The Decycling Number of Regular Graphs

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Abstract : For a graph G and $S \subseteq V(G)$, if $G - S$ is acyclic, then S is called a *decycling set* or *feedback set* of G. The *decycling number* of G is defined to be

$$\phi(G) := \min\{|S| : S \subseteq V(G) \text{ is a decycling set}\}.$$

This paper reviews some recent results on interpolation and extremal problems on the decycling number of graphs, with an emphasis on the decycling number of regular graphs.

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1 Introduction

Only finite simple graphs are considered in this paper. For the most part, our notation and terminology follows that of Bondy and Murty [3]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We will use the following notation and terminology for a typical graph G.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. We use $|S|$ to denote the cardinality of a set S and therefore we define $n = |V|$ to be the *order* of G and $m = |E|$ the *size* of G. To simplify writing, we write $e = uv$ for the edge e that joins the vertex u to the vertex v.

A *path* of length k in a graph G, denoted by P_k, is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ of G such that for all $i = 1, 2, \ldots, k-1$, $u_i u_{i+1}$ are edges of G. A u, v-path is a path which has u as its first vertex and v as its last vertex in the path.

The *degree* of a vertex v of a graph G is defined as

$$d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|.$$

The maximum degree of a graph G is usually denoted by $\Delta(G)$. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph induced by S in G.

For a graph G and $X \subseteq E(G)$, we denote $G - X$ the graph obtained from G by removing all edges in X. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$.

For a graph G and $X \subseteq V(G)$, the graph $G - X$ is the graph obtained from G by removing all vertices in X and all edges incident with vertices in X.

For a graph G and $X \subseteq E(\overline{G})$, we denote $G + X$ the graph obtained from G by adding all edges in X. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs G and H are disjoint if $V(G) \cap V(H) = \emptyset$. Any two disjoint graph G and H we define $G \cup H$, their union, by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Since the operation "$\cup$" is associative, we can extend this definition to a finite union of pairwise disjoint graphs.

A graph G is said to be *regular* if all of its vertices have the same degree. A 3-regular graph is called *cubic graph*.

Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a *degree sequence* of G, and we simply write $(d(v_1), d(v_2), \ldots, d(v_n))$ if the underlying graph G is clear from the context. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G. In this case, G is called a *realization* of $\mathbf{d}$.

An algorithm for determining whether or not a given sequence of non-negative integers is graphic was independently obtained by Havel [15] and Hakimi [7]. We state their results in the following theorem.

Theorem 1.1 *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of non-negative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'.$$

Then $\mathbf{d}$ is graphic if and only if $\mathbf{d}'$ is graphic.

Let G be a graph and $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Put $G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) + \{ac, bd\}$.

The operation $\sigma(a, b; c, d)$ is called a *switching operation*. Evidently the graph obtained from G by a switching has the same degree sequence as G. The following theorem has been shown by Havel [15] and Hakimi [7].

Theorem 1.2 *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a graphic degree sequence. If G_1 and G_2 are any two realizations of $\mathbf{d}$, then G_2 can be obtained from G_1 by a finite sequence of switchings.*

As a consequence of Theorem 1.2, Eggleton and Holton [5] defined in 1978 *the graph $\mathcal{R}(\mathbf{d})$ of realizations* of $\mathbf{d}$ whose vertices are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. They obtained the following theorem.

Theorem 1.3 *The graph $\mathcal{R}(\mathbf{d})$ is connected.*

The following theorem was shown by Taylor [22] in 1980.

Theorem 1.4 *For a graphic degree sequence $\mathbf{d}$, let $\mathcal{CR}(d)$ be the set of all connected realizations of $\mathbf{d}$. Then the induced subgraph $\mathcal{CR}(\mathbf{d})$ of $\mathcal{R}(\mathbf{d})$ is connected.*

2 The Decycling Number

The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph G is difficult even for some simply defined graphs. For a graph G, this minimum number is known as the *decycling number* of G, and is denoted by $\phi(G)$. The class of those graphs G of which $\phi(G) = 0$ consists of all forests, and $\phi(G) = 1$ if and only if G has at least one cycle and a vertex is on all of its cycles. It is also easy to see that $\phi(K_n) = n - 2$ and $K_{p,\,q} = p - 1$ if $p \leq q$, where K_n denotes the complete graph of order n and $K_{p,\,q}$ denotes the complete bipartite graph with partite sets of cardinality p and q.

Let $\mathbb{G}$ be the class of all graphs. A function $f : \mathbb{G} \to \mathbb{Z}$ is called a *graph parameter* if $G, H \in \mathbb{G}$ and $G \cong H$, then $f(G) = f(H)$. For a graph G and $S \subseteq V(G)$, if $G - S$ is acyclic, then S is called a *decycling set* or *feedback set* of G. The *decycling number* of G is defined to be

$$\phi(G) := \min\{|S| : S \subseteq V(G) \text{ is a decycling set}\}.$$

There is a counterpart graph parameter of ϕ which has been investigated in the literature, which is defined by

$$t(G) := \max\{|F| : F \subseteq V(G) \text{ is an induced forest of } G\}.$$

The graph parameter t is called the *forest number*. Thus for any graph G, $t(G)$ is the forest number of G.

It is clear that, for a graph G of order n, $\phi(G) + t(G) = n$.

Let $\mathbb{G}$ be the class of all graphs and $\mathbb{J} \subseteq \mathbb{G}$. A graph parameter f is called an *interpolation graph parameter with respect to* $\mathbb{J}$ if $G_1, G_2 \in \mathbb{J}$ and $a = f(G_1) < f(G_2) = b$, then for every integer c between a and b, there exists $G \in \mathbb{J}$ such that $f(G) = c$.

Studying interpolation theorems for graph parameters may be divided into two parts, the first part deals with the question that given a graph parameter f and a subset $\mathbb{J}$ of $\mathbb{G}$, does f interpolate over $\mathbb{J}$? If f interpolates over $\mathbb{J}$, then $\{f(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(f, \mathbb{J}) := \min\{f(G) : G \in \mathbb{J}\}$ and $\max(f, \mathbb{J}) := \max\{f(G) : G \in \mathbb{J}\}$. Thus the second part of the interpolation theorems for graph parameters is to find the values of $\min(f, \mathbb{J})$ and $\max(f, \mathbb{J})$ for the corresponding interpolation graph parameters and this part is, in fact, the extremal problem in graph theory.

The interest in the interpolation properties of graph parameters was motivated by an open question posed by Chartrand during a conference held at Kalamazoo in 1980. He posed the following question: If a graph G contains spanning trees having m and n leaves, with $m < n$, does G contain a spanning tree with k leaves for every integer k with $m < k < n$? This question (which was answered affirmatively) led to a host of papers studying the interpolation properties of invariants of spanning trees of a given graph. Details can be found in [8], [9], [10], [11], [12], [13], [14].

We proved in [17] that for a graph G and a switching σ of G, $|\phi(G) - \phi(G^\sigma)| \leq 1$. Thus we have the following results.

(i) ϕ is an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$.

(ii) ϕ is an interpolation graph parameter with respect to $\mathcal{CR}(\mathbf{d})$.

(iii) Let $\mathbb{J} \subseteq \mathcal{R}(\mathbf{d})$. If the subgraph of $\mathcal{R}(\mathbf{d})$ induced by $\mathbb{J}$ is connected, then ϕ is an interpolation graph parameter with respect to $\mathbb{J}$.

(iv) The same results are also obtained if ϕ is replaced by t.

3 Extremal Problem

An extremal problem asks for minimum and maximum values of a function over a class of objects. Consider the following simple example.

Example 3.1 In the class $\mathcal{C}_n$ of connected graphs of order n and for $G \in \mathcal{C}_n$, $f(G)$ is defined to be the number of edges of G. Thus the minimum and maximum values of f in $\mathcal{C}_n$ are $n-1$ and $\binom{n}{2}$, respectively.

It is easy to see that f is an interpolation graph parameter with respect to $\mathcal{C}_n$.

Remark 3.2 Proving that A is the minimum of $f(G)$ for graphs in a class $\mathbb{J}$ requires showing two things:

(i) $f(G) \geq A$ for all $G \in \mathbb{J}$.

(ii) $f(G) = A$ for some $G \in \mathbb{J}$.

The proof of the bound must apply to every $G \in \mathbb{J}$. For equality it suffices to obtain an example in $\mathbb{J}$ with the desired value of f.

Changing " $\geq$ " to " $\leq$ " yields the criteria for a maximum.

The following theorem is an another example of an extremal problem.

Theorem 3.3 *Every graph G with m edges has a bipartite subgraph of at least $\frac{m}{2}$ edges.*

We are now seeking for the maximum number of edges in a graph with n vertices with no triangle. Bipartite graphs have no triangles, but also many non-bipartite graphs have no triangles as well. Using extremallity, we will prove that the maximum is indeed achieved by a complete bipartite graph.

Let H be a graph. A graph G is called an *H-free graph* if G has no subgraph isomorphic to H. Let X be a set of graphs. A graph G is called *X-free graph* if G is an H-free graph for every $H \in X$.

Theorem 3.4 (Mantel (1907) cited from [23]) *The maximum number of edges in a triangle-free graph with n vertices is $\lfloor n^2/4 \rfloor$.*

$\square$

From Theorem 3.4, it is reasonable to ask the following question. What is the maximum number of edges that a K_{r+1}-free graph of order n can have? Turán (1910-1976) answered this question. This famous result of Turán generalizes Theorem 3.4 and is viewed as the origin of *Extremal Graph Theory*.

The *Turán graph $T_{n,r}$* is the complete r-partite graph with n vertices whose partite sets differ in size by at most 1. By the pigeonhole principle, some partite set has size at least $\lfloor n/r \rfloor$ and some has size at most $\lceil n/r \rceil$. Therefore, differing by at most 1 means that they all have size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.

Let $a = \lfloor n/r \rfloor$. After putting a vertices in each partite set, $b = n - ra$ remain, so $T_{n,r}$ has b partite sets of size $a + 1$ and $r - b$ partite sets of size a. Thus the defining condition on $T_{n,r}$ specifies a single isomorphism class.

Lemma 3.5 *Among r-partite graphs with n vertices, the Turán graph is the unique graph with the most edges.*

The following extremal theorem proved by Turán (1941, cited from [23]).

Theorem 3.6 *Among the graphs of order n with no $(r + 1)$-clique, $T_{n,r}$ has the maximum number of edges.*

We solved in [17] the extremal problems for ϕ in the classes $\mathcal{R}(r^n)$ and $\mathcal{CR}(r^n)$ and with these notation,

   (i) $\min(\phi, r^n) := \min\{f(G) : G \in \mathcal{R}(r^n)\}$,

   (ii) $\max(\phi, r^n) := \max\{f(G) : G \in \mathcal{R}(r^n)\}$,

   (iii) $\mathrm{Min}(\phi, r^n) := \min\{f(G) : G \in \mathcal{CR}(r^n)\}$, and

   (iv) $\mathrm{Max}(\phi, r^n) := \max\{f(G) : G \in \mathcal{CR}(r^n)\}$,

   we obtained the following results.

Theorem 3.7 *Let $r \geq 3$ and nr be even. Then*

$$\min(\phi, r^n) = \begin{cases} r - 1 & \text{if } r + 1 \leq n \leq 2r - 1, \\ \lceil \frac{nr - 2n + 2}{2(r-1)} \rceil & \text{if } n \geq 2r. \end{cases}$$

Let G be a connected r-regular graph and S be a minimum decycling set of G. Since for any $v \in S$ there is a connected component C of $G - S$ such that v is adjacent to at least two vertices of C, there exists $u \in G - S$ such that $vu = e \in E(G)$ and $G - e$ is connected. Thus for two disjoint connected r-regular graphs G and H with minimum decycling set S and T of G and H respectively, there exist $u \in S$, $v \in G - S$, $x \in T$, $y \in H - T$ such that $uv = e \in E(G)$, $xy = f \in E(H)$ and $G - e$, $H - f$ are connected. A connected r-regular graph $K = ((G - e) \cup (H - f)) + \{ux, vy\}$ satisfies

$$\phi(K) \leq \phi(G \cup H) = \phi(G) + \phi(H),$$

and the following corollary holds.

Corollary 3.8 *Let $r \geq 3$ and nr be even. Then*

$$\text{Min}(\phi, r^n) = \begin{cases} r - 1, & \text{if } r + 1 \leq n \leq 2r - 1, \\ \lceil \frac{nr - 2n + 2}{2(r-1)} \rceil, & \text{if } n \geq 2r. \end{cases}$$

Thus the values of $\text{Min}(\phi, r^n)$ for all r and n are already obtained. In particular the values of $\text{Min}(\phi, 3^{2n})$ and $\text{Max}(\phi, 3^{2n})$ are found for all n.

Determining the values of $\text{Max}(\phi, r^n)$ for $r \geq 4$ are more difficult. Note that $\mathcal{R}(r^n) = \mathcal{CR}(r^n)$ if and only if $r + 1 \leq n \leq 2r + 1$. Thus $\text{Max}(\phi, r^n) = \max(\phi, r^n)$ for all $n \in \{r + 1, r + 2, \ldots, 2r + 1\}$.

In this case we have already obtained in [17] as stated in the following theorem.

Theorem 3.9 *For $r \geq 4$, and $n = r + j$, $1 \leq j \leq r + 1$, then*

(i) $\max(\phi, r^n) = n - 2$, *if and only if $j = 1$,*

(ii) $\max(\phi, r^n) = n - 3$, *if and only if $j = 2$,*

(iii) $\max(\phi, r^n) = n - 4$, *for all even integers $n = r + j$, $3 \leq j \leq r + 1$,*

(iv) $\max(\phi, r^n) = n - 4$, *for all odd integers $n = r + j$, $3 \leq j \leq r + 1$ and $n \geq f(j)$,*

(v) $\max(\phi, r^n) = n - 5$, *for all odd integers $n = r + j$, $3 \leq j \leq r + 1$ and $n < f(j)$, where $f(j) = \frac{5}{2}(j - 1)$ if $j \equiv 3 \pmod 4$, and $f(j) = 1 + \frac{5}{2}(j - 1)$ if $j \equiv 1 \pmod 4$.*

Thus the values of $\text{Max}(\phi, r^n)$ are already obtained for all r and n with $n \leq 2r + 1$. The problem of determining the decycling number of a graph is equivalent to finding the forest number of the graph and $\text{Max}(\phi, r^n) = n - \text{Min}(t, r^n)$.

We proved the following series of results in [19] to obtain the values of $\text{Max}(\phi, r^n)$.

Let G be a K_5-free graph of order n, $\Delta(G) = 4$. Let F be a maximal induced forest of G. We denote by $c(F)$ the number of cycles in $G - F$. A pair (X, Y), where $X \subseteq F$ and $Y \subseteq G - F$, is an *interchangeable pair of vertices with respect to F* if $(F - X) \cup Y$ is a forest, $|(F - X) \cup Y| \geq |F|$, and $c((F - X) \cup Y) < c(F)$. In general we can define an interchangeable pair of vertices for a graph G with $\Delta(G) > 4$ as follows. Let G be a $K_{\Delta+1}$-free graph of order n with $\Delta(G) = \Delta > 4$. Let F be a maximal induced forest of G. We denote by $k(F)$ the number of $K_{\Delta-1}$ in $G - F$. A pair (X, Y), where $X \subseteq F$ and $Y \subseteq G - F$, is an *interchangeable pair of vertices with respect to F* if $(F - X) \cup Y$ is a forest, $|(F - X) \cup Y| \geq |F|$, and $k((F - X) \cup Y) < k(F)$.

Let G be a K_5-free graph of order n and $\Delta(G) = 4$. Thus for any maximal induced forest F of G, $G - F$ is a union of cycles and paths. We choose a maximal induced forest F of G with minimum $c(F)$. In other word, the forest F is chosen in such a way that it contains no interchangeable pair of vertices with respect to F. Suppose that $c(F) \geq 1$. Let C be a cycle in $G - F$. Then each vertex of C must be adjacent to exactly two vertices in F. Suppose that there exists a vertex

$u \in F$, $d_F(u) \geq 2$, and u is adjacent to a vertex $v \in V(C)$, then $(\{u\}, \{v\})$ is an interchangeable pair of vertices with respect to F. Thus for all cycles C of $G - F$, each vertex $v \in V(C)$, v must be adjacent to exactly two vertices $u_1, u_2 \in F$ with $d_F(u_1) = d_F(u_2) = 1$. By maximality of F, u_1 and u_2 must be in the same connected component of F. Since F is a forest, there exists a unique path in $G[F]$ from u_1 to u_2. If u_1 and u_2 are not adjacent and there is a vertex $u \in F$ in the path such that $d_F(u) \geq 3$, then $(\{u\}, \{v\})$ is an interchangeable pair of vertices with respect to F. Therefore the connected component of F containing u_1 and u_2 must be a path. Suppose that there exist exactly two vertices v, w of C adjacent to a vertex $u \in F$, then $(\{u\}, \{v\})$ is an interchangeable pair of vertices with respect to F. Finally suppose that there are three vertices v, w, z of C adjacent to a vertex $u \in F$, then the path P in $G[F]$ containing u has order at least 3 or the cycle C has order at least 4, since otherwise G would contain K_5. Let u and u' be the end vertices of P in $G[F]$. Then $\mathbb{N}(\{u, u'\}) \cap V(C) = \{v, w, z\} \subseteq V(C)$, where $\mathbb{N}(\{u, u'\}) = \{x \in V(G) : x \text{ is adjacent to } u \text{ or to } u'\}$. If P has order at least 3, then $(\{u, u'\}, \{v, w\})$ is an interchangeable pair of vertices with respect to F. If P has order 2 and C has order at least 4, then $(\{u, u'\}, \{v, w, z\})$ is an interchangeable pair of vertices with respect to F. Thus the corresponding paths in $G[F]$ of vertices in C are pairwise disjoint.

Theorem 3.10 *Let G be a K_5-free graph of order n, $\Delta(G) = 4$. Then $\phi(G) \leq \frac{n}{2}$.*

Corollary 3.11 $\text{Max}(\phi, 4^n) \leq \frac{n}{2}$.

Let G be a $K_{\Delta+1}$-free graph with $\Delta(G) = \Delta \geq 5$ and let F be a maximal induced of G with minimum $k(F)$. Then for each $v \in G - F$, there exists a connected component T of F such that v is adjacent to at least two vertices of T. Thus $\Delta(G - F) \leq \Delta - 2$. Suppose that $k(F) \geq 1$. Let K be a complete subgraph of $G - F$ of order $\Delta - 1$. Put $V(K) = \{v_1, v_2, \ldots, v_{\Delta-1}\}$. Thus for each v_i there exists a connected component $P(v_i)$ of $G[F]$ such that v_i is adjacent to exactly two vertices of $P(v_i)$. If there exists $u \in V(P(v_i))$ such that $uv_i \in E(G)$ and $d_F(u) \geq 2$, then $(\{u\}, \{v_i\})$ is an interchangeable pair of vertices with respect to F. Thus for each $i = 1, 2, \ldots, \Delta - 1$, v_i must be adjacent to exactly two vertices of degree one in $P(v_i)$. Suppose that there exists $u \in V(P(v_i))$ such that $d_F(u) \geq 3$, $(\{u\}, \{v_i\})$ is an interchangeable pair of vertices with respect to F. Thus the corresponding $P(v_i)$ of v_i in K is a path. Furthermore all such paths are pairwise disjoint.

Lemma 3.12 *Let G be a $K_{\Delta+1}$-free graph of order n with $\Delta(G) = \Delta \geq 5$. Then $t(G) \geq \frac{2n}{\Delta}$ or there exists an induced forest F of G such that $G - F$ is a $K_{\Delta-1}$-free graph.*

Lemma 3.13 *Let G be a connected K_5-free graph of order n and $\Delta(G) = 5$. Then $t(G) \geq \frac{2n}{5}$.*

Lemma 3.14 *Let G be a K_6-free graph of order n with $\Delta(G) = 5$. Then $t(G) \geq \frac{2n}{5}$.*

As a direct consequence of Lemma 3.14 we obtain the following theorem.

Theorem 3.15 *Let G be a connected 5-regular graph of order $n \geq 12$. Then $\phi(G) \leq \frac{3n}{5}$.*

Lemma 3.16 *Let G be a $K_{\Delta+1}$-free graph of order n with $\Delta(G) = \Delta \geq 5$. Then $t(G) \geq \frac{2n}{\Delta}$.*

We have the following theorem.

Theorem 3.17 *Let G be a connected r-regular graph of order $n \geq 2r + 2$. Then $\phi(G) \leq \frac{n(r-2)}{r}$ for all $r \geq 4$.*

Let G be a connected r-regular graph of order $n = rq + s$, $0 \leq s \leq r - 1$, $r \geq 4$ and $q \geq 1$. Then by Theorem 3.17, we have $t(G) \geq 2q + \lceil \frac{2s}{r} \rceil$. It is easy to construct a connected r-regular graph G of order n with $t(G) = 2q$ if $s = 0$, $t(G) = 2q + 1$ if $s = 1, 2$ and $t(G) = 2q + 2$ if $3 \leq s \leq r - 1$. Consequently, we have $\mathrm{Max}(\phi, r^n) = n - 2q$ if $s = 0$, $\mathrm{Max}(\phi, r^n) = n - 2q - 1$ if $s = 1, 2$, $\mathrm{Max}(\phi, r^n) = n - 2q - 2$ if $2s > r$ and $\mathrm{Max}(\phi, r^n) \in \{n - 2q - 2, n - 2q - 1\}$ if $3 \leq s \leq \frac{r}{2}$.

Note that we improved an upper bound for $\mathrm{Max}(\phi, r^n)$ from $\max(\phi, r^n)$, for $r \geq 4$ and n with $n \geq 2r + 2$, and we have that

$$\max(\phi, r^n) - \mathrm{Max}(\phi, r^n) = \frac{2n}{r(r+1)}, \text{ roughly speaking.}$$

Thus we can improve the upper bound for $\mathrm{Max}(\phi, r^n) \approx \frac{n(r-2)}{r}$ from $\max(\phi, r^n) \approx \frac{n(r-1)}{r+1}$.

Conjecture $\mathrm{Max}(\phi, r^n) = n - 2q - 2$ *if* $3 \leq s \leq \frac{r}{2}$, *for all* $r \geq 6$.

4   Cubic Graphs

The interest of studying the decycling number of cubic graphs was motivated by the results of Zheng and Lu [24], Alon et al. [1], Liu and Zhao [16], and Bau and Beineke [2].

Note that the values of $\min(\phi, 3^n)$ have been already obtained in all situations and $\min(\phi, 3^n) = \mathrm{Min}(\phi, 3^n)$.

The problem of finding upper bounds of $\phi(G)$, where G runs over a class of cubic graphs, have been investigated in the literature. First observe that if G is a cubic graph of order n and F is a maximum induced forest of G, then It is easy to see that $G - F$ is also a forest. Thus $|F| \geq \frac{n}{2}$. The bound is sharp if and

only if n is a multiple of 4. We proved in [17] that if $n = 4q + t$, $t = 0, 2$, then $\max(\phi, 3^n) = 2q$.

The problem of determining $\text{Max}(\phi, 3^n)$ is more difficult. We will consider the problem in term of the graph parameter t. A *cubic tree* is a tree in which its vertices consisting of degree 1 or 3. Evidently if T is a cubic tree of order n, then $n = 2k + 2$, where k is the number of vertices of degree 3 of T. Let $\mathbb{T}$ denote the family of cubic graphs obtained by taking cubic trees and replacing each vertex of degree 3 by a triangle and attaching a copy of K_4 with one subdivided edge at every vertex of degree 1.

A lower bound for the forest number in connected cubic graphs has been obtained by Liu and Zhao [16] as stated in the following theorem.

Theorem 4.1 *Let G be a connected cubic graph of order $n \geq 12$. Then $t(G) = \frac{5}{8}n - \frac{1}{4}$ if $G \in \mathbb{T}$ and $t(G) \geq \frac{5}{8}n$ if $G \notin \mathbb{T}$.*

We have determined in [19] the value of $\text{Min}(t, 3^n)$ by observing the following situation. First observe that if $G \in \mathbb{T}$, then G has order $8k + 10$, where k is the number of vertices of degree 3 in the corresponding cubic tree. Thus $t(G) = \text{Min}(t, 3^{8k+10}) = 5k+6$. We now consider a cubic graph of order $8k+8$. Let C be a cubic graph of order $8k+8$. Then by Theorem 4.1, $t(C) \geq \frac{5}{8}(8k+8) = 5(k+1)$. A cubic graph T obtained by taking cubic tree with k vertices of degree 3, replacing $k-1$ of the vertices by a triangle and attaching a copy of K at every vertex of degree 1. Thus T has order $8k + 8$ and $t(T) = 5(k + 1)$. Thus $\text{Min}(t, 3^{8k+8}) = 5(k + 1)$. The value of $\text{Min}(t, 3^n)$, $n = 8k + 4, 8k + 6$ can be obtained in the following argument. Since a switching changes the order of induced forest by at most 1, we have $\text{Min}(t, 3^{p+q}) \leq \text{Min}(t, 3^p) + \text{Min}(t, 3^q) + 1$ for all even integers p and q with $4 \leq p \leq q$. Thus $5k + 4 = \lceil \frac{5}{8}(8k + 6) \rceil \leq \text{Min}(t, 3^{8k+6}) \leq \text{Min}(t, 3^4) + \text{Min}(t, 3^{8(k-1)+10}) + 1 = 2 + 5(k-1) + 6 + 1 = 5k + 4$. Finally $5k + 3 = \lceil \frac{5}{8}(8k+4) \rceil \leq \text{Min}(t, 3^{8k+4}) \leq \text{Min}(t, 3^4) + \text{Min}(t, 3^{8(k-1)+8}) + 1 = 2 + 5k + 1 = 5k + 3$. Therefore we obtained in [19] the following theorem and corollary.

Theorem 4.2 *Let n be an even integer with $n \geq 12$. Then*

$$\text{Min}(t, 3^n) = \begin{cases} \frac{5}{8}n - \frac{1}{4}, & \text{if } n \equiv 2 (\text{mod } 8), \\[2mm] \lceil \frac{5}{8}n \rceil, & \text{otherwise.} \end{cases}$$

Corollary 4.3 *Let n be an even integer with $n \geq 12$. Then*

$$\text{Max}(\phi, 3^n) = \begin{cases} \frac{3}{8}n + \frac{1}{4}, & \text{if } n \equiv 2 (\text{mod } 8), \\[2mm] \lfloor \frac{3}{8}n \rfloor, & \text{otherwise.} \end{cases}$$

In [21], there are five connected cubic graphs of order 8, all of which having maximum induced forests of order 5. Alon et al. proved in [1] that: Let G be a $\{K_4, K_4'\}$-free graph of order n and of size m. If $\Delta(G) \leq 3$, then, then

$t(G) \geq n - \frac{m}{4}$. Consequently, if G is a cubic $\{K_4, K_4'\}$-free graph of order $n \geq 10$, then $t(G) \geq \frac{5n}{8}$. Zheng and Lu proved in [24] that $t(G) \geq \frac{2n}{3}$ for any connected cubic graph G of order n without triangles, except for two cubic graphs with $n = 8$. They also pointed out that this lower bound is best possible. It is easy to see that there exists cubic graph G of order n containing triangles and $t(G) \geq \frac{2n}{3}$. We have extended their result by proving that $t(G) \geq \frac{2n}{3}$ for any connected cubic K_4'-free graph G of order $n \geq 10$ as stated in the following results.

Theorem 4.4 *Let G be a connected triangle-free graph of order n and $\Delta(G) = 3$. If G is not a cubic graph, then $t(G) \geq \frac{2n}{3}$.*

Theorem 4.5 *Let $X = \mathcal{CR}(3^8) \cup \{K_4, K_4'\}$ and let G be an X-free graph of order n with $\Delta(G) = 3$. Then $t(G) \geq \frac{2n}{3}$.*

Theorem 4.6 *Let G be a connected cubic K_4'-free graph of order n, $n \geq 6$ and $n \neq 8$. Then $t(G) \geq \frac{2n}{3}$.*

We recently constructed in [4] a class of connected triangle-free graphs $\mathbb{J}_n$ of order n to show that $\min(t, \mathbb{J}_n) = \lceil \frac{2n}{3} \rceil$ as follows.

(i) $t(K_{3,3}) = 4$.

(ii) $t(Q_3) = 5$, where Q_3 is the 3-cube.

(iii) There is a switching σ such that $(2K_4')^\sigma$ is a connected triangle-free graph and $t((2K_4')^\sigma) = 7$. Put $K = (2K_4')^\sigma$.

(iv) If $e \in E(K_{3,3})$ and $f \in E(Q_3)$, then $t(K_{3,3} - e) = 4$ and $t(Q_3 - f) = 6$. Put $P = K_{3,3} - e$ and $Q = Q_3 - f$.

(v) Let n be an even integer with $n \geq 12$. Write $n = 6q + r$, $r = 0, 2, 4$ and construct a connected cubic triangle-free graph according to the values of r.

   (a) If $r = 0$, construct graph G of order $6q$ by taking q copies of P and joining q appropriate edges between the q copies of P.

   (b) If $r = 2$, construct graph G of order $6q + 2$ by taking $q - 1$ copies of P and a copy of Q and then joining q appropriate edges between them.

   (c) If $r = 4$, construct a graph G of order $6q + 4$ by taking $q - 1$ copies of P and a copy of K and then joining q appropriate edges between them.

It is easy to check that the graphs G constructed above satisfying $t(G) = \lceil \frac{2n}{3} \rceil$. Thus we have the following theorem.

Theorem 4.7 [4] $\min(t, \mathbb{J}_n) = \lceil \frac{2n}{3} \rceil$.

5   Cubic Planar Graphs

In the class of connected cubic graphs of order $2n$, we have already proved that in [18]

$$\phi(3^{2n}) := \{\phi(G) : G \in \mathcal{CR}(3^{2n})\} = [\mathrm{Min}(\phi, 3^{2n}), \mathrm{Max}(\phi, 3^{2n})],$$

where

$$\mathrm{Min}(\phi, 3^{2n}) = \lceil \tfrac{n+1}{2} \rceil$$

and

$$\mathrm{Max}(\phi, 3^{2n}) = \begin{cases} \tfrac{3}{4}n + \tfrac{1}{4}, & \text{if } n \equiv 1 (\mathrm{mod}\ 4), \\[2mm] \lfloor \tfrac{3}{4}n \rfloor, & \text{otherwise.} \end{cases}$$

For each $c \in \phi(3^{2n})$, let $\mathcal{CR}(3^{2n}; \ \phi = c) = \{G \in \mathcal{CR}(3^{2n}) : \phi(G) = c\}$.

We consider the problem of determining whether the subgraph of $\mathcal{CR}(3^{2n})$ induced by $\mathcal{CR}(3^{2n}; \ \phi = c)$ is connected or not. If it is connected, then for any two graphs $G_1, G_2 \in \mathcal{CR}(3^{2n}; \ \phi = c)$ there is a sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_t$ such that for each $i = 1, 2, \ldots, t$, $G_1^{\sigma_1 \sigma_2 \cdots \sigma_i} \in \mathcal{CR}(3^{2n}; \ \phi = c)$ and $G_1^{\sigma_1 \sigma_2 \cdots \sigma_t} = G_2$.

We proved in [18] that the subgraph of $\mathcal{CR}(3^{2n})$ induced by $\mathcal{CR}(3^{2n}; \ \phi = \lceil \tfrac{n+1}{2} \rceil)$ is connected is connected. Nothing has been done for other values of c.

That is the class of cubic graphs with smallest decycling number induced a connected subgraph. This provides an algorithm to finding all cubic graphs with smallest decycling number in the class of connected cubic graphs and this also leads to an answering to the Problem 1 asked by Bau and Beineke [2].

Bau and Beineke asked the following problems.

Problem 1 Which cubic graphs G of order $2n$ satisfy $\phi(G) = \lceil \tfrac{n+1}{2} \rceil$?

Problem 2 Which cubic planar graphs G of order $2n$ satisfy $\phi(G) = \lceil \tfrac{n+1}{2} \rceil$?

We will give an outline of a proof to answer Problem 2. Details can be found in [20].

Let us first consider the following example.

Example 5.1 $\mathrm{Min}(\phi, 3^{34}) = \tfrac{17+1}{2} = 9$ *and* $\mathrm{Max}(\phi, 3^{34}) = \tfrac{3 \cdot 17}{4} + \tfrac{1}{4} = 13$.

Thus $\mathcal{CR}(3^{34})$ can be partitioned into five classes C_i ($9 \le i \le 13$), where

$C_i = \{G \in \mathcal{CR}(3^{34}) : \phi(G) = i\}$ and $C_i \neq \emptyset$, for all $i = 9, 10, 11, 12, 13$.

For any integer $n \ge 2$ and for each $i \in [\mathrm{Min}(\phi, 3^{2n}), \mathrm{Max}(\phi, 3^{2n})]$ we have $C_i = \{G \in \mathcal{CR}(3^{2n}) : \phi(G) = i\} \neq \emptyset$.

Let $\mathcal{P}(3^{2n})$ be the class of all connected cubic planar graphs of order $2n$. We can ask the following questions.

1. Is ϕ an interpolation graph parameter with respect to $\mathcal{P}(3^{2n})$?

2. What is the values of $\min\{\phi(G) : G \in \mathcal{P}(3^{2n})\}$ and $\max\{\phi(G) : G \in \mathcal{P}(3^{2n})\}$?

3. Is the subgraph of the graph $\mathcal{CR}(3^{2n})$ induced by $\mathcal{P}(3^{2n})$ connected?

4. Which cubic planar graphs G of order $2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$? (Problem 2)
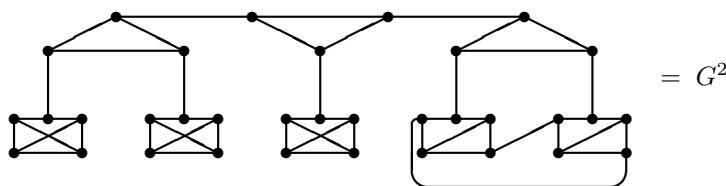
5. Is the subgraph $\mathcal{P}(3^{2n}; \phi = \lceil \frac{n+1}{2} \rceil)$ connected?

It is clear that Question 3 implies Question 1. Answering Question 3 seems to be difficult. We will answer Question 1 directly.

Let us first consider the following examples and the proofs of Question 1 and Question 2 are easily obtained.

Example 5.2 Planar graphs G^1, G^2, G^3, G^4, G^5

$$= G^1$$
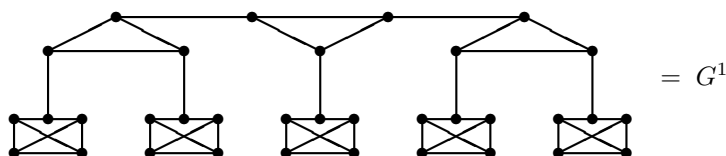
G^1 of order $34 = 8k + 10$ and $\phi(G^1) = 13 = \text{Max}(\phi, 3^{34})$.

$$= G^2$$

G^2 of order $34 = 8k + 10$ and $\phi(G^2) = 12 = \text{Max}(\phi, 3^{34}) - 1$.

$$= G^3$$
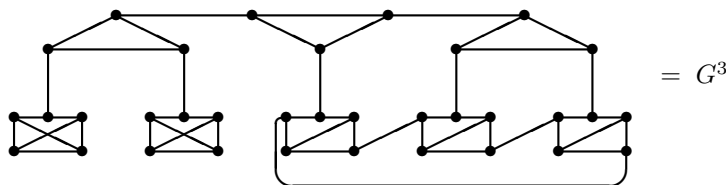
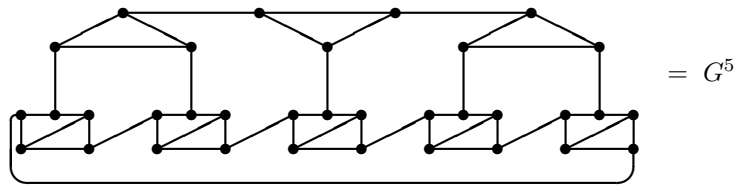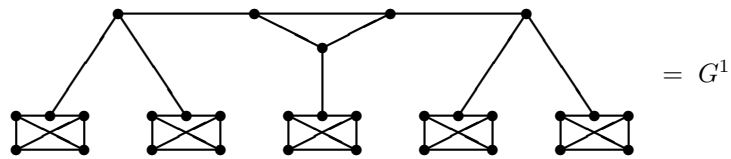G^3 of order $34 = 8k + 10$ and $\phi(G^3) = 11 = \text{Max}(\phi, 3^{34}) - 2$.
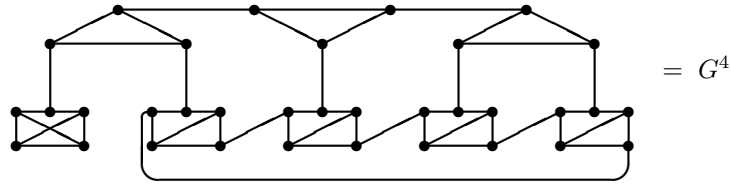
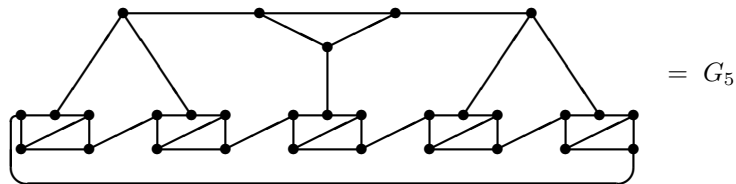G^4 of order $34 = 8k + 10$ and $\phi(G^4) = 10 = \text{Max}(\phi, 3^{34}) - 3$.

G^5 of order $34 = 8k + 10$ and $\phi(G^5) = 9 = \text{Max}(\phi, 3^{34}) - 4 = \text{Min}(\phi, 3^{34})$.

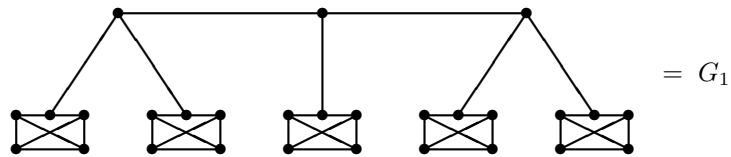**Example 5.3** Consider the following graphs G^1 and G^5.

Cubic planar graph G^1 of order $8k + 6 = 30$, $\phi(G) = 11 = \text{Max}(\phi, 3^{30})$
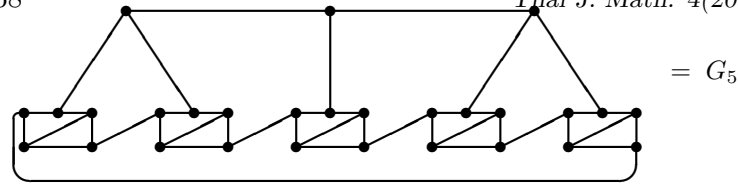

Cubic planar graph G^5 of order $8k + 6 = 30$, $\phi(G) = 8 = \text{Min}(\phi, 3^{30})$

Example 5.4 Consider the following graphs G^1 and G^5.

Cubic planar graph G^1 of order $8k + 4 = 28$, $\phi(G) = 10 = \text{Max}(\phi, 3^{28})$

$$= G_5$$

Cubic planar graph G^5 of order $8k + 4 = 28$, $\phi(G) = 8 = \text{Min}(\phi, 3^{28})$

Thus for each $i = 9, 10, 11, 12, 13$, $C_i \cap \mathcal{P}(3^{34}) \neq \emptyset$. Therefore ϕ interpolates over $\mathcal{P}(3^{34})$. The same result can be obtained in $\mathcal{P}(3^{2n})$ for $n \in \{14, 15, 16\}$.

In general if we write $2n = 8k + 2i$, $i = 2, 3, 4, 5$ and $k \geq 4$, then we can construct a cubic planar graph G_i of order $8k+2i$ by a cubic tree T_i of order $2k+2$, where k is the number of vertices of degree 3, replacing $k-5+i$ vertices of degree 3, by a triangle and attaching a copy of K_4 with one subdivided edge at every vertex of degree 1. Thus $G_i \in \mathcal{P}(3^{2n})$ and $\phi(G_i) = 3k - 1 + i = \text{Max}(\phi, 3^{2n})$. Hence $\max(\phi, \mathcal{P}(3^{2n})) = \text{Max}(\phi, 3^{2n})$. By forming appropriate sequence of switchings to G_i as described in above example and with the fact that a switching changes the value of ϕ of the resulting graph by at most 1, we have the following theorem.

Theorem 5.5 (Question 1) ϕ *is an interpolation graph parameter with respect to* $\mathcal{P}(3^{2n})$.

Theorem 5.6 (Question 2)

$$\min\{\phi(G) : G \in \mathcal{P}(3^{2n})\} = \text{Min}(\phi, 3^{2n}) \text{ and}$$

$$\min\{\phi(G) : G \in \mathcal{P}(3^{2n})\} = \text{Min}(\phi, 3^{2n}).$$

We proved in [18] the following result.

Let $G \in \mathcal{CR}(3^{2n}; \phi = \lceil \frac{n+1}{2} \rceil)$ and S be a minimum decycling set of G. Put $F = V(G) - S$. Then

  (i) $e(S) = 0$, if n is odd and $e(S) \leq 1$, if n is even,

  (ii) if n is odd, then $G[F]$ is a tree,

 (iii) if n is even and $e(S) = 1$, then $G[F]$ is a tree,

 (iv) if n is even and $e(S) = 0$, then $G[F]$ has 2 connected components.

We proved in [20] the following results.

Theorem 5.7 *There exists $G \in \mathcal{P}(3^{2n})$ such that $\phi(G) = \lceil \frac{n+1}{2} \rceil$ and G contains a path of order $\lfloor \frac{3n-1}{2} \rfloor$ as its induced forest. Furthermore the induced subgraph of the corresponding decycling set of G contains at most one edge and it contains no edge if and only if n is odd.*

Definition 5.8 Let G be a connected cubic planar graph and F is an induced forest of G. If $u, v \in F$ and $uv \in E(G)$, define a super subdivision with respect to uv of G to be a graph $G \cdot uv$ obtained from G by inserting three vertices x, y, z on the edge uv and joining x, y, z to a new vertex w. Let $u \in F$, $d_F(u) = 1$. Then there exists a unique vertex $v \in F$ that is adjacent to u and there exist two vertices $p, q \in V(G - F)$ such that p and q are adjacent to u. Define a super subdivision with respect to u of G to be a graph $G \cdot u$ obtained from G as follows: Inserting three vertices x, y, z on the edge uv to produce a path $uxyzv$ in F, deleting qu, and joining qz, and finally joining u, x, y to a new vertex w.

super subdivision
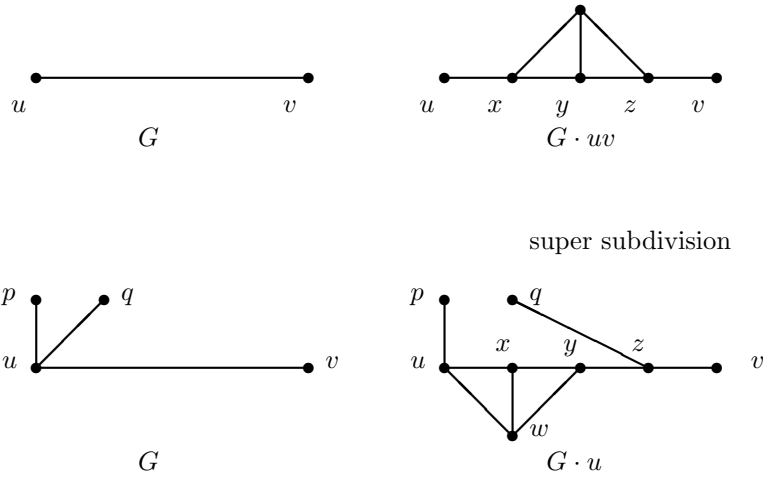
Let $\mathcal{P}(3^{2n}; \phi = \lceil \frac{n+1}{2} \rceil)$ be the class of cubic planar graphs G of order $2n$ and $\phi(G) = \lceil \frac{n+1}{2} \rceil$.

We also obtained an algorithm of finding all cubic planar graphs of order $2n$ with decycling number $\lceil \frac{n+1}{2} \rceil$. In addition we proved the following theorem.

Theorem 5.9 *The subgraph $\mathcal{P}(3^{2n}; \phi = \lceil \frac{n+1}{2} \rceil)$ is connected.*

Acknowledgement

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