa^\star SSb^\star S] \subseteq (S(a^\star Sb^\star)S] =$

$(S((Sb^\star)^\star a)^\star S] = (S(bSa)^\star S] \subseteq (S(bSa]^\star S] \subseteq (STS] \subseteq T$. Observe that $(Sa^\star S]$, $(Sb^\star S]$ are ideals, and T is weakly prime. Thus $(Sa^\star S]^\star \subseteq T$ or $(Sb^\star S]^\star \subseteq T$. Hence $(SaS] \subseteq T$ or $(SbS] \subseteq T$ by Proposition 2.2. To prove that T is prime, we just need to show that if $(SaS] \subseteq T$ then $a^\star \in T$. The other part is proved similarly.

If $(SaS] \subseteq T$ then $(I(a))^3 = (a \cup Sa \cup aS \cup SaS]^3 \subseteq ((a \cup Sa \cup aS \cup SaS)^3] \subseteq (S(a \cup Sa \cup aS \cup SaS)S] \subseteq (SaS] \subseteq T$. Thus $I(a)(I(a)I(a)] = (I(a)](I(a)I(a)] \subseteq ((I(a))^3] \subseteq (T] = T$ by Proposition 2.2. Note that T is weakly prime and $I(a)$, $(I(a)I(a)]$ are ideals. It follows that $(I(a))^\star \subseteq T$ or $(I(a)I(a)]^\star \subseteq T$. Suppose $(I(a))^\star \subseteq T$. Then $a^\star \in (I(a))^\star \subseteq T$ and we complete the proof. Suppose $(I(a)I(a)]^\star \subseteq T$. Then $a^\star a^\star \in (I(a)I(a))^\star \subseteq (I(a)I(a)]^\star \subseteq T$ because $aa \in I(a)I(a)$, whence $a = (a^\star)^\star \in T$ because T is semiprime. Now T is an ideal implies that $aa \in T$, hence $a^\star \in T$ by T is semiprime.

To prove the second statement, let T be an ideal of S. If T is prime then obviously T is weakly prime. Conversely, suppose T is weakly prime. Let $ab \in T$. Since S is commutative, we have $I(a)I(b) = (a \cup Sa \cup aS \cup SaS](b \cup Sb \cup bS \cup SbS] \subseteq ((a \cup Sa \cup aS \cup SaS)(b \cup Sb \cup bS \cup SbS)] \subseteq (ab \cup Sab]$. Observe that $(ab \cup Sab] \subseteq (T] = T$. Hence $I(a)I(b) \subseteq T$, and conclude that $(I(a))^\star \in T$ or $(I(b))^\star \in T$ by T is weakly prime. Therefore $a^\star \in T$ or $b^\star \in T$; that is, T is prime. □

Proposition 2.7. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. The following statements are equivalent:*

1. $(A^\star A^\star] = A$ *for any ideal A of S;*
2. $A^\star \cap B^\star = (AB]$ *for any ideals A, B of S;*
3. $I(a) \cap I(b) = ((I(a))^\star (I(b))^\star]$ *for any $a, b \in S$;*
4. $I(a) = (I(a^\star)I(a^\star)]$ *for any $a \in S$;*
5. $a \in (Sa^\star Sa^\star S]$ *for any $a \in S$.*

Proof. 1)$\Longrightarrow$ 2). Since $A^\star$ and $B^\star$ are ideals, by hypothesis and Proposition 2.1 we have $(AB] \subseteq (AS] \subseteq (A] = ((A^\star A^\star]] = (A^\star A^\star] \subseteq (A^\star] = A^\star$. Similarly $(AB] \subseteq (SB] \subseteq (B] = ((B^\star B^\star]] = (B^\star B^\star] \subseteq (B^\star] = B^\star$. Thus $(AB] \subseteq A^\star \cap B^\star$. Furthermore $A^\star \cap B^\star$ is an ideal implies that $A^\star \cap B^\star = ((A^\star \cap B^\star)^\star (A^\star \cap B^\star)^\star] = ((A \cap B)(A \cap B)] \subseteq (AB]$. Therefore we have $(AB] \subseteq A^\star \cap B^\star$ and $A^\star \cap B^\star \subseteq (AB]$. So $A^\star \cap B^\star = (AB]$.

2)$\Longrightarrow$ 3). Proposition 2.2 implies that $(I(a))^\star$ and $(I(b))^\star$ are ideals. Then the statement is clear by this fact.

3)$\Longrightarrow$ 4). Since $I(a) = ((I(a))^\star (I(a))^\star]$ by hypothesis, we just need to show that $(I(a))^\star = I(a^\star)$. Clearly $a^\star \in (I(a))^\star$. Hence $I(a^\star) \subseteq (I(a))^\star$ because $(I(a))^\star$ is an ideal. Now let $x \in (I(a))^\star$. We have $x^\star \in I(a) = (a \cup aS \cup Sa \cup SaS]$. This means that $x^\star \leq a$ or $x^\star \leq au$ or $x^\star \leq ua$ or $x^\star \leq uav$ for some $u, v \in S$. Thus $x \leq a^\star$ or $x \leq u^\star a^\star \in Sa^\star$ or $x \leq a^\star u^\star \in a^\star S$ or $x^\star \leq v^\star a^\star u^\star \in Sa^\star S$ for some $u^\star, v^\star \in S$, whence $x \in (a^\star]$ or $x \in (Sa^\star]$ or $x \in (a^\star S]$ or $x \in (Sa^\star S]$. Therefore $x \in (a^\star] \cup (Sa^\star] \cup (a^\star S] \cup (Sa^\star S] \subseteq (a^\star \cup Sa^\star \cup a^\star S \cup Sa^\star S] = I(a^\star)$. i.e. $(I(a))^\star \subseteq I(a^\star)$. Consequently $(I(a))^\star = I(a^\star)$.

4)$\Longrightarrow$ 5). It suffices to prove two notions. (i) $I(a) = ((I(a^\star)^6 I(a)]$. (ii) $((I(a^\star)^6 I(a)] \subseteq (Sa^\star Sa^\star S]$. Then we can conclude that $a \in I(a) \subseteq (Sa^\star Sa^\star S]$ and complete the proof.

(i) By hypothesis and Proposition 2.1, we have $I(a) = (I(a^\star)I(a^\star)] = ((I(a) I(a)](I(a)I(a)]] \subseteq ((I(a)I(a)I(a)I(a)]] = (I(a)I(a)I(a)I(a)]$. Furthermore $(I(a) I(a)I(a)I(a)] = ((I(a^\star)I(a^\star)](I(a^\star)I(a^\star)](I(a^\star)I(a^\star)](I(a)]] \subseteq ((I(a^\star))^6 I(a)] \subseteq (SI(a)] \subseteq (I(a)] = I(a)$ so that $I(a) \subseteq ((I(a^\star)^6 I(a)] \subseteq I(a)$. Thus $I(a) = ((I(a^\star)^6 I(a)]$.

(ii) Since $(I(a))^3 \subseteq (SaS]$ (has shown in Theorem 2.6), we have $(I(a))^5 = (I(a))^3 I(a)I(a) \subseteq (SaS](a \cup aS \cup Sa \cup SaS](S] \subseteq (SaS(a \cup aS \cup Sa \cup SaS)S]$. Clearly $S(a \cup aS \cup Sa \cup SaS)S \subseteq SaS$, whence $(SaS(a \cup aS \cup Sa \cup SaS)S] \subseteq (SaSSaS] \subseteq (SaSaS]$. Therefore $(I(a))^5 \subseteq (SaSaS]$ and hence $(I(a^\star))^5 \subseteq (Sa^\star Sa^\star S]$. Finally we conclude that $((I(a^\star))^6 I(a)] \subseteq ((Sa^\star Sa^\star S]I(a^\star)I(a)] \subseteq ((Sa^\star Sa^\star S](S]] \subseteq (Sa^\star Sa^\star SS] \subseteq (Sa^\star Sa^\star S]$, and hence $((I(a^\star)^6 I(a)] \subseteq (Sa^\star Sa^\star S]$.

5)$\Longrightarrow$ 1). If $x \in (A^\star A^\star]$, then $x \le yz$ for some $y, z \in A^\star$. By hypothesis $y \in (Sy^\star Sy^\star S]$, then $y \le u_1 y^\star u_2 y^\star u_3$ for some $u_i \in S$, $i = 1, 2, 3$. Similarly, $z \le v_1 z^\star v_2 z^\star v_3$ for some $v_i \in S$, $i = 1, 2, 3$. Consequently, $yz \le u_1 y^\star u_2 y^\star u_3 v_1 z^\star v_2 z^\star v_3 \in Sy^\star S \subseteq SAS \subseteq A$. Therefore $x \in (A]$ because $x \le yz$, whence $(A^\star A^\star] \subseteq (A] = A$. If $x \in A$, then we have $x \le w_1 x^\star w_2 x^\star w_3$ for some $w_i \in S$, $i = 1, 2, 3$ because $x \in (Sx^\star Sx^\star S]$. Clearly $w_1 x^\star w_2 \in A^\star$ and $x^\star w_3 \in A^\star$ since $A^\star$ is an ideal of S by Proposition 2.2. Therefore $x \le w_1 x^\star w_2 x^\star w_3 \in A^\star A^\star$, whence $A \subseteq (A^\star A^\star]$. Hence $A = (A^\star A^\star]$. □

Theorem 2.8. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. The ideals of S are weakly prime if and only if $A^\star = (AA]$ for any ideal A of S and any two ideals are comparable under the inclusion relation $\subseteq$.*

Proof. Suppose that the ideals of S are weakly prime. Let A, B be any ideals of S. Note that $B^\star$ is an ideal and $(AB^\star]$ is weakly prime. Thus $AB^\star \subseteq (AB^\star]$ implies that $A^\star \subseteq (AB^\star]$ or $B \subseteq (AB^\star]$. If $A^\star \subseteq (AB^\star]$, then $A^\star \subseteq (SB^\star] \subseteq (B^\star] = B^\star$ and hence $(A^\star)^\star \subseteq (B^\star)^\star$; that is, $A \subseteq B$. If $B \subseteq (AB^\star]$, then $B \subseteq (AS] \subseteq (A] = A$. The conclusion now follows that A and B are comparable.

Next we claim that $A^\star = (AA]$. Since $(AA]$ is weakly prime and $AA \subseteq (AA]$, we have $A^\star \subseteq (AA]$. On the other hand let $x \in (AA]$. Then $x \le a_1 a_2 \in AA$ for some $a_1, a_2 \in A$. Since $A^\star \subseteq (AA]$, we have $a_1^\star \le u_1 v_1 \in AA$ and $a_2^\star \le u_2 v_2 \in AA$ for some $u_1, u_2, v_1, v_2 \in A$. Thus $a_1 \le (u_1 v_1)^\star$ and $a_2 \le (u_2 v_2)^\star$. This implies that $x \le a_1 a_2 \le (u_1 v_1)^\star (u_2 v_2)^\star \in (AA)^\star (AA)^\star = A^\star A^\star A^\star A^\star \subseteq A^\star$ because $A^\star$ is an ideal. Consequently $x \in (A^\star] = A^\star$. Therefore $(AA] \subseteq A^\star$.

Conversely, let A, B and T be ideals of S with $AB \subseteq T$. Since $A^\star = (AA]$, we have $A^\star \cap B^\star = (AB]$ by Proposition 2.7. Since A and B are comparable, there are two cases. If $A \subseteq B$, then $A^\star \subseteq B^\star$, whence $A^\star = A^\star \cap B^\star = (AB] \subseteq (T] = T$ by Proposition 2.7. On the other hand if $B \subseteq A$, then $B^\star \subseteq A^\star$, whence $B^\star = A^\star \cap B^\star = (AB] \subseteq (T] = T$. Thus T is weakly prime. □

Definition 2.9. An ordered $\star$-semigroup S is called *intra-regular* if $a \in (Sa^\star a^\star S]$ for any $a \in S$.

Proposition 2.10. *Let S be an ordered $\star$-semigroup. Then S is intra-regular if and only if the ideals of S are semiprime.*

Proof. Suppose that I is an ideal of S with $aa \in I$ for some $a \in S$. Since S is intra-regular, we have $a^\star \in (SaaS] \subseteq (SIS] \subseteq (I] = I$ and hence I is semiprime.

Conversely, suppose that a is an element of S. Clearly $(Sa^\star a^\star S]$ is an ideal. So $(Sa^\star a^\star S]$ is semiprime by hypothesis. This implies $aa = (a^\star a^\star)^\star \in (Sa^\star a^\star S]$ because $(a^\star a^\star)(a^\star a^\star) \in Sa^\star a^\star S \subseteq (Sa^\star a^\star S]$. Therefore $a^\star \in (Sa^\star a^\star S]$ whence it follows that $a^\star a^\star \in (Sa^\star a^\star S]$. Hence $a \in (Sa^\star a^\star S]$ and we conclude that S is intra-regular. □

Proposition 2.11. *Let S be an ordered $\star$-semigroup. If S is intra-regular, then $(SxyS] = (Sx^\star y^\star S]$ for any $x, y \in S$.*

Proof. Let x, $y \in S$. Since S is intra-regular, we have $xy \in (S(xy)^\star(xy)^\star S] = (Sy^\star x^\star y^\star x^\star S] \subseteq (Sx^\star y^\star S]$. Thus $xy \le u_1x^\star y^\star u_2$ for some $u_1, u_2 \in S$. Hence $u_3xyu_4 \le u_3u_1x^\star y^\star u_2u_4 \in Sx^\star y^\star S \subseteq (Sx^\star y^\star S]$ for any $u_3, u_4 \in S$. This implies $SxyS \subseteq (Sx^\star y^\star S]$, so $(SxyS] \subseteq ((Sx^\star y^\star S]] = (Sx^\star y^\star S]$ by Proposition 2.1. By symmetry we have $(Sx^\star y^\star S] \subseteq (SxyS]$. Therefore $(SxyS] = (Sx^\star y^\star S]$. □

Proposition 2.12. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. If the ideals of S are semiprime, then*

1. *$I(x) = (SxS]$ for any $x \in S$, and*
2. *$I(xy) = I(x) \cap I(y)$ for any $x, y \in S$.*

Proof. 1) Let x be an element in S. Note that $(SxS]$ is an ideal whence is semiprime. Applying this fact and $x^2x^2 = x^4 \in (SxS]$ yields $x^\star x^\star = (x^2)^* \in (SxS]$. Similarly $x \in (SxS]$ so that $I(x) \subseteq (SxS]$. Furthermore $(SxS] \subseteq (x \cup xS \cup Sx \cup SxS] = I(x)$. Hence $I(x) = (SxS]$.

2) Since $xy \in I(x)S \subseteq I(x)$, we have $I(xy) \subseteq I(x)$. Also $I(xy) \subseteq I(y)$ because $xy \in SI(y) \subseteq I(y)$. Thus $I(xy) \subseteq I(x) \cap I(y)$.

We now show that $I(x) \cap I(y) \subseteq I(xy)$. If $z \in I(x) \cap I(y)$, then $z \in (SxS] \cap (SyS]$ by 1), whence $z \le u_1xu_2$ and $z \le v_1yv_2$ for some $u_1, u_2, v_1, v_2 \in S$. Note that $(yv_2u_1x)^2 = yv_2u_1xyv_2u_1x \in (SxyS] = I(xy)$ and that $I(xy)$ is semiprime. Thus $(yv_2u_1x)^\star \in I(xy)$. Therefore $z^\star z^\star \le (u_1xu_2)^\star(v_1yv_2)^\star = u_2^\star(yv_2u_1x)^\star v_1^\star \in I(xy)$, whence $z^\star z^\star \in (I(xy)] = I(xy)$. It follows that $z \in I(xy)$, then $I(x) \cap I(y) \subseteq I(xy)$. □

Theorem 2.13. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. The ideals of S are prime if and only if S is intra-regular and any two ideals are comparable under the inclusion relation $\subseteq$.*

Proof. If the ideals are prime, then they are weakly prime, and hence Theorem 2.8 implies that any two ideals are comparable. Let $a \in S$. Note that $(Sa^\star a^\star S]$ is an ideal by Proposition 2.1, whence is prime. Therefore $a^4 \in (Sa^\star a^\star S]$ because $(a^\star)^4(a^\star)^4 \in (Sa^\star a^\star S]$. A similar argument shows $(a^\star)^2 \in (Sa^\star a^\star S]$ and $a \in (Sa^\star a^\star S]$; that is, S is intra-regular.

Conversely suppose that S is intra-regular and any two ideals are comparable under the inclusion relation $\subseteq$. Let T be an ideal of S and $ab \in T$. We claim that $a^\star \in T$ or $b^\star \in T$. By virtue of Proposition 2.10, $I(a)$ is semiprime. Thus $aa \in I(a)$ implies $a^\star \in I(a)$. $b^\star \in I(b)$ is proved similarly. Furthermore by hypothesis we have either $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$, then $a^\star \in I(a) = I(a) \cap I(b) = I(ab) \subseteq T$ by Proposition 2.12. If $I(b) \subseteq I(a)$, then $b^\star \in I(b) = I(a) \cap I(b) = I(ab) \subseteq T$. □

3 Characterization of Intra-Regular Ordered $\star$-Semigroups

In Section 2 we considered ideals. In this section we shall introduce the notion of filters which will be used to establish some congruence. Once some properties are well made it is not difficult to establish the characterization. For convenience we define $a\mathcal{I}b$ if and only if $I(a) = I(b)$.

Definition 3.1. Let S be an ordered $\star$-semigroup. A subsemigroup F of S is called a *filter* if

1. for any $a, b \in S$, $ab \in F$ implies $a^\star \in F$ and $b^\star \in F$,
2. for any $a \in F$, $c \in S$, $c \geq a$ implies $c \in F$.

Let $N(x)$ be the least filter of S containing x and $\mathcal{N}$ be defined by $\mathcal{N} := \{(x, y) \in S \times S \mid N(x) = N(y)\}$. A congruence on ordered $\star$-semigroup S is an equivalence relation σ on S which preserves both $\cdot$ and $\star$. In other words, if $(a, b) \in \sigma$, then $(a^\star, b^\star) \in \sigma$ [1].

Definition 3.2. A congruence σ on ordered $\star$-semigroup S is called *semilattice congruence* if $(a^\star a^\star, a) \in \sigma$ and $(ab, ba) \in \sigma$ for any $a, b \in S$. A semilattice congruence σ on S is called *complete* if $a \leq b$ implies $(a, ab) \in \sigma$.

Proposition 3.3. *Let S be an ordered $\star$-semigroup. Then the relation $\mathcal{N}$ is a complete semilattice congruence on S.*

Proof. Trivially $\mathcal{N}$ is an equivalence relation on S. Let $(a, b) \in \mathcal{N}$. In order to show that $\mathcal{N}$ is a congruence, it suffices to prove that $(ac, bc) \in \mathcal{N}$ for any $c \in S$ since $(ca, cb) \in \mathcal{N}$ can be proved similarly. If $N(ac) = N(ab)$, $N(bc) = N(ba)$ and $N(ab) = N(ba)$ for any $c \in S$, then $N(ac) = N(bc)$, whence $(ac, bc) \in \mathcal{N}$.

We first show that $N(ac) = N(ab)$. Obviously $ac \in N(ac)$. Thus $a^\star \in N(ac)$ and hence $a^\star a^\star \in N(ac)$. It follows that $a \in N(ac)$, whence $N(a) \subseteq N(ac)$. Therefore $b \in N(b) = N(a) \subseteq N(ac)$ because $(a, b) \in \mathcal{N}$. Consequently $ab \in N(ac)$ and $N(ab) \subseteq N(ac)$. $N(ac) \subseteq N(ab)$ is proved similarly.

$N(bc) = N(ba)$ is completed by similar arguments.

Next we show that $N(ab) = N(ba)$. Since $ab \in N(ab)$, we have $a^\star, b^\star \in N(ab)$ by Definition 3.1. Then $a^\star a^\star, b^\star b^\star \in N(ab)$ because $N(ab)$ is a subsemigroup.

Again $a, b \in N(ab)$ follows directly from Definition 3.1. Hence $ba \in N(ab)$ and $N(ba) \subseteq N(ab)$. Similarly $N(ab) \subseteq N(ba)$.

Now we turn to prove that $\mathcal{N}$ is a semilattice congruence. In addition to the fact that $N(ab) = N(ba)$ we shall need to show that $(a, a^\star a^\star) \in \mathcal{N}$. Clearly $aa \in N(a)$, and hence $a^\star \in N(a)$. Therefore $N(a^\star a^\star) \subseteq N(a)$ because $a^\star a^\star \in N(a)$. On the other hand $a^\star a^\star \in N(a^\star a^\star)$, whence $a \in N(a^\star a^\star)$. This implies $N(a) \subseteq N(a^\star a^\star)$. Consequently $N(a) = N(a^\star a^\star)$; that is, $(a, a^\star a^\star) \in \mathcal{N}$.

To complete the proof we claim that $a \leq b$ implies $(a, ab) \in \mathcal{N}$. Observe that $ab \in N(ab)$, whence $a^\star, b^\star \in N(ab)$ and $a^\star a^\star \in N(ab)$. Therefore $a \in N(ab)$, whence $N(a) \subseteq N(ab)$. Furthermore since $a \leq b$ and $a \in N(a)$, this implies that $b \in N(a)$ by Definition 3.1. Thus $ab \in N(a)$, whence $N(ab) \subseteq N(a)$. We conclude that $N(a) = N(ab)$, and $(a, ab) \in \mathcal{N}$. □

Proposition 3.4. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. Then S is intra-regular if and only if $N(x) = \{y \in S \mid x \in (Sy^\star S]\}$.*

Proof. Suppose S is intra-regular. Let $T_x = \{y \in S \mid x \in (Sy^\star S]\}$ for any $x \in S$. We shall show that T_x is a filter, then claim that $T_x \subseteq F$ for any filter F containing x. To show that T_x is a filter, we first prove that T_x is a subsemigroup. Let a, $b \in T_x$. Then $x \in (Sx^\star x^\star S]$ since S is intra-regular. Thus $x \leq v_1 x^\star x^\star v_2$ for some $v_1, v_2 \in S$, and $x \in (Sx^\star S]$. By definition $x \in T_x$, whence $T_x \neq \emptyset$. Let $a, b \in T_x$. Then $x \in (Sa^\star S]$ and $x \in (Sb^\star S]$, hence $x \leq u_1 a^\star u_2$ and $x \leq u_3 b^\star u_4$ for some $u_i \in S$, $i = 1, \ldots, 4$. This implies that $x^\star \leq u_2^\star a u_1^\star$ and $x^\star \leq u_4^\star b u_3^\star$ because $\star$ is an order preserving involution. Note that $x \in (Sx^\star x^\star S]$ and therefore $x \leq u_5 x^\star x^\star u_6$ for some $u_5, u_6 \in S$. Consequently $x \leq u_5(u_2^\star a u_1^\star)(u_4^\star b u_3^\star)u_6 = u_5 u_2^\star(a u_1^\star u_4^\star b)u_3^\star u_6$. Furthermore S is intra-regular implies that $a u_1^\star u_4^\star b \leq u_7(a u_1^\star u_4^\star b)^\star(a u_1^\star u_4^\star b)^\star u_8 = u_7(b^\star u_4 u_1 a^\star)(b^\star u_4 u_1 a^\star)u_8 = u_7 b^\star u_4 u_1 (ba)^\star u_4 u_1 a^\star u_8$ for some $u_7, u_8 \in S$. We finally obtain that $x \leq u_5 u_2^\star(u_7 b^\star u_4 u_1 (ba)^\star u_4 u_1 a^\star u_8)u_3^\star u_6 = u_5 u_2^\star u_7 b^\star u_4 u_1 (ba)^\star u_4 u_1 a^\star u_8 u_3^\star u_6 \in S(ba)^\star S$. This means that $x \in (S(ba)^\star S]$, and $ba \in T_x$. The similar proof shows that $ab \in T_x$.

To complete the proof that T_x is a filter, we prove (i) for any $a, b \in S$, $ab \in T_x$ implies $a^\star \in T_x$ and $b^\star \in T_x$, (ii) for any $a \in T_x$, $c \in S$, $c \geq a$ implies $c \in T_x$.

(i) Since $ab \in T_x$, then $x \in (S(ab)^\star S] = (Sb^\star a^\star S]$, so $x \leq u_1 b^\star a^\star u_2$ for some $u_1, u_2 \in S$. Also $b^\star \leq u_3 bbu_4$ for some $u_3, u_4 \in S$ because $b^\star \in S$ and S is intra-regular. Hence $x \leq u_1(u_3 bbu_4)a^\star u_2 = u_1 u_3 bbu_4 a^\star u_2 \in SbS = S(b^\star)^\star S$. So $x \in (S(b^\star)^\star S]$. Thus $b^\star \in T_x$. Similarly $a^\star \in T_x$. (ii) Since $a \in T_x$, we have $x \in (Sa^\star S]$. Thus $x \leq u_1 a^\star u_2$ for some $u_1, u_2 \in S$. Therefore $x \leq u_1 a^\star u_2 \leq u_1 b^\star u_2 \in Sb^\star S$ because $a^\star \leq b^\star$. So $x \in (Sb^\star S]$, and $b \in T_x$.

Now we claim that T_x is the least filter containing x, i.e. $T_x = N(x)$. Let F be a filter of S containing x and t be an element of T_x. By definition $x \in (St^\star S]$, then $x \leq u_1 t^\star u_2$ for some $u_1, u_2 \in S$. Since S is intra-regular, this implies $t \leq u_3 t^\star t^\star u_4$ for some $u_3, u_4 \in S$. Thus $t^\star \leq u_4^\star t^2 u_3^\star$ and $x \leq u_1 t^\star u_2 \leq u_1(u_4^\star t^2 u_3^\star)u_2 = u_1 u_4^\star t^2 u_3^\star u_2$. Therefore $u_1 u_4^\star t^2 u_3^\star u_2 = (u_1 u_4^\star t^2)(u_3^\star u_2) \in F$ because F is a filter containing x. Definition 3.1 implies $(u_1 u_4^\star t^2)^\star = t^\star t^\star u_4 u_1^\star = (t^\star)(t^\star u_4 u_1^\star) \in F$. Again $t = (t^\star)^\star \in F$ by the same reason. We conclude that $T_x \subseteq F$, whence T_x is the filter generated by x.

Conversely, suppose $N(x) = \{y \in S \mid x \in (Sy^\star S]\}$. Let $x \in S$. Observe that $N(x)$ is a subsemigroup and $x \in N(x)$. Thus $x^2 \in \{y \in S \mid x \in (Sy^\star S]\}$. This implies $x \in (S(x^2)^\star S] = (Sx^\star x^\star S]$, i.e. S is intra-regular. □

Theorem 3.5. *Let S be an ordered $\star$-semigroup with order preserving involution $\star$. Then S is intra-regular if and only if $\mathcal{N} = \mathcal{I}$.*

Proof. Suppose that S is intra-regular. To show that $\mathcal{I} \subseteq \mathcal{N}$ we let $(a,b) \in \mathcal{I}$ and $x \in N(a)$. Since $I(a) = I(b)$, we have $a \le u_1x^\star u_2$ for some $u_1, u_2 \in S$ by Proposition 3.4. Furthermore $b \in (a \cup aS \cup Sa \cup SaS]$ because $b \in I(b)$. Thus $b \le a$ or $b \le au_3$ or $b \le u_3a$ or $b \le u_3au_4$ for some $u_3, u_4 \in S$. This implies that $b \le u_1x^\star u_2 \in Sx^\star S$ or $b \le u_1x^\star u_2u_3 \in Sx^\star S$ or $b \le u_3u_1x^\star u_2 \in Sx^\star S$ or $b \le u_3u_1x^\star u_2u_4 \in Sx^\star S$. Hence $b \in (Sx^\star S]$, whence $x \in \{y \in S \mid b \in (Sy^\star S]\} = N(b)$ by Proposition 3.4. We conclude that $N(a) \subseteq N(b)$. Similarly $N(b) \subseteq N(a)$. This means that $N(a) = N(b)$, hence $(a,b) \in \mathcal{N}$.

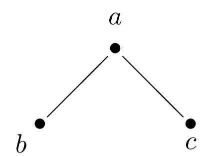
To show that $\mathcal{N} \subseteq \mathcal{I}$ we let $(a,b) \in \mathcal{N}$ and $x \in I(a)$. Note that $N(a) = N(b)$ and $I(a) = (a \cup aS \cup Sa \cup SaS]$. Then $x \le a$ or $x \le au_1$ or $x \le u_1a$ or $x \le u_1au_2$ for some $u_1, u_2 \in S$. Since $b \in N(b) = N(a) = \{y \in S \mid a \in (Sy^\star S]\}$, we get $a \le u_3b^\star u_4$ for some $u_3, u_4 \in S$. This implies that $x \le u_3b^\star u_4$ or $x \le u_3b^\star u_4u_1$ or $x \le u_1u_3b^\star u_4$ or $x \le u_1u_3b^\star u_4u_2$. Also $b^\star \le u_5b^2u_6$ for some $u_5, u_6 \in S$ because S is intra-regular. Therefore $x \le u_3(u_5b^2u_6)u_4 \in SbS$ or $x \le u_3(u_5b^2u_6)u_4u_1 \in SbS$ or $x \le u_1u_3(u_5b^2u_6)u_4 \in SbS$ or $x \le u_1u_3(u_5b^2u_6)u_4u_2 \in SbS$. Thus $x \in (SbS] \subseteq I(b)$, hence $I(a) \subseteq I(b)$. Similarly $I(b) \subseteq I(a)$. We conclude that $I(a) = I(b)$ and $(a,b) \in \mathcal{I}$.

To prove the converse let $a \in S$. Observe that $(a, a^\star a^\star) \in \mathcal{N}$ by Definition 3.2 and Proposition 3.3. Thus $\mathcal{N} = \mathcal{I}$ implies that $(a, a^\star a^\star) \in \mathcal{I}$. Therefore $a \in I(a) = I(a^\star a^\star)$, hence $a \in (a^\star a^\star \cup a^\star a^\star S \cup Sa^\star a^\star \cup Sa^\star a^\star S]$. We now consider the four possibilities: (i) $a \le a^\star a^\star$; (ii) $a \le a^\star a^\star u_1$; (iii) $a \le u_1a^\star a^\star$; (iv) $a \le u_1a^\star a^\star u_2$ for some u_1, $u_2 \in S$. In case (i) clearly $a \le a^\star a^\star \le a^\star(a^\star a^\star)^\star \le a^\star(a^\star a^\star)a \in Sa^\star a^\star S$. In case (ii) $a \le a^\star a^\star u_1 \le a^\star(a^\star a^\star u_1)^\star u_1 = a^\star u_1^\star aau_1 \le a^\star u_1^\star(a^\star a^\star u_1)au_1 \in Sa^\star a^\star S$. In case (iii) it is easy to see that $a \le u_1a^\star a^\star \le u_1a^\star(u_1a^\star a^\star)^\star = u_1a^\star aau_1^\star \le u_1a^\star(u_1a^\star a^\star)au_1^\star \in Sa^\star a^\star S$. In case (iv) $a \le u_1a^\star a^\star u_2 \in Sa^\star a^\star S$ trivially. Therefore $a \in (Sa^\star a^\star S]$; that is, S is intra-regular. □

Example 3.6. *Let $S = \{a,b,c\}$ be an ordered semigroup. The multiplication "$\cdot$", the order "$\le$" and the corresponding Hasse diagram are given below. Define the involution $\star$ by $a^\star = a$ and $b^\star = c$ (hence $c^\star = b$). It is easy to check that S is an ordered $\star$-semigroup with order preserving involution $\star$.*

$$\le := \{(a,a),(b,a),(b,b),(c,a),(c,c)\}$$

$\cdot$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c



By Definition 2.9, S is intra-regular because $(Sa^\star a^\star S] = (Sb^\star b^\star S] = (Sc^\star c^\star S] = S$. *Also, by Definition 3.1,* $N(a) = N(b) = N(c) = S$, *thus* $\mathcal{N} := \{(a,a),(b,b),(c,c),(a,b),(b,c),(a,c)\}$. *Furthermore* $I(a) = (a \cup Sa \cup aS \cup SaS]$ *implies* $I(a) = S$. *Similarly,* $I(b) = I(c) = S$. *Therefore* $\mathcal{I} := \{(a,a),(b,b),(c,c),(a,b),(b,c),(a,c)\}$, *whence* $\mathcal{N} = \mathcal{I}$.

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