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Coupled Common Fixed Point Theorems under Weak Contractions in Cone Metric Type Spaces

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Abstract : In this paper we define the concept of a coupled common fixed point for contractive conditions in a cone metric type space and prove some coupled common fixed point theorems. In the sequel, we obtain a general approach for our theorems. These results extend, unify and generalize several well known comparable results in the existing literature.

Keywords : cone metric type space; coupled common fixed point; w-compatible mapping; coupled coincidence point.
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1 Introduction and Preliminaries

The symmetric space, as metric-like spaces lacking the triangle inequality was introduced in 1931 by Wilson [1]. In the sequel, a new type of spaces which they

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called metric type spaces are defined by Boriceanu [2] and Khamsi and Hussain [3]. Also, Jovanović et al. [4], Rahimi and Soleimani Rad [5, 6], Bota et al. [7], Pavlović et al. [8], Singh et al. [9] and Hussain et al. [10] generalized and unified some fixed point theorems of metric spaces by considering metric type spaces.

On the other hand, the cone metric space was introduced in 2007 by Huang and Zhang [11] and several fixed and common fixed point results in cone metric spaces were proved in [5, 12–24] and the references contained therein. Recently, analogously with definition of metric type space, Radenović and Kadelburg [25], Ćvetković et al. [26], Rahimi et al. [27] considered cone metric type spaces and proved several fixed and common fixed point theorems.

In 2006, Bhaskar and Lakshmikantham [28] considered the concept of coupled fixed point theorems in partially ordered metric spaces. Afterward, some other authors generalized this concept and proved several common coupled fixed and coupled fixed point theorems in ordered metric and ordered cone metric spaces (see [29–42] and the references contained therein).

In this paper we introduce the concept of coupled fixed point in a cone metric type space and prove some coupled fixed point theorems. Our results extend well known comparable results in the literature.

Let us start by defining some important definitions.

Definition 1.1 (See [1]). Let X be a nonempty set and the mapping $D : X \times X \to [0, \infty)$ satisfies

$$(S1)\ D(x, y) = 0 \Longleftrightarrow x = y;$$
$$(S2)\ D(x, y) = D(y, x),$$

for all $x, y \in X$. Then D is called a symmetric on X and (X, D) is called a symmetric space.

Definition 1.2 (See [11, 43]). Let E be a real Banach space and P be a subset of E. Then P is called a cone if and only if

(a) P is closed, non-empty and $P \neq \{\theta\}$;

(b) $a, b \in R, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$;

(c) if $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is the interior of P). The cone P is named normal if there is a number $K > 0$ such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number satisfying the above is called the normal constant of P.

Definition 1.3 (See [11]). Let X be a nonempty set and the mapping $d : X \times X \to E$ satisfies

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then, d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.4 (See [3]). Let X be a nonempty set, and $K \geq 1$ be a real number. Suppose the mapping $D_m : X \times X \to [0, \infty)$ satisfies

(D1) $D_m(x, y) = 0$ if and only if $x = y$;

(D2) $D_m(x, y) = D_m(y, x)$ for all $x, y \in X$;

(D3) $D_m(x, z) \leq K(D_m(x, y) + D_m(y, z))$ for all $x, y, z \in X$.

(X, D_m, K) is called metric type space. Obviously, for $K = 1$, metric type space is a metric space.

Definition 1.5 (See [25, 26]). Let X be a nonempty set, $K \geq 1$ be a real number and E a real Banach space with cone P. Suppose that the mapping $d : X \times X \to E$ satisfies

(cd1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(cd2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(cd3) $d(x, z) \preceq K(d(x, y) + d(y, z))$ for all $x, y, z \in X$.

(X, d, K) is called cone metric type space. Obviously, for $K = 1$, cone metric type space is a cone metric space.

Example 1.6 (See [26]). *Let $B = \{e_i | i = 1, \ldots, n\}$ be orthonormal basis of $\mathbf{R}^n$ with inner product $(\cdot, \cdot)$ and $p > 0$. Define*

$$X_p = \Big\{[x] | x : [0, 1] \to \mathbf{R}^n, \int_0^1 |(x(t), e_j)|^p dt \in \mathbf{R}, j = 1, 2, \ldots, n\Big\},$$

where $[x]$ represents class of element x with respect to equivalence relation of functions equal almost everywhere. Let $E = \mathbf{R}^n$ and

$$P_B = \big\{y \in \mathbf{R}^n | (y, e_i) \geq 0, i = 1, 2, \ldots, n\big\}$$

be a solid cone. Define $d : X_p \times X_p \to P_B \subset \mathbf{R}^n$ by

$$d(f, g) = \sum_{i=1}^n e_i \int_0^1 |((f - g)(t), e_i)|^p dt, \qquad f, g \in X_p.$$

Then (X_p, d, K) is cone metric type space with $K = 2^{p-1}$.

Similarly, we define convergence in cone metric type spaces.

Definition 1.7 (See [25, 26])**.** Let (X, d, K) be a cone metric type space, $\{x_n\}$ a sequence in X and $x \in X$.

(i) $\{x_n\}$ converges to x if for every $c \in E$ with $\theta \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $\lim_{n\to\infty} x_n = x$.

(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$.

Lemma 1.8 (See [25, 26])**.** *Let (X, d, K) be a cone metric type space over ordered real Banach space E. Then the following properties are often used, particularly when dealing with cone metric type spaces in which the cone need not be normal.*

(P_1) *If $u \preceq v$ and $v \ll w$, then $u \ll w$.*

(P_2) *If $\theta \preceq u \ll c$ for each $c \in intP$, then $u = \theta$.*

(P_3) *If $u \preceq \lambda u$ where $u \in P$ and $0 \leq \lambda < 1$, then $u = \theta$.*

(P_4) *Let $x_n \to \theta$ in E and $\theta \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.*

2 Main Results

At the first, we define the concept of the coupled common fixed point under contractive conditions in a cone metric type space for w-compatible mappings. Then, we prove some coupled common fixed point theorems as generalization of Abbas et al.'s works in [29], Sabetghadam et al.'s theorems in [42] and Bhaskar and Lakshmikantham's results in [28].

Definition 2.1. Let (X, d, K) be a cone metric type space with constant $K \geq 1$.

(i) An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$;

(ii) An element $(x, y) \in X \times X$ is said to be a coupled coincidence fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$, and (gx, gy) is called coupled point of coincidence;

(iii) An element $(x, y) \in X \times X$ is said to be a coupled common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x) = x$ and $F(y, x) = g(y) = y$;

(iv) The mappings $F : X \times X \to X$ and $g : X \to X$ are called w-compatible if $g(F(x, y)) = F(gx, gy)$ whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Note that if (x, y) is a coupled common fixed point of F then (y, x) is coupled common fixed point of F too.

Theorem 2.2. *Let (X, d, K) be a cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ and $g : X \to X$ satisfy the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$\begin{aligned} d(F(x,y), F(x^*,y^*)) \preceq \alpha_1 d(gx, gx^*) &+ \alpha_2 d(F(x,y), gx) + \alpha_3 d(gy, gy^*) \\ &+ \alpha_4 d(F(x^*,y^*), gx^*) + \alpha_5 d(F(x,y), gx^*) \\ &+ \alpha_6 d(F(x^*,y^*), gx), \end{aligned} \tag{2.1}$$

where α_i for $i = 1, 2, \ldots, 6$ are nonnegative constants with

$$2K(\alpha_1 + \alpha_3) + (K+1)(\alpha_2 + \alpha_4) + (K^2 + K)(\alpha_5 + \alpha_6) < 2. \tag{2.2}$$

If $F(X \times X) \subset g(X)$ and $g(X)$ is complete subset of X, then F and g have a coupled coincidence point in X.

Proof. Let $x_0, y_0 \in X$ and set

$$g(x_1) = F(x_0, y_0), g(y_1) = F(y_0, x_0), \ldots, g(x_{n+1}) = F(x_n, y_n), g(y_{n+1}) = F(y_n, x_n).$$

This can be done because $F(X \times X) \subset g(X)$. From (2.1), we have

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\preceq \alpha_1 d(gx_n, gx_{n-1}) + \alpha_2 d(F(x_n, y_n), gx_n) + \alpha_3 d(gy_n, gy_{n-1}) \\ &\quad + \alpha_4 d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + \alpha_5 d(F(x_n, y_n), gx_{n-1}) + \\ &\quad + \alpha_6 d(F(x_{n-1}, y_{n-1}), gx_n) \\ &\preceq \alpha_1 d(gx_n, gx_{n-1}) + \alpha_2 d(gx_{n+1}, gx_n) + \alpha_3 d(gy_n, gy_{n-1}) \\ &\quad + \alpha_4 d(gx_n, gx_{n-1}) + \alpha_5 d(gx_{n+1}, gx_{n-1}) + \alpha_6 d(gx_n, gx_n) \\ &\preceq \alpha_1 d(gx_n, gx_{n-1}) + \alpha_2 d(gx_{n+1}, gx_n) + \alpha_3 d(gy_n, gy_{n-1}) \\ &\quad + \alpha_4 d(gx_n, gx_{n-1}) + K\alpha_5 [d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})]. \end{aligned} \tag{2.3}$$

It follows

$$(1 - \alpha_2 - K\alpha_5) d(gx_{n+1}, gx_n) \preceq (\alpha_1 + \alpha_4 + K\alpha_5) d(gx_n, gx_{n-1}) + \alpha_3 d(gy_n, gy_{n-1}). \tag{2.4}$$

Similarly,

$$(1 - \alpha_2 - K\alpha_5) d(gy_{n+1}, gy_n) \preceq (\alpha_1 + \alpha_4 + K\alpha_5) d(gy_n, gy_{n-1}) + \alpha_3 d(gx_n, gx_{n-1}). \tag{2.5}$$

Because of the symmetry in (2.1), we get

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\preceq \alpha_1 d(gx_{n-1}, gx_n) + \alpha_2 d(F(x_{n-1}, y_{n-1}), gx_{n-1}) \\ &\quad + \alpha_3 d(gy_{n-1}, gy_n) + \alpha_4 d(F(x_n, y_n), gx_n) \\ &\quad + \alpha_5 d(F(x_{n-1}, y_{n-1}), gx_n) + \alpha_6 d(F(x_n, y_n), gx_{n-1}) \\ &\preceq \alpha_1 d(gx_{n-1}, gx_n) + \alpha_2 d(gx_n, gx_{n-1}) + \alpha_3 d(gy_{n-1}, gy_n) \\ &\quad + \alpha_4 d(gx_{n+1}, gx_n) + \alpha_5 d(gx_n, gx_n) + \alpha_6 d(gx_{n+1}, gx_{n-1}) \\ &\preceq \alpha_1 d(gx_{n-1}, gx_n) + \alpha_2 d(gx_n, gx_{n-1}) + \alpha_3 d(gy_{n-1}, gy_n) \\ &\quad + \alpha_4 d(gx_{n+1}, gx_n) + K\alpha_6 [d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})]. \end{aligned} \tag{2.6}$$

It follows

$$(1-\alpha_4-K\alpha_6) d(gx_n, gx_{n+1}) \preceq (\alpha_1+\alpha_2+K\alpha_6) d(gx_{n-1}, gx_n) + \alpha_3 d(gy_{n-1}, gy_n). \tag{2.7}$$

Similarly,

$$(1-\alpha_4-K\alpha_6) d(gy_n, gy_{n+1}) \preceq (\alpha_1+\alpha_2+K\alpha_6) d(gy_{n-1}, gy_n) + \alpha_3 d(gx_{n-1}, gx_n). \tag{2.8}$$

Now, adding up (2.4) and (2.5), we get

$$\begin{aligned} &(1-\alpha_2-K\alpha_5)[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)] \\ &\qquad \preceq (\alpha_1+\alpha_3+\alpha_4+K\alpha_5)[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})]. \end{aligned} \tag{2.9}$$

Similarly, adding up (2.7) and (2.8), we get

$$\begin{aligned} &(1-\alpha_4-K\alpha_6)[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)] \\ &\qquad \preceq (\alpha_1+\alpha_2+\alpha_3+K\alpha_6)[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})]. \end{aligned} \tag{2.10}$$

Let $D_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$. Then, adding up (2.9) and (2.10), we have

$$(2-\alpha_2-\alpha_4-K(\alpha_5+\alpha_6))D_n \preceq (2\alpha_1+\alpha_2+2\alpha_3+\alpha_4+K(\alpha_5+\alpha_6))D_{n-1}. \tag{2.11}$$

Thus, for all n,

$$\theta \preceq D_n \preceq \lambda D_{n-1} \preceq \lambda^2 D_{n-2} \preceq \cdots \preceq \lambda^n D_0, \tag{2.12}$$

where

$$\lambda = \frac{2\alpha_1+\alpha_2+2\alpha_3+\alpha_4+K(\alpha_5+\alpha_6)}{2-\alpha_2-\alpha_4-K(\alpha_5+\alpha_6)} < \frac{1}{K}. \tag{2.13}$$

If $D_0 = \theta$ then (x_0, y_0) is a coupled fixed point of F. Now, let $D_0 > \theta$. If $m > n$, we have

$$\begin{aligned} d(gx_n, gx_m) &\preceq K[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\ &\preceq Kd(gx_n, gx_{n+1}) + K^2[d(gx_{n+1}, gx_{n+2}) + d(gx_{n+2}, gx_m)] \\ &\vdots \\ &\preceq Kd(gx_n, gx_{n+1}) + K^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\ &\quad + K^{m-n-1}d(gx_{m-2}, gx_{m-1}) + K^{m-n}d(gx_{m-1}, gx_m), \end{aligned} \tag{2.14}$$

and similarly,

$$\begin{aligned} d(gy_n, gy_m) &\preceq Kd(gy_n, gy_{n+1}) + K^2 d(gy_{n+1}, gy_{n+2}) + \cdots \\ &\quad + K^{m-n-1}d(gy_{m-2}, gy_{m-1}) + K^{m-n}d(gy_{m-1}, gy_m). \end{aligned} \tag{2.15}$$

Adding up (2.14) and (2.15) and using (2.12). Since $\lambda < 1/K$, we have

$$\begin{aligned} d(gx_n, gx_m) + d(gy_n, gy_m) &\preceq KD_n + K^2 D_{n+1} + \cdots + K^{m-n}D_{m-1} \\ &\preceq (K\lambda^n + K^2\lambda^{n+1} + \cdots + K^{m-n}\lambda^{m-1})D_0 \\ &\preceq \frac{K\lambda^n}{1-K\lambda}D_0 \to \theta \quad \text{as } n \to \infty. \end{aligned}$$

Now, by (P_1) and (P_4), it follows that for every $c \in intP$ there exist positive integer N such that $d(gx_n, gx_m) + d(gy_n, gy_m) \ll c$ for every $m > n > N$, so $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. Since $g(X)$ is complete subset of cone metric type space X, there exist $x, y \in X$ such that $gx_n \to gx$ and $gy_n \to gy$ as $n \to \infty$. Now, we prove that $F(x, y) = gx$ and $F(y, x) = gy$. From $(cd3)$ and (2.1), we have

$$\begin{aligned} d(F(x,y), gx) &\preceq K[d(F(x,y), gx_{n+1}) + d(gx_{n+1}, gx)] \\ &= K[d(F(x,y), F(x_n, y_n)) + d(gx_{n+1}, gx)] \\ &\preceq K[\alpha_1 d(gx, gx_n) + \alpha_2 d(F(x,y), gx) + \alpha_3 d(gy, gy_n) \\ &\quad + K\alpha_4[d(gx_{n+1}, gx) + d(gx, gx_n)] \\ &\quad + K\alpha_5[d(F(x,y), gx) + d(gx, gx_n)] \\ &\quad + \alpha_6 d(gx_{n+1}, gx) + d(gx_{n+1}, gx)]. \end{aligned} \tag{2.16}$$

Therefore,

$$\begin{aligned} d(F(x,y), gx) &\preceq \frac{K\alpha_1 + K^2(\alpha_4+\alpha_5)}{1-K\alpha_2-K^2\alpha_5}d(gx_n, gx) \\ &\quad + \frac{K + K^2\alpha_4 + K\alpha_6}{1-K\alpha_2-K^2\alpha_5}d(gx_{n+1}, gx) \\ &\quad + \frac{K\alpha_3}{1-K\alpha_2-K^2\alpha_5}d(gy_n, gy). \end{aligned} \tag{2.17}$$

Since $gx_n \to gx$ and $gy_n \to gy$, by using Lemma 1.8 we have $d(F(x,y), gx) = \theta$; that is, $F(x,y) = gx$. Similarly, we can get $d(F(y,x), gy) = \theta$; that is, $F(y,x) = gy$. Therefore, (x,y) coupled coincidence point of the mappings F and g. This completes the proof. □

Theorem 2.3. *Let $F : X \times X \to X$ and $g : X \to X$ be two mappings which satisfy all the conditions of Theorem 2.2. If F and g are w-compatible, then F and g have a unique coupled common fixed point. Moreover, common fixed point of F and g is of the form (z,z) for some $z \in X$.*

Proof. At the first, we prove that coupled point of coincidence is unique. Suppose that $(x,y), (x',y') \in X \times X$ with $g(x) = F(x,y)$, $g(y) = F(y,x)$, $g(x') = F(x',y')$ and $g(y') = F(y',x')$. From (2.1), we have

$$\begin{aligned} d(gx, gx') &= d(F(x,y), F(x',y')) \\ &\preceq (\alpha_1 + \alpha_5 + \alpha_6) d(gx, gx') + \alpha_3 d(gy, gy'). \end{aligned} \tag{2.18}$$

Similarly

$$\begin{aligned} d(gy, gy') &= d(F(y,x), F(y',x')) \\ &\preceq (\alpha_1 + \alpha_5 + \alpha_6) d(gy, gy') + \alpha_3 d(gx, gx'). \end{aligned} \tag{2.19}$$

Adding up (2.18) and (2.19), we get

$$d(gx, gx') + d(gy, gy') \preceq (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)[d(gx, gx') + d(gy, gy')]. \tag{2.20}$$

Since $2K(\alpha_1 + \alpha_3) + (K+1)(\alpha_2 + \alpha_4) + (K^2 + K)(\alpha_5 + \alpha_6) < 2$, by using Lemma 1.8, we have $d(gx, gx') + d(gy, gy') = \theta$. It follows that $gx = gx'$ and $gy = gy'$. Similarly, we can prove $gx = gy'$ and $gy = gx'$. Thus $gx = gy$ and (gx, gx) is unique coupled point of coincidence of F and g. Now, let $g(x) = z$. Then we have $z = g(x) = F(x,x)$. By w-compatibility of F and g, we have

$$g(z) = g(g(x)) = g(F(x,x)) = F(gx, gx) = F(z,z).$$

Thus (gz, gz) is coupled point of coincidence of F and g. Therefore $z = gz = F(z,z)$. Consequently (z,z) is unique coupled common fixed point of F and g. □

Corollary 2.4. *Let (X, d, K) be a cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ and $g : X \to X$ satisfy the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$\begin{aligned} d(F(x,y), F(x^*,y^*)) \preceq\ & \alpha[d(gx, gx^*) + d(F(x,y), gx)] \\ &+ \beta[d(gy, gy^*) + d(F(x^*,y^*), gx^*)] \\ &+ \gamma[d(F(x,y), gx^*) + d(F(x^*,y^*), gx)], \end{aligned} \tag{2.21}$$

where α, β and γ are nonnegative constants with

$$(3K+1)(\alpha + \beta) + 2(K^2 + K)\gamma < 2. \tag{2.22}$$

If $F(X \times X) \subset g(X)$ and $g(X)$ is complete subset of X, then F and g have a coupled coincidence point in X. Also, if F and g are w-compatible, then F and g have a unique coupled common fixed point. Moreover, common fixed point of F and g is of the form (z, z) for some $z \in X$.

Proof. Corollary 2.4 follows from Theorems 2.2 and 2.3 by setting $\alpha_1 = \alpha_2 = \alpha$, $\alpha_3 = \alpha_4 = \beta$ and $\alpha_5 = \alpha_6 = \gamma$. □

Corollary 2.5. *Let (X, d, K) be a cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ and $g : X \to X$ satisfy the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$d(F(x, y), F(x^*, y^*)) \preceq \alpha d(gx, gx^*) + \beta d(gy, gy^*), \tag{2.23}$$

where α, β are nonnegative constants with $\alpha + \beta < 1/K$. If $F(X \times X) \subset g(X)$ and $g(X)$ is complete subset of X, then F and g have a coupled coincidence point in X. Also, if F and g are w-compatible, then F and g have a unique coupled common fixed point. Moreover, common fixed point of F and g is of the form (z, z) for some $z \in X$.

Proof. Corollary 2.5 follows from Theorems 2.2 and 2.3 by setting $\alpha_1 = \alpha$, $\alpha_3 = \beta$ and $\alpha_2 = \alpha_4 = \alpha_5 = \alpha_6 = 0$. □

Corollary 2.6. *Let (X, d, K) be a cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ and $g : X \to X$ satisfy the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$d(F(x, y), F(x^*, y^*)) \preceq \alpha d(F(x, y), gx^*) + \beta d(F(x^*, y^*), gx), \tag{2.24}$$

where α, β are nonnegative constants with $\alpha + \beta < 2/(K^2 + K)$. If $F(X \times X) \subset g(X)$ and $g(X)$ is complete subset of X, then F and g have a coupled coincidence point in X. Also, if F and g are w-compatible, then F and g have a unique coupled common fixed point. Moreover, common fixed point of F and g is of the form (z, z) for some $z \in X$.

Proof. Corollary 2.5 follows from Theorems 2.2 and 2.3 by setting $\alpha_i = 0$ for $i = 1, \ldots, 4$, $\alpha_5 = \alpha$ and $\alpha_6 = \beta$. □

Remark 2.7.

(i) *The Theorems* 2.2 *and* 2.3*, and the Corollary* 2.4 *generalized some common fixed point theorems of cone metric spaces of Abbas et al.'s works in* [29] *by considering cone metric type spaces.*

(ii) *In Corollaries* 2.5 *and* 2.6*, set $K = 1$ and $g = i_x$. Also, suppose X is a complete cone metric space. Then, we get the results of Sabetghadam et al.'s work in* [42]*. Also, our corollaries extend and unify the results of Bhaskar and Lakshmikantham's theorems on a cone metric space in* [28].

Example 2.8. *Let $E = \mathbf{R}$, $P = [0, \infty)$, $X = [0, 1]$ and $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = |x - y|^2$. Then (X, d) is a cone metric type space, but it is not a cone metric space since the triangle inequality is not satisfied. Starting with Minkowski inequality, we get $|x - z|^2 \leq 2(|x - y|^2 + |y - z|^2)$. Here $K = 2$. Define the mappings $F : X \times X \to X$ by $F(x, y) = (x + y)/4$ and $g : X \to X$ by $g = i_X$, where i_X is a identity mapping. Therefore, F and g satisfies the contractive condition (2.23) for $\alpha = \beta = 1/8$ with $\alpha + \beta = 1/4 \in [0, 1/K)$ with $K = 2 \geq 1$; that is,*

$$d(F(x, y), F(x^*, y^*)) \preceq \frac{1}{8}[d(x, x^*) + d(y, y^*)].$$

According to Corollary 2.5, F has a unique coupled fixed point with $g = i_X$. $(0, 0)$ is a unique coupled fixed point of F.

Remark 2.9. *Similar to previous example, one can get many examples of other coupled fixed point theorems in cone metric type spaces.*

3 General Approach

We start with following Lemma.

Lemma 3.1.

(1) Suppose that (X, d, K) is a cone metric type space with $K \geq 1$. Then, (X^2, d_1, K) is a cone metric type space with

$$d_1((x, y), (u, v)) = d(x, u) + d(y, v). \tag{3.1}$$

Further, (X, d, K) is complete if and only if (X^2, d_1, K).

(2) Mappings $F : X^2 \to X$ and $g : X \to X$ have a coupled fixed point if and only if mapping $T_F : X^2 \to X^2$ defined by $T_F(x, y) = (F(x, y), F(y, x))$ and $g : X \to X$ have a coupled common fixed point in X^2.

Proof. The proof of the Lemma is easy and left to reader. □

Totally, there exists a method of reducing some coupled fixed point results to the respective results for mappings with one variable, even obtaining (in some cases) more general theorems. Now, we prove a general version of our theorems and corollaries in previous section.

Theorem 3.2. *Let (X, d, K) be a cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ and $g : X \to X$ satisfy the following* *contractive condition for all $x, y, x^*, y^* \in X$:*

$$\begin{aligned} d(F(x,y),&F(x^*,y^*)) + d(F(y,x),F(y^*,x^*)) \\ &\preceq a_1[d(gx,gx^*) + d(gy,gy^*)] + a_2[d(F(x,y),gx) + d(F(y,x),gy)] \\ &\quad + a_3[d(F(x^*,y^*),gx^*) + d(F(y^*,x^*),gy^*)] \\ &\quad + a_4[d(F(x,y),gx^*) + d(F(y,x),gy^*)] \\ &\quad + a_5[d(F(x^*,y^*),gx) + d(F(y^*,x^*),gy)], \end{aligned} \tag{3.2}$$

where a_i for $i = 1, 2, \ldots, 5$ are nonnegative constants with

$$2Ka_1 + (K+1)(a_2 + a_3) + (K^2 + K)(a_4 + a_5) < 2. \tag{3.3}$$

If $F(X \times X) \subset g(X)$ and $g(X)$ is complete subset of X, then F and g have a coupled coincidence point in X. If F and g are w-compatible, then F and g have a unique coupled common fixed point. Moreover, common fixed point of F and g is of the form (z, z) for some $z \in X$.

Proof. According to (3.1) and Lemma 3.1(2), the contractive condition (3.2) for all $Y = (x, y), V = (x^*, y^*), g(Y) = (gx, gy), g(V) = (gx^*, gy^*) \in X^2$ become

$$\begin{aligned} d_1(T_F(Y), T_F(V)) \preceq a_1 d_1(g(Y), g(V)) + a_2 d_1(T_F(Y), g(Y)) + a_3 d_1(T_F(V), g(V)) \\ + a_4 d_1(T_F(Y), g(V)) + a_5 d_1(T_F(V), g(Y)). \end{aligned}$$

Since $2Ka_1 + (K+1)(a_2 + a_3) + (K^2 + K)(a_4 + a_5) < 2$, the proof further follows by [4, Theorem 3.7]. □

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