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Commutators of Strongly Singular Convolution Operators on Weighted Herz-type Hardy Spaces

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Abstract : In this paper, it is proved that the commutators generated by strongly singular integral operators T_b associated with BMO functions g are bounded from the homogeneous weighted Herz-type Hardy spaces to the homogeneous weighted Herz spaces when $\alpha = n(1 - 1/q)$.

Keywords : Strongly singular convolution operator, commutator, boundedness, Herz space, Hardy space.
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1 Introduction

Many results about the boundedness of singular convolution operators and their commutators have been obtained, see [1],[2],[3],[5] and [6]. But discussions on questions about the boundedness of some strongly singular convolution operators and their commutators are still needed. The commutators of strongly singular convolution operators on weighted Herz-Hardy spaces have been studied by S.chanillo in [4], but there are still some questions which are worth study, such as their boundedness on the Herz-type Hardy spaces. Xiaochun Li and Shanzhen Lu have obtained the boundedness of the strongly singular convolution operators on the weighted Herz-type Hardy spaces in [10].

Although the strongly singular convolution operators are very singular, the commutators of them should be bounded on many function spaces. What we are interested in is the boundedness of commutators on Herz-type Hardy space. In this paper, we show that the commutators $[g, T_b]$ generated by the strongly singular convolution operators T_b associated with BMO functions g are bounded from the homogeneous weighted Herz-type Hardy space $H\dot{K}_{q,b}^{\alpha,p,s}(\omega_1, \omega_2)$ to the homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$.

For convenience, we always use the letter C to denote a positive constant, which may change from one step to the next and only depend on some fixed parameters.

First of all, we introduce the strongly singular convolution operator. Suppose that $\theta(\xi)$ is a smooth radial function, and $\theta(\xi) = 1$ when $|\xi| \geq 1$, $\theta(\xi) = 0$ when $|\xi| \leq 1/2$, the operator T_b is defined by Fourier transform(cf. [10]):

$$(T_b f)^\wedge(\xi) = \theta(\xi)\frac{e^{i|\xi|^b}}{|\xi|^{nb/2}}\hat{f}(\xi), \qquad 0 < b < 1. \tag{1.1}$$

According to the definition, we can see that the kernel of T_b is very singular, briefly, it approximates $K_{b'}(x) = \frac{e^{i|\xi|^{-b'}}}{|x|^n}$, where $b' = b/(1-b)$. In fact, if $|x| \geq 2|y|$, by simple calculation, we can get

$$|K_{b'}(x-y) - K_{b'}(x)| \leq C\frac{|y|}{|x|^{n+b'+1}}. \tag{1.2}$$

Analogous to the usual, we introduce the commutators $[g, T_b]$ as follows.

Definition 1.1 Let g be a BMO function, T_b be a strongly singular operator as (1.1). The commutator $[g, T_b]$ is defined by

$$[g, T_b]f(x) = g(x)T_b f(x) - T_b(gf)(x). \tag{1.3}$$

Now, we write the weighted Herz space and weighted Herz-type Hardy space (cf.[12]) as follows.

Definition 1.2 Let $0 < \alpha < \infty$, $0 < p < \infty$, $1 \leq q < \infty$, and ω_1, ω_2 be nonnegative weight functions. Write $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $E_k = B_k \setminus B_{k-1}$, and denote by χ_k the character function of E_k. Homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ is defined by

$$\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) = \left\{f : f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}, \omega_2) : \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} < \infty\right\}, \tag{1.4}$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \left\{\sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \|f\chi_k\|^p_{L^q(\omega_2)}\right\}^{1/p}.$$

Definition 1.3 Suppose $0 < \alpha < \infty$, $0 < p < \infty$, $1 \leq q < \infty$, and $\omega_1, \omega_2 \in A_1$. Homogeneous weighted Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ is defined by

$$H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) = \left\{f \in S'(\mathbb{R}^n) : Gf \in \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)\right\}. \tag{1.5}$$

And

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \|Gf\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)},$$

where Gf is the Grand Maximal function of f (cf. [12] and [13]).

The $A(p, q)$ class and the weight function class A_p are as usual(cf. [8] and [9]). In fact that if $\omega \in A_p$, then there exist constant $C > 0$ and $\delta_\omega \in (0, 1)$ such that

$$\omega(E)/\omega(Q) \leq C(|E|/|Q|)^{\delta_\omega} \quad \text{and} \quad \omega(Q)/\omega(E) \leq C(|Q|/|E|)^p,$$

for all measurable subset $E \subset Q$, where $\omega(Q) = \int_Q \omega(x)dx$.

2 Main Result

First, we state the main result of this paper as follows.

Theorem 2.1 *Let $0 < p \le 1 < q < \infty$, $\alpha = n(1 - 1/q)$, $g \in BMO(\mathbb{R}^n)$, and $\omega_1, \omega_2 \in A_1$. Then*

$$\|[g, T_b]f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \le C\|f\|_{H\dot{K}_{q,g}^{\alpha,p,s}(\omega_1,\omega_2)}, \tag{2.1}$$

where C is independent of f.

To proof Theorem 2.1 we need two lemmas as follows (cf. [4] and [14]).

Lemma 2.2 *The kernel of $T_b f(x)$ is*

$$C\frac{e^{i\alpha_b|x|^{-b'}}}{|x|^n}\chi(|x| \le 1) + h(x), \tag{2.2}$$

where $b' = b/(1 - b)$, $\alpha_b = b^{b/(1-b)} - b^{1/(1-b)}$, and $|h(x)| \le C(1 + |x|)^{-(n+1)} + C|x|^{-n+\varepsilon}\chi(|x| \le 1)$, for some $\varepsilon > 0$, and ε only depends on b.

Lemma 2.3 *Let $\tilde{K}_{b',s}(x) = \dfrac{e^{i\alpha_b|x|^{-b'}}}{|x|^{n(b'+2)/s}}$, and $(b'+2)/s < 1$. Then $\|\tilde{K}_{b',s} * f\|_s \le C\|f\|_{s'}$, where $1/s + 1/s' = 1$.*

Now, we recall the $(\alpha, q, \omega_1, \omega_2)$-atom, the $(\alpha, q, \omega_1, \omega_2, s, b)$-atom, and atomic decomposition of the Herz-type Hardy spaces.

Definition 2.4 Let $0 < p < \infty$, $1 \le q < \infty$, $s \ge [\alpha+n(1/q-1)]$, and $\omega_1, \omega_2 \in A_1$. A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha, q, \omega_1, \omega_2)$-atom, if it satisfies the following three conditions:

(a) $\operatorname{supp} a \subset B(0, r)$, $r > 0$, where $B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$;

(b) $\|a\|_{L^q(\omega_2)} \le [\omega_1(B(0, r))]^{-\alpha/n}$;

(c) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0, \qquad |\beta| \le s.$

Definition 2.1′ Let $0 < p < \infty$, $1 \le q < \infty$, $s \ge [\alpha + n(1/q - 1)]$, and $\omega_1, \omega_2 \in A_1$. A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha, q, \omega_1, \omega_2, s, b)$-atom, if it satisfies (a),(b) in Definition 2.1 and

(c′) $\int_{\mathbb{R}^n} a(x)x^\beta dx = \int_{\mathbb{R}^n} a(x)b(x)x^\beta dx = 0, \qquad |\beta| \le s.$

Definition 2.5 Let $0 < p < \infty$, $1 \le q < \infty$, $n(1-1/q) \le \alpha < \infty$, and $\omega_1, \omega_2 \in A_1$. Then f is said to be in $H\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ (or $H\dot{K}_{q,b}^{\alpha,p,s}(\omega_1, \omega_2)$), if in the sense of distribution

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

where a_k is a central $(\alpha, q, \omega_1, \omega_2)$-atom(or $(\alpha, q, \omega_1, \omega_2, s, b)$-atom) with the support B_k, $k \in \mathbb{Z}$, and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. Moreover

$$\|f\|_{H\dot{K}_q^{\alpha,p}} = \inf\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \qquad (\|f\|_{H\dot{K}_{q,b}^{\alpha,p,s}} = \inf\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p}),$$

where the infimum is taken over all the decompositions of f.

For commutators $[g, T_b]$, in[1,8], it is easy to see that the following lemma holds.

Lemma 2.6 *Suppose $g \in BMO(\mathbb{R}^n)$, $1 < q < \infty$, $\omega \in A_1$. Then $[g, T_b]$ is bounded on $L^q(\omega)$.*

Proof of Theorem 2.1 We only need to prove that (2.1) holds for any $f \in H\dot{K}_{q,g}^{\alpha,p,0}(\omega_1, \omega_2)$. By Definition 2.2, for $f \in H\dot{K}_{q,g}^{\alpha,p,0}(\omega_1, \omega_2)$, there exists decomposition $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where a_j is a central $(\alpha, q, \omega_1, \omega_2, 0, g)$-atom with the support B_j, and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. Write

$$\begin{aligned}\|[g, T_b]f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} &= \sum_{k=-\infty}^{+\infty} [\omega_1(B_k)]^{\alpha p/n} \|[g, T_b]f\chi_k\|^p_{L^q(\omega_2)} \\ &= \sum_{k=-\infty}^{+\infty} [\omega_1(B_k)]^{\alpha p/n} \left\| \sum_{j=-\infty}^{\infty} \lambda_j [g, T_b] a_j \chi_k \right\|^p_{L^q(\omega_2)}. \qquad (2.3)\end{aligned}$$

For $0 < p \leq 1$, by the Jensen inequality, we get that

$$\begin{aligned}\|[g, T_b]f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} &\leq \sum_{k=-\infty}^{+\infty} [\omega_1(B_k)]^{\alpha p/n} \sum_{j=-\infty}^{\infty} |\lambda_j|^p \|[g, T_b]a_j\chi_k\|^p_{L^q(\omega_2)} \\ &\leq \sum_{j=-\infty}^{+\infty} |\lambda_j|^p \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \|[g, T_b]a_j\chi_k\|^p_{L^q(\omega_2)}. \qquad (2.4)\end{aligned}$$

So, we only need to prove that for any central $(\alpha, q, \omega_1, \omega_2, 0, g)$-atom a, there exists a constant $C > 0$ such that

$$\|[g, T_b]a\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \leq C,$$

where C is independent of a.

Let $\operatorname{supp} a \subset B(0,r)$. And for any $k_0 \in Z$, let $r = 2^{k_0-1}$. Then

$$\begin{aligned}
\|[g,T_b]a\|^p_{\dot K_q^{\alpha,p}(\omega_1,\omega_2)} &= \sum_{k=-\infty}^{+\infty} [\omega_1(B_k)]^{\alpha p/n} \|[g,T_b]a\chi_k\|^p_{L^q(\omega_2)} \\
&= \sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{\alpha p/n} \|[g,T_b]a\chi_k\|^p_{L^q(\omega_2)} \\
&\quad + \sum_{k=k_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \|[g,T_b]a\chi_k\|^p_{L^q(\omega_2)} \\
&:= I + II. \qquad (2.5)
\end{aligned}$$

For I, since $\omega_1, \omega_2 \in A_1$, by the weighted boundedness of $[g,T_b]$ in Lemma 2.4, we have

$$\begin{aligned}
I &\le \sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{\alpha p/n} \|a\|^p_{L^q(\omega_2)} \\
&\le \sum_{k=-\infty}^{k_0-1} \left[\frac{\omega_1(B_k)}{\omega_1(B(0,r))}\right]^{\alpha p/n} + \left[\frac{\omega_1(B_{k_0})}{\omega_1(B(0,r))}\right]^{\alpha p/n} \\
&\le \sum_{k=-\infty}^{k_0-1} \left(\frac{|B_k|}{|B(0,r)|}\right)^{\delta\alpha p/n} + \left(\frac{|B_{k_0}|}{|B(0,r)|}\right)^{\alpha p/n} \\
&\le \sum_{k=-\infty}^{k_0-1} 2^{\delta\alpha p(k+1-k_0)} + 2^{\alpha p} \le C, \qquad (2.6)
\end{aligned}$$

where δ only depends on ω_1.

Suppose $K_{b'}(x) = C\dfrac{e^{i\alpha_b|x|^{-b'}}}{|x|^n}\chi(|x| \le 1)$. By Lemma 2.2, one can get

$$\begin{aligned}
II &\le \sum_{k=k_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \left(\int_{E_k}\left|\int_{B(0,r)} K_{b'}(x-y)(g(x)-g(y))a(y)dy\right|^q \omega_2(x)dx\right)^{p/q} \\
&\quad + \sum_{k=k_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \left(\int_{E_k}\left|\int_{B(0,r)} h(x-y)(g(x)-g(y))a(y)dy\right|^q \omega_2(x)dx\right)^{p/q} \\
&:= J_1 + J_2. \qquad (2.7)
\end{aligned}$$

By the Minkowski inequality, we get that

$$\begin{aligned} J_2 \leq & \sum_{k=k_0+1}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\Big(\int_{E_k}\Big|\int_{B(0,r)}h(x-y)(g(x)-g_r)a(y)dy\Big|^q\omega_2(x)dx\Big)^{p/q} \\ & + \sum_{k=k_0+1}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\Big(\int_{E_k}\Big|\int_{B(0,r)}h(x-y)(g_r-g(y))a(y)dy\Big|^q\omega_2(x)dx\Big)^{p/q} \\ := & J_{21}+J_{22}. \end{aligned} \tag{2.8}$$

To estimate J_{21}, we first give the pointwise estimation of $h*a(x)$. If $|x|\geq 2r$, then

$$\begin{aligned} |h*a(x)| & \leq C\int_{B(0,r)}|a(t)|\cdot\Big(\frac{1}{(1+|x-t|)^{n+1}}+\frac{\chi(|x-t|\leq 1)}{|x-t|^{n-\varepsilon}}\Big)dt \\ & \leq C\Big(\int_{B(0,r)}|a(t)|dt\Big)\Big(\frac{1}{(1+|x|)^{n+1}}+\frac{\chi(|x|\leq 2)}{|x|^{n-\varepsilon}}\Big) \\ & \leq C\|a\|_{L^q_{\omega_2}}\Big(\int_{B(0,r)}\omega_2^{-q'/q}(t)dt\Big)^{1/q'}\Big(\frac{1}{(1+|x|)^{n+1}}+\frac{\chi(|x|\leq 2)}{|x|^{n-\varepsilon}}\Big) \\ & \leq C[\omega_1(B(0,r))]^{-\alpha/n}r^{n/q'}\Big(\mathrm{essinf}_{B(0,r)}\omega_2\Big)^{-1/q}\Big(\frac{1}{(1+|x|)^{n+1}}+\frac{\chi(|x|\leq 2)}{|x|^{n-\varepsilon}}\Big). \end{aligned} \tag{2.9}$$

Thus

$$\begin{aligned} J_{21} \leq & C\sum_{k=k_0+1}^{\infty}[\sup_{x\in E_k}\omega_2(x)]^{p/q}[\omega_1(B_k)]^{\alpha p/n} \\ & \cdot\Big(\int_{E_k}|g(x)-g_r|^q\cdot\Big|\int_{B(0,r)}h(x-y)a(y)dy\Big|^q dx\Big)^{p/q} \\ \leq & CC_1 r^{np/q'}\Big(\mathrm{essinf}_{B(0,r)}\omega_2\Big)^{-p/q}\sum_{k=k_0+1}^{\infty}[\inf_{x\in E_k}\omega_2(x)]^{p/q}\Big[\frac{\omega_1(B_k)}{\omega_1(B(0,r))}\Big]^{\alpha p/n} \\ & \cdot\Big(\int_{E_k}|g(x)-g_r|^q\cdot\Big(\frac{1}{(1+|x|)^{n+1}}+\frac{\chi(|x|\leq 2)}{|x|^{n-\varepsilon}}\Big)^q dx\Big)^{p/q} \\ \leq & C\|g\|^p_{BMO}r^{np/q'}\Big(\mathrm{essinf}_{B(0,r)}\omega_2\Big)^{-p/q} \\ & \cdot\sum_{k=k_0+1}^{\infty}(k+1-k_0)\Big(\frac{\omega_2(B_k)}{|B_k|}\Big)^{p/q}\Big(\frac{|B_k|}{|B(0,r)|}\Big)^{\alpha p/n}\frac{2^{knp/q}}{(1+2^k)^{(n+1)p}} \\ & +C\|g\|^p_{BMO}r^{np/q'}\Big(\mathrm{essinf}_{B(0,r)}\omega_2\Big)^{-p/q} \end{aligned}$$

$$\begin{aligned}
&\cdot \sum_{k=k_0+1}^{1}(k+1-k_0)\Big(\frac{\omega_2(B_k)}{|B_k|}\Big)^{p/q}\Big(\frac{|B_k|}{|B(0,r)|}\Big)^{\alpha p/n}\frac{2^{knp/q}}{2^{k(n-\varepsilon)p}}\\
\leq & C\|g\|_{BMO}^p r^{p(n/q'-\alpha)}\Big(\mathrm{essinf}_{B(0,r)}\omega_2\Big)^{-p/q}\Big(\frac{\omega_2(B(0,r))}{|B(0,r)|}\Big)^{p/q}\\
&\cdot \sum_{k=k_0+1}^{\infty}(k+1-k_0)\frac{2^{knp/q}2^{k\alpha p}}{(1+2^k)^{(n+1)p}}\\
&+C\|g\|_{BMO}^p r^{p(n/q'-\alpha)}\Big(\mathrm{essinf}_{B(0,r)}\omega_2\Big)^{-p/q}\Big(\frac{\omega_2(B(0,r))}{|B(0,r)|}\Big)^{p/q}\\
&\cdot \sum_{k=k_0+1}^{1}(k+1-k_0)\frac{2^{knp/q}2^{k\alpha p}}{2^{k(n-\varepsilon)p}}\\
\leq & C\|g\|_{BMO}^p r^{p(n/q'-\alpha)}\Big(\sum_{k=k_0+1}^{\infty}(k+1-k_0)\frac{2^{kp(n/q+\alpha)}}{(1+2^k)^{(n+1)p}}\\
&+\sum_{k=k_0+1}^{1}(k+1-k_0)\frac{2^{kp(n/q+\alpha)}}{2^{k(n-\varepsilon)p}}\Big)\\
\leq & C\|g\|_{BMO}^p r^{p(n/q'-\alpha)}\Big(\sum_{k=1}^{\infty}(k+1-k_0)\frac{1}{2^{kp(n/q'+1-\alpha)}}\\
&+\sum_{k=k_0+1}^{1}2^{kp(n/q+\alpha)}+\sum_{k=k_0+1}^{1}2^{kp(\alpha-n/q'+\varepsilon)}\Big).
\end{aligned}$$

Since $\alpha=n(1-1/q)$, we get

$$J_{21}\leq C\|g\|_{BMO}^p\Big(\sum_{k=1}^{\infty}(k+1-k_0)\frac{1}{2^{kp}}+\sum_{k=k_0+1}^{1}2^{kpn}+\sum_{k=k_0+1}^{1}2^{kp\varepsilon}\Big)\leq C. \tag{2.10}$$

For J_{22}, by the Hölder inequality, we have

$$\begin{aligned}
J_{22}\leq C&\sum_{k=k_0+1}^{\infty}[\omega_1(B_k)]^{\alpha p/n}[\sup_{x\in E_k}\omega_2(x)]^{p/q}\Big(\int_{B(0,r)}|g(y)-g_r|^{q'}dy\Big)^{p/q'}\\
&\cdot\Big(\int_{E_k}\int_{B(0,r)}|h(x-y)a(y)|^q dydx\Big)^{p/q}.
\end{aligned}\tag{2.11}$$

Similarly, by the estimation of $h*a(x)$, if $|x|\geq 2r$, then

$$\int_{B(0,r)}|h(x-y)a(y)|^q dy\leq C\Big(\frac{1}{(1+|x|)^{n+1}}+\frac{\chi(|x|\leq 2)}{|x|^{n-\varepsilon}}\Big)^q\int_{B(0,r)}|a(y)|^q dy. \tag{2.12}$$

Hence

$$J_{22} \leq CC_1 r^{np/q'} \sum_{k=k_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} [\inf_{x\in E_k} \omega_2(x)]^{p/q} \Big(\int_{B(0,r)} |a(y)|^q dy \Big)^{p/q}$$

$$\cdot \Big(\int_{E_k} (\frac{1}{(1+|x|)^{(n+1)q}} + \frac{\chi(|x| \leq 2)}{|x|^{(n-\varepsilon)q}}) dx \Big)^{p/q}$$

$$\leq C\|g\|^p_{BMO} r^{pn/q'} \sum_{k=k_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \Big[\frac{\omega_2(B_k)}{|B_k|} \Big]^{p/q}$$

$$\cdot \Big(\int_{B(0,r)} |a(y)|^q dy \Big)^{p/q} \frac{2^{knp/q}}{(1+2^k)^{(n+1)p}}$$

$$+ C\|g\|^p_{BMO} r^{pn/q'} \sum_{k=k_0+1}^{1} [\omega_1(B_k)]^{\alpha p/n} \Big[\frac{\omega_2(B_k)}{|B_k|} \Big]^{p/q} \Big(\int_{B(0,r)} |a(y)|^q dy \Big)^{p/q} \frac{2^{knp/q}}{2^{k(n-\varepsilon)p}}$$

$$\leq C\|g\|^p_{BMO} r^{pn/q'} \sum_{k=k_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \Big[\frac{\omega_2(B(0,r))}{|B(0,r)|} \Big]^{p/q}$$

$$\cdot \Big(\int_{B(0,r)} |a(y)|^q dy \Big)^{p/q} \frac{2^{knp/q}}{(1+2^k)^{(n+1)p}}$$

$$+ C\|g\|^p_{BMO} r^{pn/q'} \sum_{k=k_0+1}^{1} [\omega_1(B_k)]^{\alpha p/n} \Big[\frac{\omega_2(B(0,r))}{|B(0,r)|} \Big]^{p/q} \Big(\int_{B(0,r)} |a(y)|^q dy \Big)^{p/q} \frac{2^{knp/q}}{2^{k(n-\varepsilon)p}}$$

$$\leq C\|g\|^p_{BMO} r^{pn/q'} \sum_{k=k_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \Big(\int_{B(0,r)} |a(y)|^q \omega_2(y) dy \Big)^{p/q} \frac{2^{knp/q}}{(1+2^k)^{(n+1)p}}$$

$$+ C\|g\|^p_{BMO} r^{pn/q'} \sum_{k=k_0+1}^{1} [\omega_1(B_k)]^{\alpha p/n} \Big(\int_{B(0,r)} |a(y)|^q \omega_2(y) dy \Big)^{p/q} \frac{2^{knp/q}}{2^{k(n-\varepsilon)p}}$$

$$\leq C\|g\|^p_{BMO} r^{pn/q'} \sum_{k=k_0+1}^{\infty} \Big[\frac{\omega_1(B_k)}{\omega_1(B(0,r))} \Big]^{\alpha p/n} \frac{2^{knp/q}}{(1+2^k)^{(n+1)p}}$$

$$+ C\|g\|^p_{BMO} r^{pn/q'} \sum_{k=k_0+1}^{1} \Big[\frac{\omega_1(B_k)}{\omega_1(B(0,r))} \Big]^{\alpha p/n} \frac{2^{knp/q}}{2^{k(n-\varepsilon)p}}$$

$$\leq C\|g\|^p_{BMO} r^{p(n/q'-\alpha)} \Big(\sum_{k=k_0+1}^{\infty} \frac{2^{kp(\alpha+n/q)}}{(1+2^k)^{(n+1)p}} + \sum_{k=k_0+1}^{1} \frac{2^{kp(\alpha+n/q)}}{2^{k(n-\varepsilon)p}} \Big)$$

$$\leq C\|g\|^p_{BMO} r^{p(n/q'-\alpha)} \Big(\sum_{k=1}^{\infty} \frac{1}{2^{kp(n/q'+1-\alpha)}} + \sum_{k=k_0+1}^{1} 2^{kp(\alpha+n/q)} + \sum_{k=k_0+1}^{1} 2^{kp(\alpha-n/q'+\varepsilon)} \Big).$$

Since $\alpha = n(1-1/q)$, we get

$$J_{22} \le C\|g\|_{BMO}^p \Bigg(\sum_{k=1}^{\infty} \frac{1}{2^{kp}} + \sum_{k=k_0+1}^{1} 2^{knp} + \sum_{k=k_0+1}^{1} 2^{kp\varepsilon} \Bigg) \le C. \tag{2.13}$$

To estimate J_1, if $2^{j_0-1} < r^{1-b} \le 2^{j_0}$, for some $j_0 \in Z$, then

$$\begin{aligned} J_1 \le & \sum_{k=k_0+1}^{j_0} [\omega_1(B_k)]^{\alpha p/n} \Bigg(\int_{E_k} \bigg| \int_{B(0,r)} K_{b'}(x-y)(g(x)-g(y))a(y)dy \bigg|^q \omega_2(x)dx \Bigg)^{p/q} \\ & + \sum_{k=j_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \Bigg(\int_{E_k} \bigg| \int_{B(0,r)} K_{b'}(x-y)(g(x)-g(y))a(y)dy \bigg|^q \omega_2(x)dx \Bigg)^{p/q} \\ & := J_{11} + J_{12}. \end{aligned} \tag{2.14}$$

For J_{12}, by the Minkowski inequality, we get that

$$\begin{aligned} J_{12} \le & \sum_{k=j_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \Bigg(\int_{E_k} \bigg| \int_{B(0,r)} K_{b'}(x-y)(g(x)-g_r)a(y)dy \bigg|^q \omega_2(x)dx \Bigg)^{p/q} \\ & + \sum_{k=j_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \Bigg(\int_{E_k} \bigg| \int_{B(0,r)} K_{b'}(x-y)(g_r-g(y))a(y)dy \bigg|^q \omega_2(x)dx \Bigg)^{p/q} \\ & := J_{121} + J_{122}. \end{aligned} \tag{2.15}$$

To estimate $K_{b'} * a(x)$, let $|x| \ge 2r$. By the cancelling condition of $a(x)$, we can get

$$|K_{b'} * a(x)| \le \int_{B(0,r)} |K_{b'}(x-y) - K_{b'}(x)| \cdot |a(y)|dy.$$

And by the property

$$|K_{b'}(x-y) - K_{b'}(x)| \le C\frac{|y|}{|x|^{n+b'+1}}, \quad |x| \ge 2|y|,$$

we get that

$$\begin{aligned} |K_{b'} * a(x)| & \le C\frac{r}{|x|^{n+b'+1}} \int_{B(0,r)} |a(y)|dy \\ & \le C\frac{r}{|x|^{n+b'+1}} \|a\|_{L^q(\omega_2)} \Bigg(\int_{B(0,r)} \omega_2^{-q'/q}(x)dx \Bigg)^{1/q'} \\ & \le Cr^{n/q'+1}[\omega_1(B(0,r))]^{-\alpha/n} \Bigg(\mathrm{essinf}_{B(0,r)}\omega_2 \Bigg)^{-1/q} \frac{1}{|x|^{n+b'+1}}. \end{aligned} \tag{2.16}$$

Therefore

$$\begin{aligned}
J_{121} \leq\ & C \sum_{k=j_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} [\sup_{x\in E_k} \omega_2(x)]^{p/q} \\
& \cdot\bigg(\int_{E_k} |g(x)-g_r|^q \bigg| \int_{B(0,r)} K_{b'}(x-y)a(y)dy \bigg|^q dx \bigg)^{p/q} \\
\leq\ & CC_1 r^{(n/q'+1)p} \bigg(\mathrm{essinf}_{B(0,r)}\omega_2 \bigg)^{-p/q} \sum_{k=j_0+1}^{\infty} [\inf_{x\in E_k} \omega_2(x)]^{p/q} \bigg[\frac{\omega_1(B_k)}{\omega_1(B(0,r))} \bigg]^{\alpha p/n} \\
& \cdot\bigg(\int_{E_k} |g(x)-g_r|^q \cdot \bigg(\frac{1}{|x|^{n+b'+1}} \bigg)^q dx \bigg)^{p/q} \\
\leq\ & C\|g\|_{BMO}^p r^{(n/q'+1)p} \bigg(\mathrm{essinf}_{B(0,r)}\omega_2 \bigg)^{-p/q} \\
& \cdot \sum_{k=j_0+1}^{\infty} (k+1-k_0)^p \bigg(\frac{\omega_2(B_k)}{|B_k|} \bigg)^{p/q} \bigg[\frac{|B_k|}{|B(0,r)|} \bigg]^{\alpha p/n} \frac{2^{knp/q}}{2^{kp(n+b'+1)}} \\
\leq\ & C\|g\|_{BMO}^p r^{p(n/q'+1-\alpha)} \bigg(\mathrm{essinf}_{B(0,r)}\omega_2 \bigg)^{-p/q} \\
& \cdot \sum_{k=j_0+1}^{\infty} (k+1-k_0)^p \bigg(\frac{\omega_2(B(0,r))}{|B(0,r)|} \bigg)^{p/q} \frac{2^{knp/q}2^{k\alpha p}}{2^{kp(n+b'+1)}}.
\end{aligned}$$

Since $\alpha = n(1-1/q)$, then

$$\begin{aligned}
J_{121} &\leq C\|g\|_{BMO}^p r^{p(n/q'+1-\alpha)} \bigg(\sum_{k=j_0+1}^{\infty} (k+1-k_0)^p \frac{1}{2^{kp(b'+1+n/q'-\alpha)}} \bigg) \\
&= C\|g\|_{BMO}^p r^p \bigg(\sum_{k=j_0+1}^{\infty} (k+1-k_0)^p \frac{1}{2^{kp(b'+1)}} \bigg) \\
&\leq C\|g\|_{BMO}^p r^p \frac{1}{2^{j_0 p(b'+1)}} \\
&\leq \|g\|_{BMO}^p r^p \frac{1}{r^{(1-b)(b'+1)p}} \leq C. \qquad (2.17)
\end{aligned}$$

For J_{122}, by cancelling condition and the Hölder inequality, we get that

$$\begin{aligned}
J_{122} \leq & C \sum_{k=j_0+1}^{\infty} [\sup_{x\in E_k} \omega_2(x)]^{p/q} [\omega_1(B_k)]^{\alpha p/n} \bigg(\int_{B(0,r)} |g(y)-g_r|^{q'} dy \bigg)^{p/q'} \\
& \cdot \bigg(\int_{E_k} \int_{B(0,r)} |(K_{b'}(x-y)-K_{b'}(x))a(y)|^q dydx \bigg)^{p/q}. \qquad (2.18)
\end{aligned}$$

Similarly, if $|x| \geq 2r$, then

$$\int_{B(0,r)} |(K_{b'}(x-y)-K_{b'}(x))a(y)|^q dy \leq C \frac{r^q}{|x|^{(n+b'+1)q}} \int_{B(0,r)} |a(y)|^q dy. \qquad (2.19)$$

So

$$J_{122} \leq C\|g\|^p_{BMO} r^{(n/q'+1)p} \sum_{k=j_0+1}^{\infty} [\inf_{x\in E_k} \omega_2(x)]^{p/q} [\omega_1(B_k)]^{\alpha p/n}$$

$$\cdot \Big(\int_{B(0,r)} |a(y)|^q dy \Big)^{p/q} \Big(\int_{E_k} \frac{1}{|x|^{(n+b'+1)q}} \Big)^{p/q}$$

$$\leq C\|g\|^p_{BMO} r^{(n/q'+1)p} \sum_{k=j_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \Big[\frac{\omega_2(B_k)}{|B_k|} \Big]^{p/q}$$

$$\cdot \Big(\int_{B(0,r)} |a(y)|^q dy \Big)^{p/q} \frac{2^{knp/q}}{2^{kp(n+b'+1)}}$$

$$\leq C\|g\|^p_{BMO} r^{(n/q'+1)p} \sum_{k=j_0+1}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \Big(\int_{B(0,r)} |a(y)|^q dy \Big)^{p/q} \frac{2^{knp/q}}{2^{kp(n+b'+1)}}$$

$$\leq C\|g\|^p_{BMO} r^{(n/q'+1)p} \sum_{k=j_0+1}^{\infty} \Big[\frac{\omega_1(B_k)}{\omega_1(B(0,r))} \Big]^{\alpha p/n} \frac{2^{knp/q}}{2^{kp(n+b'+1)}}$$

$$\leq C\|g\|^p_{BMO} r^{(n/q'+1)p} \sum_{k=j_0+1}^{\infty} \Big[\frac{|B_k|}{|B(0,r)|} \Big]^{\alpha p/n} \frac{2^{knp/q}}{2^{kp(n+b'+1)}}$$

$$\leq C\|g\|^p_{BMO} r^{(n/q'+1-\alpha)p} \sum_{k=j_0+1}^{\infty} \frac{1}{2^{kp(b'+1+n/q'-\alpha)}}$$

$$\leq C\|g\|^p_{BMO} r^{p} \sum_{k=j_0+1}^{\infty} \frac{1}{2^{kp(b'+1)}} \leq C\|g\|^p_{BMO} r^{p} \frac{1}{2^{j_0 p(b'+1)}}$$

$$\leq C\|g\|^p_{BMO} r^{p} \frac{1}{r^{p(1-b)(b'+1)}} \leq C.$$

Now, to estimate J_{11}. Decomposing $K_{b'} * a(x)$ as

$$K_{b'} * a(x) = C \int \frac{e^{i\alpha_b |x-y|^{-b'}}}{|x-y|^{n(b'+2)/s}} \Big(\frac{1}{|x-y|^{n[1-(b'+2)/s]}} - \frac{1}{|x|^{n[1-(b'+2)/s]}} \Big) a(y) dy$$

$$+ C(\tilde{K}_{b',s} * a(x)) \frac{1}{|x|^{n[1-(b'+2)/s]}}$$

$$:= E(x) + F(x). \tag{2.20}$$

Applying Intermediate Value Theorem to the bracket part in the integrand of $E(x)$, for $|x| \geq 2r$, we can get the pointwise estimate of $E(x)$ as follows.

$$E(x) \leq C \int_{B(0,r)} \frac{|y|}{|x|^{n+1}} |a(y)| dy$$

$$\leq C[\omega_1(B(0,r))]^{-\alpha/n} \Big(\operatorname{essinf}_{B(0,r)} \omega_2 \Big)^{-1/q} r^{n/q'+1} \frac{1}{|x|^{n+1}}.$$

Write

$$E'(x,y)=\frac{e^{i\alpha_b|x-y|^{-b'}}}{|x-y|^{n(b'+2)/s}}\left(\frac{1}{|x-y|^{n[1-(b'+2)/s]}}-\frac{1}{|x|^{n[1-(b'+2)/s]}}\right).$$

Thus

$$\begin{aligned}J_{11}\leq&\sum_{k=k_0+1}^{j_0}[\omega_1(B_k)]^{\alpha p/n}\left(\int_{E_k}\left|\int_{B(0,r)}E'(x,y)(g(x)-g(y))a(y)dy\right|^q\omega_2(x)dx\right)^{p/q}\\&+\sum_{k=k_0+1}^{j_0}[\omega_1(B_k)]^{\alpha p/n}\\&\cdot\left(\int_{E_k}\left|\frac{1}{|x|^{n[1-(b'+2)/s]}}\int_{B(0,r)}\tilde{K}_{b',s}(x-y)(g(x)-g(y))a(y)dy\right|^q\omega_2(x)dx\right)^{p/q}\\&:=J_{111}+J_{112}.\end{aligned}\tag{2.21}$$

For J_{111}, by the Minkowski inequality, we get that

$$\begin{aligned}J_{111}\leq&\sum_{k=k_0+1}^{j_0}[\omega_1(B_k)]^{\alpha p/n}\left(\int_{E_k}\left|\int_{B(0,r)}E'(x,y)(g(x)-g_r)a(y)dy\right|^q\omega_2(x)dx\right)^{p/q}\\&+\sum_{k=k_0+1}^{j_0}[\omega_1(B_k)]^{\alpha p/n}\left(\int_{E_k}\left|\int_{B(0,r)}E'(x,y)(g_r-g(y))a(y)dy\right|^q\omega_2(x)dx\right)^{p/q}\\&:=J_{1111}+J_{1112}.\end{aligned}\tag{2.22}$$

By the pointwise estimate of $E(x)$, we can show that

$$\begin{aligned}J_{1111}\leq& C\sum_{k=k_0+1}^{j_0}[\omega_1(B_k)]^{\alpha p/n}[\sup_{x\in E_k}\omega_2(x)]^{p/q}\\&\cdot\left(\int_{E_k}|g(x)-g_r|^q\cdot\left|\int_{B(0,r)}E'(x,y)a(y)dy\right|^q dx\right)^{p/q}\\\leq& CC_1r^{(n/q'+1)p}\left(\operatorname{essinf}_{x\in B(0,r)}\omega_2\right)^{-p/q}\\&\cdot\sum_{k=k_0+1}^{j_0}[\inf_{x\in E_k}\omega_2(x)]^{p/q}\left[\frac{\omega_1(B_k)}{\omega_1(B(0,r))}\right]^{\alpha p/n}\left(\int_{E_k}\frac{|g(x)-g_r|^q}{|x|^{(n+1)q}}dx\right)^{p/q}\\\leq& C\|g\|_{BMO}^p r^{(n/q'+1-\alpha)p}\left(\operatorname{essinf}_{x\in B(0,r)}\omega_2\right)^{-p/q}\\&\cdot\sum_{k=k_0+1}^{j_0}(k+1-k_0)^p\frac{2^{kp(\alpha+n/q)}}{2^{k(n+1)p}}\left[\frac{\omega_2(B_k)}{|B_k|}\right]^{p/q}\end{aligned}$$

$$\begin{aligned}
&\le C\|g\|_{BMO}^p r^{(n/q'+1-\alpha)p}\Big(\text{essinf}_{x\in B(0,r)}\omega_2\Big)^{-p/q} \\
&\quad\cdot\sum_{k=k_0+1}^{j_0}(k+1-k_0)^p\frac{1}{2^{kp(1+n/q'-\alpha)}}\left[\frac{\omega_2(B(0,r))}{|B(0,r)|}\right]^{p/q} \\
&\le C\|g\|_{BMO}^p r^{(n/q'+1-\alpha)p}\sum_{k=k_0+1}^{j_0}(k+1-k_0)^p\frac{1}{2^{kp(1+n/q'-\alpha)}}.
\end{aligned}$$

Note that $\alpha = n(1-1/q)$, hence

$$J_{1111} \le C\|g\|_{BMO}^p r^p\frac{1}{2^{K_0 p}} \le C. \tag{2.23}$$

For J_{1112}, by the Hölder inequality, one can get

$$\begin{aligned}
J_{1112} \le\ & C\sum_{k=k_0+1}^{j_0}[\omega_1(B_k)]^{\alpha p/n}[\sup_{x\in E_k}\omega_2(x)]^{p/q}\Big(\int_{B(0,r)}|g(y)-g_r|^{q'}dy\Big)^{p/q'} \\
&\cdot\Big(\int_{E_k}\int_{B(0,r)}|E'(x,y)a(y)|^q dydx\Big)^{p/q}.
\end{aligned}$$

Similarly , if $|x|\ge 2r$, then

$$\int_{B(0,r)}|E'(x,y)a(y)|^q dy \le C\frac{r^q}{|x|^{(n+1)q}}\int_{B(0,r)}|a(y)|^q dy.$$

So

$$\begin{aligned}
J_{1112} \le\ & CC_1\|g\|_{BMO}^p r^{(n/q'+1)p}\sum_{k=k_0+1}^{j_0}[\omega_1(B_k)]^{\alpha p/n}[\inf_{x\in E_k}\omega_2(x)]^{p/q} \\
&\cdot\Big(\int_{B(0,r)}|a(y)|^q dy\Big)^{p/q}\Big(\int_{E_k}\frac{1}{|x|^{(n+1)q}}dx\Big)^{p/q} \\
\le\ & C\|g\|_{BMO}^p r^{(n/q'+1)p}\sum_{k=k_0+1}^{j_0}[\omega_1(B_k)]^{\alpha p/n}\left[\frac{\omega_2(B_k)}{|B_k|}\right]^{p/q} \\
&\cdot\Big(\int_{B(0,r)}|a(y)|^q dy\Big)^{p/q}\frac{2^{knp/q}}{2^{kp(n+1)}} \\
\le\ & C\|g\|_{BMO}^p r^{(n/q'+1)p}\sum_{k=k_0+1}^{j_0}[\omega_1(B_k)]^{\alpha p/n}\Big(\int_{B(0,r)}|a(y)|^q\omega_2(y)dy\Big)^{p/q}\frac{2^{knp/q}}{2^{kp(n+1)}} \\
\le\ & C\|g\|_{BMO}^p r^{(n/q'+1)p}\sum_{k=k_0+1}^{j_0}\left[\frac{\omega_1(B_k)}{\omega_1(B(0,r))}\right]^{\alpha p/n}\frac{2^{knp/q}}{2^{kp(n+1)}} \\
\le\ & C\|g\|_{BMO}^p r^{(n/q'+1)p}\sum_{k=k_0+1}^{j_0}\left[\frac{|B_k|}{|B(0,r)|}\right]^{\alpha p/n}\frac{2^{knp/q}}{2^{kp(n+1)}} \\
\le\ & C\|g\|_{BMO}^p r^{(n/q'+1-\alpha)p}\sum_{k=k_0+1}^{j_0}\frac{1}{2^{kp(1+n/q'-\alpha)}}.
\end{aligned}$$

Since $\alpha = n(1-1/q)$, then

$$J_{1112} \le C\|g\|^p_{BMO} r^p \frac{1}{2^{K_0 p}} \le C. \tag{2.24}$$

For J_{112}, by the Minkowski inequality, we get

$$\begin{aligned} J_{112} \le & \sum_{k=k_0+1}^{j_0} [\omega_1(B_k)]^{\alpha p/n} \\ & \cdot \left(\int_{E_k} \frac{\omega_2(x)}{|x|^{nq[1-(b'+2)/s]}} \left| \int_{B(0,r)} \tilde{K}_{b',s}(x-y)(g(x)-g_r)a(y)dy \right|^q dx \right)^{p/q} \\ & + \sum_{k=k_0+1}^{j_0} [\omega_1(B_k)]^{\alpha p/n} \\ & \cdot \left(\int_{E_k} \frac{\omega_2(x)}{|x|^{nq[1-(b'+2)/s]}} \left| \int_{B(0,r)} \tilde{K}_{b',s}(x-y)(g_r-g(y))a(y)dy \right|^q dx \right)^{p/q} \\ := & J_{1121} + J_{1122}. \end{aligned} \tag{2.25}$$

Now, for J_{1121}, one can get

$$\begin{aligned} J_{1121} \le & C \sum_{k=k_0+1}^{j_0} [\omega_1(B_k)]^{\alpha p/n} [\sup_{x\in E_k} \omega_2(x)]^{p/q} 2^{-knp[1-(b'+2)/s]} \\ & \cdot \left(\int_{E_k} |g(x)-g_r|^q \cdot \left| \int_{B(0,r)} \tilde{K}_{b',s}(x-y)a(y)dy \right|^q dx \right)^{p/q} \\ \le & C \sum_{k=k_0+1}^{j_0} [\omega_1(B_k)]^{\alpha p/n} [\inf_{x\in E_k} \omega_2(x)]^{p/q} 2^{-knp[1-(b'+2)/s]} \\ & \cdot \left(\int_{E_k} |g(x)-g_r|^{sq/(s-q)} \right)^{p(s-q)/sq} \cdot \left(\int_{E_k} |\tilde{K}_{b',s} * a(x)|^s \right)^{p/s}. \end{aligned}$$

By Lemma 2.3, we know that $\|\tilde{K}_{b',s} * a(x)\|^p_s \le C\|a\|^p_{s'}$. Thus

$$J_{1121} \le C\|g\|^p_{BMO}\|a\|^p_{s'} \sum_{k=k_0+1}^{j_0} (k+1-k_0)^p [\omega_1(B_k)]^{\alpha p/n} \left[\frac{\omega_2(B_k)}{|B_k|}\right]^{p/q} \frac{2^{knp(1/q-1/s)}}{2^{knp[1-(b'+2)/s]}}.$$

Since

$$\|a\|_{s'} = \left(\int_{B(0,r)} |a(x)|^{s'} \frac{\omega_2^{s'/q}(x)}{\omega_2^{s'/q}(x)} \right)^{1/s'} \le \|a\|_{L^q(\omega_2)} \left(\int_{B(0,r)} \frac{1}{(\omega_2^{s'/q}(x))^{(q/s')}} \right)^{1/s'(q/s')'}$$

and $\dfrac{1}{(q/s')} + \dfrac{1}{(q/s')'} = 1$, then

$$\|a\|_{s'} \leq C r^{n(q-s')/qs'} [\omega_1(B(0,r))]^{-\alpha/n} \Big(\text{essinf}_{x\in E_k} \omega_2(x)\Big)^{-1/q}.$$

Therefore

$$\begin{aligned}
J_{1121} \leq\ & C\|g\|_{BMO}^p \Big(\text{essinf}_{x\in E_k}\omega_2(x)\Big)^{-p/q} \Big[\tfrac{\omega_2(B(0,r))}{|B(0,r)|}\Big]^{p/q} \\
&\cdot \sum_{k=k_0+1}^{j_0} (k+1-k_0)^p \Big(\frac{|B_k|}{|B(0,r)|}\Big)^{\alpha p/n} \frac{r^{np(q-s')/qs'}}{2^{knp[1/q'-(b'+1)/s]}} \\
\leq\ & C\|g\|_{BMO}^p \sum_{k=k_0+1}^{j_0} (k+1-k_0)^p \Big(\frac{|B_k|}{|B(0,r)|}\Big)^{\alpha p/n} \frac{r^{np(q-s')/qs'}}{2^{knp[1/q'-(b'+1)/s]}} \\
\leq\ & C\|g\|_{BMO}^p \sum_{k=k_0+1}^{j_0} (k+1-k_0)^p \frac{r^{np(q-s')/qs'-\alpha p}}{2^{knp[1/q'-(b'+1)/s]-k\alpha p}} \\
\leq\ & C\|g\|_{BMO}^p \sum_{k=k_0+1}^{j_0} (k+1-k_0)^p \frac{2^{knp(b'+1)/s}}{r^{np/s}} \\
\leq\ & C\|g\|_{BMO}^p \frac{2^{j_0 pn(b'+1)/s}}{r^{np/s}} \leq C\frac{r^{np(1-b)(b'+1)/s}}{r^{np/s}} \leq C.
\end{aligned}$$

For J_{1122}, by the Hölder inequality, we get

$$\begin{aligned}
J_{1122} \leq\ & C\sum_{k=k_0+1}^{j_0} [\omega_1(B_k)]^{\alpha p/n} [\sup_{x\in E_k}\omega_2(x)]^{p/q} 2^{-knp[1-(b'+2)/s]} \\
&\cdot \Big(\int_{E_k}\Big|\int_{B(0,r)} \tilde{K}_{b',s}(x-y)(g_r-g(y))a(y)dy\Big|^q dx\Big)^{p/q} \\
\leq\ & C\sum_{k=k_0+1}^{j_0} [\omega_1(B_k)]^{\alpha p/n} [\inf_{x\in E_k}\omega_2(x)]^{p/q} 2^{-knp[1-(b'+2)/s]} \\
&\cdot \Big(\int_{B(0,r)} |g(y)-g_r|^{q'} dy\Big)^{p/q'} \Big(\int_{E_k}\int_{B(0,r)} \Big|\tilde{K}_{b',s}(x-y)a(y)\Big|^q dydx\Big)^{p/q}.
\end{aligned}$$

By the estimate of $\tilde{K}_{b',s} * a(x)$ ($|x|\geq 2r$), we have

$$\int_{B(0,r)} \Big|\tilde{K}_{b',s}(x-y)a(y)\Big|^q dy \leq C\frac{r^q}{|x|^{[n(b'+2)/s+b'+1]q}} \int_{B(0,r)} \Big|a(y)\Big|^q dy.$$

So, we get that

$$\begin{aligned}
J_{1122} \leq\ & C\|g\|^p_{BMO} \sum_{k=k_0+1}^{j_0} [\omega_1(B_k)]^{\alpha p/n} \left[\frac{\omega_2(B_k)}{|B_k|}\right]^{p/q} \frac{r^{p(n/q'+1)}}{2^{kp[1+b'+n/q']}} \|a\|^p_{L^q} \\
\leq\ & C\|g\|^p_{BMO} \sum_{k=k_0+1}^{j_0} [\omega_1(B_k)]^{\alpha p/n} \left[\frac{\omega_2(B(0,r))}{|B(0,r)|}\right]^{p/q} \frac{r^{p(n/q'+1)}}{2^{kp[1+b'+n/q']}} \|a\|^p_{L^q} \\
\leq\ & C\|g\|^p_{BMO} \sum_{k=k_0+1}^{j_0} [\omega_1(B_k)]^{\alpha p/n} \frac{r^{p(n/q'+1)}}{2^{kp[1+b'+n/q']}} \|a\|^p_{L^q(\omega_2)} \\
\leq\ & C\|g\|^p_{BMO} \sum_{k=k_0+1}^{j_0} \left[\frac{\omega_1(B_k)}{\omega_1(B(0,r))}\right]^{\alpha p/n} \frac{r^{p(n/q'+1)}}{2^{kp[1+b'+n/q']}} \\
\leq\ & C\|g\|^p_{BMO} \sum_{k=k_0+1}^{j_0} \left[\frac{|B_k|}{|B(0,r)|}\right]^{\alpha p/n} \frac{r^{p(n/q'+1)}}{2^{kp[1+b'+n/q']}} \\
\leq\ & C\|g\|^p_{BMO} r^{p(n/q'+1-\alpha)} \sum_{k=k_0+1}^{j_0} \frac{1}{2^{kp[1+b'+n/q'-\alpha]}}.
\end{aligned}$$

Since $\alpha = n(1-1/q)$, we get

$$\begin{aligned}
J_{1122} &\leq C\|g\|^p_{BMO} r^p \sum_{k=k_0+1}^{j_0} \frac{1}{2^{kp[1+b']}} \\
&\leq C\|g\|^p_{BMO} \frac{r^p}{2^{p(k_0+1)(1+b')}} \\
&\leq C\|g\|^p_{BMO} \frac{1}{r^{b'} 4^{p(1+b')}} \leq C. \qquad (2.26)
\end{aligned}$$

This ends the proof of Theorem 2.1. □

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