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Existence Results for Vector Saddle Point Problems in the n-direction[1]

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Abstract : In this paper, the author introduce the n-vectorial saddle point problem (for short VSP_n) which defined on n-dimensional saddle point where ($n > 2$) by focusing only on the saddle point of order one. For that matter, the results are proved existence saddle point of (VSP_n) under assuming compactness and uncompactness by using Fan-KKM Theorem. These results improve and extend some literatures on the existence theorems of saddle point problems.

Keywords : saddle point; vector saddle point; Fan-KKM Theorem.
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1 Introduction

To determine the stationary points (maximum, minimum and saddle points) of energy surfaces is important in chemical physics because they indicate the equilibrium geometries and transition states which can be used to describe reaction dynamic by classical equation using these stationary points. In n-dimensional real space, the saddle point of order m is the point is maximum point in directions of m degree of freedoms and minimum point in directions of n-m degree of freedoms [1]. However, the saddle point of order one is a maximum point in a direction and minimum point in the other directions which is generally defined and used in

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literatures. An example of utilising saddle point order one is calculation of reaction rate in chemical physics by harmonic transition state theory (hTST) equation [2, 3], which requires the saddle point order one on its general formula.

In this paper, we will consider properties of the n-dimensional saddle point where (n > 2) by focuing only on the saddle point of order one. We suppose that C is a closed convex cone in the topological vector space E such that $intC \neq \emptyset$ and $0 \notin intC$ where $intC$ denotes the interior of C. For each $i = 1, 2, ..., n$, we let K_i be nonempty convex subsets of Hausdorff topological vector spaces X_i and let $f : \prod_{i=1}^{n} K_i \to E$ be a vector valued mapping. Consider the following n-vectorial saddle point problem is to find $\bar{z} := (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$ such that

$$VSP_n : \begin{cases} f(\bar{z}) - f(x_1, \bar{x}_2, \bar{x}_3, ..., \bar{x}_n) & \notin -intC \quad \forall x_1 \in K_1, \\ f(\bar{x}_1, x_2, \bar{x}_3, ..., \bar{x}_n) - f(\bar{z}) & \notin -intC \quad \forall x_2 \in K_2, \\ f(\bar{x}_1, \bar{x}_2, x_3, ..., \bar{x}_n) - f(\bar{z}) & \notin -intC \quad \forall x_3 \in K_3, \\ \vdots & \\ f(\bar{x}_1, \bar{x}_2, \bar{x}_3, ..., x_n) - f(\bar{z}) & \notin -intC \quad \forall x_n \in K_n. \end{cases}$$

A point $\bar{z}$ is said to be a saddle point of f on $\prod_{i=1}^{n} K_i$, if it is a solution for (VSP_n). Note that when $E = \mathbb{R}$ and $C = [0, +\infty)$, problem (VSP_n) is reduced to the saddle point problem of a real valued function, i.e., find $\bar{z} := (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$ such that

$$SP_n : \begin{cases} f(\bar{z}) - f(x_1, \bar{x}_2, \bar{x}_3, ..., \bar{x}_n) & \geq 0 \quad \forall x_1 \in K_1, \\ f(\bar{x}_1, x_2, \bar{x}_3, ..., \bar{x}_n) - f(\bar{z}) & \geq 0 \quad \forall x_2 \in K_2, \\ f(\bar{x}_1, \bar{x}_2, x_3, ..., \bar{x}_n) - f(\bar{z}) & \geq 0 \quad \forall x_3 \in K_3, \\ \vdots & \\ f(\bar{x}_1, \bar{x}_2, \bar{x}_3, ..., x_n) - f(\bar{z}) & \geq 0 \quad \forall x_n \in K_n. \end{cases}$$

Studies on saddle points of scalar functions have been generalized to studies of saddle points, with respect to a cone, of vector valued functions under necessary and sufficient conditions; see, for example, [4–7]. In particular, many literature studies (VSP_n) for case $n = 2$; see, for example, [8–10], which is called vector saddle point problem (for short, VSP); find $(\bar{x}, \bar{y}) \in K_1 \times K_2$ such that

$$VSP : \begin{cases} f(\bar{x}, \bar{y}) - f(x, \bar{y}) & \notin -intC \quad \forall x \in K_1, \\ f(\bar{x}, y) - f(\bar{x}, \bar{y}) & \notin -intC \quad \forall y \in K_2, \end{cases}$$

where $f : K_1 \times K_2 \to E$. Moreover, if we let $E = \mathbb{R}$ and $C = [0, +\infty)$, then (VSP) can be reduced to the saddle point problem of a real valued function to find $(\bar{x}, \bar{y}) \in K_1 \times K_2$ such that

$$f(\bar{x}, y) \geq f(\bar{x}, \bar{y}) \geq f(x, \bar{y}) \quad for\ all\ (x, y) \in K_1 \times K_2.$$

The aim of this paper is to introduce the n-vectorial saddle point problem and prove the existence saddle point of (VSP_n) under assuming compactness and uncompactness by using Fan-KKM Theorem.

2 Preliminaries

In this section, we recall some definitions about continuities and convexities concerned with respect to ordering cone C. For a given nonempty subset K of topological space X, we shall denote by $int(K)$ the interior of K, $co(K)$ the convex hull of K and $\overline{K}$ the closure of K. Though this paper, we suppose that C is a closed convex cone in the topological vector space E such that $intC \neq \emptyset$ and $0 \notin intC$.

Definition 2.1 ([7])**.** Let X be a topological vector space, K a nonempty convex subset of X. A vector-valued mapping $f : K \to E$ is said to be C-properly quasiconvex on K if

$$\begin{array}{rrcl} & f(tu_1 + (1-t)u_2) & \in & f(u_1) - C \\ or & f(tu_1 + (1-t)u_2) & \in & f(u_2) - C \end{array}$$

for all $u_1, u_2 \in K$ and $t \in [0, 1]$.
It is also said to be C-properly quasiconcave on K if $(-f)$ is C-properly quasiconvex.

Definition 2.2 ([11])**.** Let X be a topological space. A vector-valued mapping $f : X \to E$ is said to be C-lower semicontinuous (for short, C-l.s.c.) on X if it satisfies one of the following three equivalent conditions:

(i) For all $a \in E$, $f^{-1}(a + intC)$ is open.

(ii) For each $x_0 \in X$ and any open neighborhood V of $f(x_0)$, there exists an open neighborhood U of x_0 such that $f(x) \in V + C$ for all $x \in U$.

(iii) For each $x_0 \in X$ and any $d \in intC$, there exists an open neighborhood U of x_0 such that $f(x) \in f(x_0) - d + intC$ for all $x \in U$.

It is also said to be C-upper semicontinuous (for short, C-u.s.c.) on X if $(-f)$ is C-lower semicontinuous (for short, C-l.s.c.) on X.

Definition 2.3. Let X be a topological space and K a nonempty convex subset of X. A vector-valued mapping $f : K \to E$ has (CL)-property if and only if it satisfies the condition: for any $u \in K$ and $\{u_\alpha\} \subseteq X$ such that $u_\alpha \to \bar{u}$, if

$$f(tu + (1-t)\bar{u}) - f(u_\alpha) \notin -intC \quad \forall t \in [0, 1] \quad \Rightarrow \quad f(u) - f(\bar{u}) \notin -intC.$$

Moreover, f also has (CU)-property if $(-f)$ has (CL)-property.

Proposition 2.1. *Let K be a nonempty convex subset of a topological space X. If a vector-valued function $f : K \to E$ is C-upper semicontinuous, then it has (CU)-property.*

Proof. Let $u \in K$ and for any $\{u_\alpha\}_{\alpha \in I} \subseteq K$ such that $u_\alpha \to \tilde{u}$,

$$f(u_\alpha) - f(\lambda u + (1-\lambda)\tilde{u}) \notin -intC \quad \text{for all } \lambda \in [0,1].$$

Then, by the assumption we have $f(u_\alpha) - f(u) \notin -intC$. We will prove by contrary. Assume that $d := f(\tilde{u}) - f(u) \in -intC$. Since f is C-u.s.c. at $\tilde{u}$, there exists an open neighbourhood U of u_0 such that

$$f(\tilde{u}) \in f(x) + d + intC \quad \text{for all} \quad x \in U.$$

Since $\{u_\alpha\}_{\alpha \in I}$ converges to $\tilde{u}$, there exists $\gamma \in I$ such that

$$\gamma \leq \alpha \Rightarrow f(\tilde{u}) \in f(u_\alpha) + d + intC \quad \text{and} \quad u_\alpha \in U.$$

Then, we have $f(u_\alpha) - f(u) \in -intC$ for all $u_\alpha \in U$ and $\alpha \in U$ and $\alpha \geq \gamma$. □

Moreover, we can prove that if f is C-lower semicontinuous, then it has (CL)-property.

Definition 2.4 (KKM-mapping)**.** Let K be a nonempty subset of a topological vector space X and $f : K \to 2^X$ be a set-valued mapping. We say that f is a KKM mapping (or the family of sets $\{f(x)\}_{x \in K}$ satisfies the KKM principle) if for any nonempty finite set $A \subset K$ one has

$$co(A) \subset \bigcup_{x \in A} f(x). \tag{2.1}$$

Definition 2.5 (Fan-KKM Theorem [12])**.** Let K be a nonempty subset of a topological vector space X, and let $f : K \to 2^X$ be a KKM-mapping. If $f(x)$ is closed in X for every x, and if $f(x_0)$ is compact for some $x_0 \in K$, then $\bigcap_{x \in K} f(x)$ is nonempty.

3 Main theorem

At the beginning in this section, we consider and show the existence theorems for (VSP_n) that is the problem to find $(\bar{x}, \bar{y}, \bar{z}) \in K_1 \times K_2 \times K_3$ such that

$$VSP_3 : \begin{cases} f(\bar{x}, \bar{y}, \bar{z}) - f(x, \bar{y}, \bar{z}) & \notin -intC \quad \forall x \in K_1, \\ f(\bar{x}, y, \bar{z}) - f(\bar{x}, \bar{y}, \bar{z}) & \notin -intC \quad \forall y \in K_2, \\ f(\bar{x}, \bar{y}, z) - f(\bar{x}, \bar{y}, \bar{z}) & \notin -intC \quad \forall z \in K_3. \end{cases}$$

The following Lemma is useful for our main result.

Lemma 3.1. *Let K be a nonempty convex subset of a topological vector space X, let a vector valued map $\phi : K \to E$ be C-properly quasiconcave and let A be a nonempty finite subset of K. For any $e \in E$, if $e - \phi(\hat{x}) \notin -intC$ for some $\hat{x} \in co(A)$. Then, there exists $x \in A$ such that $e - \phi(x) \notin -intC$.*

Proof. Let $e \in E$ and $A = \{x_1, x_2, ..., x_n\}$ be a finite subset of K and $\hat{x} = \sum_{i=1}^{n} \alpha_i x_i$ where $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i \geq 0$ for all i = 1,2,...,n. We will prove it by the mathematical induction. Let $n = 2$. Since $\hat{\phi}$ is C-properly quasiconcave on K, we have

$$\begin{array}{rrcl} & e - \phi(\alpha_1 x_1 + \alpha_2 x_2) & \in & e - \phi(x_1) - C \\ \text{or} & e - \phi(\alpha_1 x_1 + \alpha_2 x_2) & \in & e - \phi(x_2) - C. \end{array}$$

If $e - \phi(x_1) \in -intC$ and $e - \phi(x_2) \in -intC$, then $e - \phi(\hat{x}) = e - \phi(\alpha_1 x_1 + \alpha_2 x_2) \in -intC$, which is a contradiction with assumption. So $e - \phi(x_1) \notin -intC$ or $e - \phi(x_2) \notin -intC$. This completes the proof of case n=2. Assume that the statement is true for $n \in \mathbb{N}$ and for each $e \in E$,

$$e - \phi(\hat{x}) = e - \phi\left(\sum_{i=1}^{n+1} \alpha_i x_i\right) \notin -intC \tag{3.1}$$

where $\sum_{i=1}^{n+1} \alpha_i = 1$ and $\alpha_i \geq 0$ for all i = 1, 2, ..., n+1. Let $\alpha := \sum_{i=1}^{n} \alpha_i = 1 - \alpha_{n+1}$ and $x := \sum_{i=1}^{n} \frac{\alpha_i}{\alpha} x_i$, thus $\hat{x} := \sum_{i=1}^{n+1} \alpha_i x_i = \alpha x + \alpha_{n+1} x_{n+1}$. Since ϕ is C-properly quasiconcave and by (3.1), we have

$$\begin{array}{rrcl} & e - \phi(\alpha x + \alpha_{n+1} x_{n+1}) & \in & e - \phi(x) - C \\ \text{or} & e - \phi(\alpha x + \alpha_{n+1} x_{n+1}) & \in & e - \phi(x_{n+1}) - C. \end{array}$$

By the same argument in case n=2, we have

$$e - \phi(x) \notin -intC \quad \text{or} \quad e - \phi(x_{n+1}) - C.$$

If $e - \phi(x) \notin -intC$, then by the induction hypothesis, there exists $x_i \in A$ such that $e - \phi(x_i) \notin -intC$, which completes the proof. □

Remark 3.2. *If we replace the assumption of the map ϕ in Lemma 3.1 by C-properly quasiconvex then we have the result: for any $e \in E$, if $\phi(\hat{x}) - e \notin -intC$ for some $\hat{x} \in co(A)$, then there exists $x \in A$ such that $\phi(x) - e \notin -intC$.*

Lemma 3.3. *For each i=1,2,3, let X_i be Hausdorff topological vector spaces, $K_i \subset X_i$ be nonempty convex subsets and $f : K_1 \times K_2 \times K_3 \to E$ be a vector valued function satisfying the conditions (i) and (ii).*

(i) f is C-properly quasiconcave and C-u.s.c. in the first argument on the convex hull of every nonempty finite subset of K_1

(ii) f is C-properly quasiconvex and C-l.s.c. in the second and third argument on the convex hull of every nonempty finite subset of K_2 and K_3 respectively.

Then, for each finite subset A_i of K_i where i=1,2,3, there exist $\hat{x} \in co(A_1)$, $\hat{y} \in co(A_2)$, and $\hat{z} \in co(A_3)$ such that

$$\begin{array}{lll} f(\hat{x},\hat{y},\hat{z}) - f(u,\hat{y},\hat{z}) & \notin -intC & \forall u \in co(A_1), \\ f(\hat{x},v,\hat{z}) - f(\hat{x},\hat{y},\hat{z}) & \notin -intC & \forall v \in co(A_2), \\ f(\hat{x},\hat{y},w) - f(\hat{x},\hat{y},\hat{z}) & \notin -intC & \forall w \in co(A_3). \end{array}$$

Proof. Take $K := K_1 \times K_2 \times K_3$ and for $(u,v,w) \in K$, we define the following subsets

$$\begin{array}{ll} L(u,v,w) & = \{x \in K_1 : f(x,v,w) - f(u,v,w) \notin -intC\}, \\ M(u,v,w) & = \{y \in K_2 : f(u,v,w) - f(u,y,w) \notin -intC\}, \\ N(u,v,w) & = \{z \in K_3 : f(u,v,w) - f(u,v,z) \notin -intC\}. \end{array}$$

By the definition of three sets, they are nonempty sets because $(u,v,w) \in P(u,v,w)$ $:= L(u,v,w) \times M(u,v,w) \times N(u,v,w)$. For each $i = 1,2,3$, we suppose A_i is the finite subset of K_i and set $A := A_1 \times A_2 \times A_3$. Define the set-valued mapping $Q : co(A) \to 2^{co(A)}$ by

$$Q(u,v,w) = \{(x,y,z) \in co(A) : (x,y,z) \in P(u,v,w)\} \quad \forall (u,v,w) \in co(A).$$

We will show that Q is a KKM mapping. Asuume that there exists a finite set $(\{u_1,...,u_l\} \times \{v_1,...,v_m\} \times \{w_1,...,w_n\}) \subset co(A)$ such that

$$co(\{u_1,...,u_l\} \times \{v_1,...,v_m\} \times \{w_1,...,w_n\}) \not\subset \bigcup_{i=1,j=1,k=1}^{l,m,n} Q(u_i,v_j,w_k).$$

Then, there exists

$$\begin{array}{ll} (u_0,v_0,w_0) & = (\sum_{i=1}^{l} \alpha_i u_i, \sum_{j=1}^{m} \beta_j v_j, \sum_{k=1}^{n} \gamma_k w_k) \\ & \in co(\{u_1,...,u_l\} \times \{v_1,...,v_m\} \times \{w_1,...,w_n\}) \end{array}$$

such that $u_0 \notin L(u_i,v_j,w_k)$ or $v_0 \notin M(u_i,v_j,w_k)$ or $w_0 \notin N(u_i,v_j,w_k)$ for $i = 1,...,l$, $j = 1,...,m$ and $k = 1,...,n$. We consider the case $u_0 \notin L(u_i,v_j,z_k)$ for $i = 1,...,l$, $j = 1,...,m$ and $k = 1,...,n$. Let $j \in \{1,...,m\}$ and $k \in \{1,...,n\}$ be fixed. Clearly,

$$f(u_0,v_j,w_k) - f(u_0,v_j,w_k) \notin -intC.$$

Since f is C-properly quasiconcave in the first argument and by Lemma 3.1, there exists $u_i \in \{u_1,...,u_l\}$ such that

$$f(u_0,v_j,w_k) - f(u_i,v_j,w_k) \notin -intC,$$

it is a contradiction with $u_0 \notin L(u_i, v_j, w_k)$. Similarly on other cases, we also obtain a contradiction and so we have Q is a KKM mapping.

Next, we will show that $Q(u, v, w)$ is closed for each $(u, v, w) \in co(A)$. Let $\{(u_\lambda, v_\lambda, w_\lambda)\}_{\lambda \in I} \subseteq Q(u, v, w)$ such that $(u_\lambda, v_\lambda, w_\lambda) \to (u', v', w') \in co(K)$. Assume that $(u', v', w') \notin Q(u, v, w)$, then we have $u' \notin L(u, v, w)$ or $v' \notin M(u, v, w)$ or $w' \notin N(u, v, w)$. Consider the case $u' \notin L(u, v, w)$. Then, we have $f(u', v, w) - f(u, v, w) \in -intC$, it follows that there is $-c' \in intC$ such that

$$-c' = f(u', v, w) - f(u, v, w) \in -intC. \tag{3.2}$$

Since f is C-u.s.c. in the first argument, there exists an open neighbourhood U of u' such that for any $c \in intC$ there is an $\lambda_0 \in I$ such that

$$f(u', v, w) \in f(u_\lambda, v, w) - C + intC \quad \forall u_\lambda \in U \quad where \quad \lambda \geq \lambda_0.$$

Set $c = c'$ and by (3.2), we obtain that

$$f(u_\lambda, v, w) - f(u, v, w) \in -intC.$$

Then $u_\lambda \notin L(u, v, w)$ which is a contradiction. For other cases, the proof is similar by using the C-lower semicontinility of f. This implies that $Q(u, v, w)$ is closed for each $(u, v, w) \in co(A)$. Since $X_1 \times X_2 \times X_3$ is a Hausdorff space, $co(A)$ is compact and also we have $Q(u, v, w)$ is compact. By the *Fan-KKM* Theorem, we obtain that

$$\bigcap_{(u,v,w) \in co(A)} Q(u, v, w) \neq \emptyset.$$

Hence there exist $(\hat{x}, \hat{y}, \hat{z}) \in co(A)$ such that $(\hat{x}, \hat{y}, \hat{z}) \in P(u, v, w)$ for all $(u, v, w) \in co(A)$. Then $\hat{x} \in L(u, v, w), \hat{y} \in M(u, v, w)$ and $\hat{z} \in N(u, v, w)$ for all $(u, v, w) \in co(A)$. Therefore $\hat{x} \in L(u, \hat{y}, \hat{z}) \ \ \forall u \in co(A_1), \ \ \hat{y} \in M(\hat{x}, v, \hat{z}) \ \ \forall v \in co(A_2)$ and $\hat{z} \in N(\hat{x}, \hat{y}, w) \ \ \forall w \in co(A_3)$. This completes the proof. □

Remark 3.4. *Lemma 3.3 is the generalization of Lemma 3.1 in [8], moreover the idea of the proof in Lemma 3.3 similar to that obtained by Chadli and Mahdioui [8]. In the same way of the proof in Lemma 3.3, we can extend this result to n-tuples.*

Theorem 3.5. *For each $i = 1, 2, 3$, let X_i be Hausdorff topological vector spaces, $K_i \subseteq X_i$ be nonempty compact convex subsets and $f : K_1 \times K_2 \times K_3 \to E$ be a vector valued mapping satisfying the conditions (i) and (ii) in Lemma 3.3. Then, (VSP_3) has a saddle point.*

Proof. Let $\mathscr{K}$ be the family of all nonempty finite subsets of $K := K_1 \times K_2 \times K_3$ and for each $A := A_1 \times A_2 \times A_3 \in \mathscr{K}$, we suppose the following set

$$\mathscr{L}_A = \{(x, y, z) \in K : x \in L(u, v, w), y \in M(u, v, w), z \in N(u, v, w) \ \forall (u, v, w) \in co(A)\}.$$

By Lemma 3.3, we have $\mathscr{L}_A$ is nonempty for each A $\in \mathscr{K}$. Next, we will show that the family $\{\overline{\mathscr{L}_A}\}_{A \in \mathscr{K}}$ has the finite intersection property. Suppose that

$A' := A'_1 \times A'_2 \times A'_3$ and $A'' := A''_1 \times A''_2 \times A''_3$ are two finite subsets of K. Setting $A := A' \cup A''$, by the definition of the set $\mathscr{L}_A$, we obtain that $\mathscr{L}_A \subset \mathscr{L}_{A'} \cap \mathscr{L}_{A''}$ and so we have

$$\emptyset \neq \overline{\mathscr{L}_A} \subset \overline{\mathscr{L}_{A'}} \cap \overline{\mathscr{L}_{A''}}.$$

This leads to $\{\overline{\mathscr{L}_A}\}_{A \in \mathscr{K}}$ has finite intersection property. Since K is compact, $\bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A} \neq \emptyset$. Let $(x, y, z) \in K$ be an arbitrary and $(\tilde{x}, \tilde{y}, \tilde{z}) \in \bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A}$ be fixed. Set $D = \{(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})\}$, then we have $D \in \mathscr{K}$. Since $(\tilde{x}, \tilde{y}, \tilde{z}) \in \overline{\mathscr{L}_D}$, there exists a generalized sequence $\{(x_\alpha, y_\alpha, z_\alpha)\}_{\alpha \in I} \subset \overline{\mathscr{L}_D}$ such that $\{x_\alpha, y_\alpha, z_\alpha\} \to (\tilde{x}, \tilde{y}, \tilde{z})$. Since $(x_\lambda, y_\lambda, z_\lambda) := (\lambda x + (1-\lambda)\tilde{x}, \lambda y + (1-\lambda)\tilde{y}, \lambda z + (1-\lambda)\tilde{z}) \in co(D)$ and by the definition of $\mathscr{L}_D$, for $\alpha \in I$ and $\lambda \in [0, 1]$, we note that

$$x_\alpha \in L(x_\lambda, y_\lambda, z_\lambda), \quad y_\alpha \in M(x_\lambda, y_\lambda, z_\lambda), \quad z_\alpha \in N(x_\lambda, y_\lambda, z_\lambda).$$

Then, for all $\alpha \in I$ and $\lambda \in [0, 1]$,

$$\begin{aligned} f(x_\alpha, y_\lambda, z_\lambda) - f(x_\lambda, y_\lambda, z_\lambda) &\notin -intC, \\ f(x_\lambda, y_\lambda, z_\lambda) - f(x_\lambda, y_\alpha, z_\lambda) &\notin -intC, \\ f(x_\lambda, y_\lambda, z_\lambda) - f(x_\lambda, y_\lambda, z_\alpha) &\notin -intC. \end{aligned}$$

By Proposition 2.1, we conclude that for all $\lambda \in [0, 1]$,

$$\begin{aligned} f(\tilde{x}, y_\lambda, z_\lambda) - f(x, y_\lambda, z_\lambda) &\notin -intC, \\ f(x_\lambda, y, z_\lambda) - f(x_\lambda, \tilde{y}, z_\lambda) &\notin -intC, \\ f(x_\lambda, y_\lambda, z) - f(x_\lambda, y_\lambda, \tilde{z}) &\notin -intC. \end{aligned}$$

Therefore, we have

$$\begin{aligned} f(\tilde{x}, \tilde{y}, \tilde{z}) - f(x, \tilde{y}, \tilde{z}) &\notin -intC, \\ f(\tilde{x}, y, \tilde{z}) - f(\tilde{x}, \tilde{y}, \tilde{z}) &\notin -intC, \\ f(\tilde{x}, \tilde{y}, z) - f(\tilde{x}, \tilde{y}, \tilde{z}) &\notin -intC. \end{aligned}$$

Since (x, y, z) is an arbitrary element in $K_1 \times K_2 \times K_3$, we complete the proof. □

In Theorem 3.5, we set for each $z \in K_3$, $f(x, y, z) = g(x, y)$ for all $(x, y) \in K_1 \times K_2$, where $g : K_1 \times K_2 \to E$. Then we have the following Corollary.

Corollary 3.6. *For each $i = 1, 2$, let X_i be Hausdorff topological vector spaces, $K_i \subseteq X_i$ be nonempty compact convex subsets and the vector valued mapping $g : K_1 \times K_2 \to E$ is C-properly quasiconcave and C-u.s.c. in the first argument on the convex hull of every nonempty finite subset of K_1 and C-properly quasiconvex and C-l.s.c. in the second argument on the convex hull of every nonempty finite subset of K_2. Then, there exists $(\bar{x}, \bar{y}) \in K_1 \times K_2$ such that*

$$VSP : \begin{cases} f(\bar{x}, \bar{y}) - f(x, \bar{y}) \notin -intC & \forall x \in K_1, \\ f(\bar{x}, y) - f(\bar{x}, \bar{y}) \notin -intC & \forall y \in K_2. \end{cases}$$

Setting $E = \mathbb{R}$ and $C = [0, +\infty)$. Then Corollary 3.6 can be reduced to the following Corollary.

Corollary 3.7. *For each $i = 1, 2$, let X_i be Hausdorff topological vector spaces, $K_i \subseteq X_i$ be nonempty compact convex subsets and the vector valued mapping $g : K_1 \times K_2 \to \mathbb{R}$ is quasiconcave and u.s.c. in the first argument on the convex hull of every nonempty finite subset of K_1 and quasiconvex and l.s.c. in the second argument on the convex hull of every nonempty finite subset of K_2. Then, there exists $(\bar{x}, \bar{y}) \in K_1 \times K_2$ such that*

$$f(\bar{x}, y) \geq f(\bar{x}, \bar{y}) \geq f(x, \bar{y}) \quad for\ all\ (x, y) \in K_1 \times K_2.$$

The next theorem presents the existence solution for (VSP_3) without assuming compactness of the subsets.

Theorem 3.8. *For each $i = 1, 2, 3$, let X_i be Hausdorff topological vector spaces, $K_i \subseteq X_i$ be nonempty convex subsets and $f : K_1 \times K_2 \times K_3 \to E$ be a vector value mapping satisfying the conditions $(i) - (ii)$ and if it satisfies the following condition:*

(iii) (The coercivity) there is a nonempty compact set $B := B_1 \times B_2 \times B_3 \subseteq K := K_1 \times K_2 \times K_3$ and there is a nonempty compact convex set $\tilde{B} := \tilde{B_1} \times \tilde{B_2} \times \tilde{B_3} \subseteq K$ such that if $(x, y, z) \in K \cap B^C$, then

$$\begin{aligned} f(\tilde{x}, \tilde{y}, \tilde{z}) - f(x, \tilde{y}, \tilde{z}) &\in -intC, \\ f(\tilde{x}, y, \tilde{z}) - f(\tilde{x}, \tilde{y}, \tilde{z}) &\in -intC, \\ f(\tilde{x}, \tilde{y}, z) - f(\tilde{x}, \tilde{y}, \tilde{z}) &\in -intC. \end{aligned}$$

for some $(\tilde{x}, \tilde{y}, \tilde{z}) \in \tilde{B}$.Then VSP_3 has a saddle point.

Proof. Let $\mathscr{K}$ be the family of all nonempty finite subsets of $K := K_1 \times K_2 \times K_3$ and for each $A := A_1 \times A_2 \times A_3 \in \mathscr{K}$, we suppose the following set

$$\mathscr{L}_A = \{(x, y, z) \in B : x \in L(u, v, w), y \in M(u, v, w), z \in N(u, v, w)\ \forall (u, v, w) \in co(A \cup \tilde{B})\}.$$

It is easy to see that $co(A \cup \tilde{B})$ is compact for every $A \in \mathscr{K}$. By Theorem 3.5, there exists $(\tilde{x}, \tilde{y}, \tilde{z}) \in co(A \cup \tilde{B})$ such that

$$\begin{aligned} f(\tilde{x}, \tilde{y}, \tilde{z}) - f(x, \tilde{y}, \tilde{z}) &\notin -intC \quad \text{for all } x \in co(A_1 \cup \tilde{B}_1), \\ f(\tilde{x}, y, \tilde{z}) - f(\tilde{x}, \tilde{y}, \tilde{z}) &\notin -intC \quad \text{for all } y \in co(A_2 \cup \tilde{B}_2), \\ f(\tilde{x}, \tilde{y}, z) - f(\tilde{x}, \tilde{y}, \tilde{z}) &\notin -intC \quad \text{for all } z \in co(A_3 \cup \tilde{B}_3). \end{aligned}$$

By the contrary of coercivity condition (iii) and since $\tilde{B} \subset co(AU\tilde{B})$, we deduce that $(\tilde{x}, \tilde{y}, \tilde{z}) \in B$. This means that $\mathscr{L}_A$ is nonempty for all $A \in \mathscr{K}$. Similarly proved in Theorem 3.5, it implies that the family $\{\overline{\mathscr{L}_A}\}_{A \in \mathscr{K}}$ has the finite intersection property and hence $\bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A}$ is also nonempty by the compactness of B. Let $(x, y, z) \in K$ be an arbitary and $(\bar{x}, \bar{y}, \bar{z}) \in \bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A}$ be fixed. Setting $D = \{(x, y, z), (\bar{x}, \bar{y}, \bar{z})\}$, then we have $D \in \mathscr{K}$. Since $(\bar{x}, \bar{y}, \bar{z}) \in \overline{\mathscr{L}_D}$, there exists a generalized sequence $\{(x_\alpha, y_\alpha, z_\alpha)\}_{\alpha\in I} \subset \overline{\mathscr{L}_D}$ such that $(x_\alpha, y_\alpha, z_\alpha) \to (\bar{x}, \bar{y}, \bar{z})$. By the same argument of Theorem 3.5, we conclude that

$$\begin{aligned} f(\bar{x}, \bar{y}, \bar{z}) - f(x, \bar{y}, \bar{z}) &\notin -intC, \\ f(\bar{x}, y, \bar{z}) - f(\bar{x}, \bar{y}, \bar{z}) &\notin -intC, \\ f(\bar{x}, \bar{y}, z) - f(\bar{x}, \bar{y}, \bar{z}) &\notin -intC. \end{aligned}$$

for all $(x, y, z) \in K$, which impiles that VSP_3 has a saddle point and completes the proof. □

Remark 3.9. *In Theorem 3.8, if we set for each $z \in K_3$, $f(x, y, z) = g(x, y)$ for all $(x, y) \in K_1 \times K_2$, where $g : K_1 \times K_2 \to E$ then we have Theorem 3.2 in [8]. In addition to this, if we let $E = \mathbb{R}$ and $C = [0, +\infty)$ then we also have Corollary 3.1 in [8].*

The following theorem presents the existence solution for (VSP_n) which generalizes Theorem 3.5.

Theorem 3.10. *For each i =1, 2, ..., n, let $K_i \subseteq X_i$ be a nonempty compact convex subsets and $f : \prod_{i=1}^{n} K_i \to E$ be a vector valued mapping satisfying the following conditions:*

(I) f is C-properly quasiconcave, C-u.s.c. in the first argument and C-properly quasiconvex, C-l.s.c. in the other arguments on the convex hull of every nonempty finite subset of $\prod_{i=1}^{n} K_i$.

Then, VSP_n has a saddle point.

Proof. For each $(u_1, u_2, ..., u_n) \in \prod_{i=1}^{n} K_i$, we define the following subsets

$$\begin{aligned} L_1(u_1, u_2, ..., u_n) &= \{x_1 \in K_1 : f(x_1, u_2, ..., u_n) - f(u_1, u_2, u_3, ..., u_n) \notin -intC\}, \\ L_2(u_1, u_2, ..., u_n) &= \{x_2 \in K_2 : f(u_1, u_2, ..., u_n) - f(u_1, x_2, u_3, ..., u_n) \notin -intC\}, \\ L_3(u_1, u_2, ..., u_n) &= \{x_3 \in K_3 : f(u_1, u_2, ..., u_n) - f(u_1, u_2, x_3, ..., u_n) \notin -intC\}, \\ &\vdots \\ L_n(u_1, u_2, ..., u_n) &= \{x_n \in K_n : f(u_1, u_2, ..., u_n) - f(u_1, u_2, u_3, ..., x_n) \notin -intC\}. \end{aligned}$$

By the definition of these sets, they are nonempty sets because $(u_1, u_2, ..., u_n) \in \prod_{i=1}^{n} L_i(u_1, u_2, ..., u_n)$. Let $\mathscr{K}$ be the family of all nonempty finite subsets of $\prod_{i=1}^{n} K_i$ and for each $A = \prod_{i=1}^{n} A_i \in \mathscr{K}$, we suppose the following set

$$\mathscr{L}_A = \{(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} K_i : x_i \in L_i(w_1, w_2, ..., w_n)\ \forall (w_1, w_2, ..., w_n) \in co(A)\}.$$

By Remark 3.4, we have $\mathscr{L}_A$ is nonempty for each $A \in \mathscr{K}$. Using the similar idea of the proof in the Theorem 3.5, we have $\bigcap_{A\in\mathscr{K}} \overline{\mathscr{L}_A} \neq \emptyset$. Let $(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} K_i$ be an arbitrary and $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in \bigcap_{A\in\mathscr{K}} \overline{\mathscr{L}_A}$ be fixed. In the same way as the proof in the Theorem 3.5 once more, we conclude that

$$VSP_n : \begin{cases} f(\bar{z}) - f(x_1, \bar{x}_2, \bar{x}_3, ..., \bar{x}_n) & \notin -intC \quad \forall x_1 \in K_1, \\ f(\bar{x}_1, x_2, \bar{x}_3, ..., \bar{x}_n) - f(\bar{z}) & \notin -intC \quad \forall x_2 \in K_2, \\ f(\bar{x}_1, \bar{x}_2, x_3, ..., \bar{x}_n) - f(\bar{z}) & \notin -intC \quad \forall x_3 \in K_3, \\ \vdots & \\ f(\bar{x}_1, \bar{x}_2, \bar{x}_3, ..., x_n) - f(\bar{z}) & \notin -intC \quad \forall x_n \in K_n. \end{cases}$$

□

If we set $E = \mathbb{R}$ and $C = [0, +\infty)$, then Theorem 3.10 is reduced to the following corollary.

Corollary 3.11. *For each i =1, 2, ..., n, let $K_i \subseteq X_i$ be a nonempty compact convex subsets and $f : \prod_{i=1}^{n} K_i \to \mathbb{R}$ is quasiconcave, u.s.c. in the first argument and quasiconvex, l.s.c. in the other arguments on the convex hull of every nonempty finite subset of $\prod_{i=1}^{n} K_i$. Then, there exists a saddle point $\bar{z} \in \prod_{i=1}^{n} K_i$ for (SP_n).*

The following theorem presents the existence solution for (VSP_n) which generalizes Theorem 3.8.

Theorem 3.12. *For each i=1, 2, ..., n, let $K_i \subseteq X_i$ be a nonempty convex subsets and $f : \prod_{i=1}^{n} K_i \to E$ be a vector valued mapping satisfying the condition (I) and if it satisfies the following condition:*

(II) (The coercivity) there is a nonempty compact set $B = \prod_{i=1}^{n} B_i \subseteq \prod_{i=1}^{n} K_i$ and there is a nonempty compact convex set $\widetilde{B} = \prod_{i=1}^{n} \widetilde{B}_i \subseteq \prod_{i=1}^{n} K_i$ such that if $(x_1, ..., x_n) \in \prod_{i=1}^{n} K_i \cap B^C$, then

$$\begin{aligned} f(\tilde{z}) - f(x_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n) &\in -intC, \\ f(\tilde{x}_1, x_2, \tilde{x}_3, ..., \tilde{x}_n) - f(\tilde{z}) &\in -intC, \\ f(\tilde{x}_1, \tilde{x}_2, x_3, ..., \tilde{x}_n) - f(\tilde{z}) &\in -intC, \\ &\vdots \\ f(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_{n-1}, x_n) - f(\tilde{z}) &\in -intC. \end{aligned}$$

for some $\tilde{z} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n) \in \widetilde{B}$.

Then VSP_n *has a saddle point.*

Proof. For each $(u_1, u_2, ..., u_n) \in \prod_{i=1}^{n} K_i$, we define $L_i(u_1, u_2, ..., u_n)$ same as Theorem 3.10. Let $\mathscr{K}$ be the family of all nonempty finite subsets of $\prod_{i=1}^{n} K_i$ and for each $A = \prod_{i=1}^{n} A_i \in \mathscr{K}$, we consider the following set

$$\mathscr{L}_A = \{(x_1, x_2, ..., x_n) \in B : x_i \in L_i(w_1, w_2, ..., w_n) \in co(A \cup \tilde{B})\}.$$

It is easy to see that $co(A \cup \tilde{B})$ is compact for every $A \in \mathscr{K}$. By Theorem 3.10, there exists $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n) \in co(A \cup \widetilde{B})$ such that

$$\begin{array}{llll} f(\tilde{z}) - f(x_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n) & \notin & -intC & \forall x_1 \in co(A_1 \cup \tilde{B}_1), \\ f(\tilde{x}_1, x_2, \tilde{x}_3, ..., \tilde{x}_n) - f(\tilde{z}) & \notin & -intC & \forall x_2 \in co(A_2 \cup \tilde{B}_2), \\ f(\tilde{x}_1, \tilde{x}_2, x_3, ..., \tilde{x}_n) - f(\tilde{z}) & \notin & -intC & \forall x_3 \in co(A_3 \cup \tilde{B}_3), \\ & \vdots & & \\ f(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_{n-1}, x_n) - f(\tilde{z}) & \notin & -intC & \forall x_n \in co(A_n \cup \tilde{B}_n). \end{array}$$

Since $\widetilde{B} \subseteq co(A \cup \widetilde{B})$ and by the contrapositive coercivity condition (II), we conclude that $(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n) \in B$. This implies that $\mathscr{L}_A \neq \emptyset$ for all $A \in \mathscr{K}$. By the compactness of B, we now follow an idea similar to that in Theorem 3.5 which implies that $\bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A} \neq \emptyset$. Let $(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} K_i$ be an arbitary and $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in \bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A}$ be fixed. Setting $D = \{(x_1, x_2, ..., x_n), (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)\}$, then we have $D \in \mathscr{K}$. By the same argument of Theorem 3.8 applying to n-tuples, it implies that $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ is a saddle point for VSP_n. □

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