Special M-Hyperidentities in (x(yz))z with Opposite Loop and Reverse Arc Graph Varieties of Type (2,0)[1]

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Abstract : Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies a term equation $s \approx t$ if the corresponding graph algebra $A(G)$ satisfies $s \approx t$. A class of graph algebras $\mathcal{V}$ is called a graph variety if $\mathcal{V} = \overline{Mod_g\Sigma}$ where Σ is a subset of $T(X) \times T(X)$. A graph variety $\mathcal{V}' = Mod_g\Sigma'$ is called an (x(yz))z with opposite loop and reverse arc graph variety if Σ' is a set of (x(yz))z with opposite loop and reverse arc term equations. A term equation $s \approx t$ is called an identity in a variety $\mathcal{V}$ if $A(G)$ satisfies $s \approx t$ for all $G \in \mathcal{V}$. An identity $s \approx t$ of a variety $\mathcal{V}$ is called a hyperidentity of a graph algebra $A(G)$, $G \in \mathcal{V}$ whenever the operation symbols occuring in s and t are replaced by any term operations of $A(G)$ of the appropriate arity, the resulting identities hold in $A(G)$. An identity $s \approx t$ of a variety $\mathcal{V}$ is called an M-hyperidentity of a graph algebra $A(G)$, $G \in \mathcal{V}$ whenever the operation symbols occuring in s and t are replaced by any term operations in a subgroupoid M of term operations of $A(G)$ of the appropriate arity, the resulting identities hold in $A(G)$.

In this paper we characterize special M-hyperidentities in each (x(yz))z with opposite loop and reverse arc graph variety.

Keywords : varieties; (x(yz))z with opposite loop and reverse arc graph varieties; identities; term; hyperidentity; M-hyperidentity; binary algebra; graph algebra.

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1　Introduction

An identity $s \approx t$ of terms s, t of any type τ is called a *hyperidentity* (*M-hyperidentity*) of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations (any term operations in a subgroupoid M of term operations) of $\underline{A}$ of the appropriate arity (see the meaning in [1] page 3), the resulting identity holds in $\underline{A}$. Hyperidentities can be defined more precisely by using the concept of a hypersubstitution, which was introduced by Denecke, Lau, Pöschel and Schweigert in [2].

We fix a type $\tau = (n_i)_{i \in I}$, $n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is $n_i - ary$. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $Alg(\tau)$ be the class of all algebras of type τ. Then a mapping

$$\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$$

which assigns to every $n_i - ary$ operation symbol f_i an $n_i - ary$ term will be called a *hypersubstitution* of type τ (for short, a hypersubstitution). By $\hat{\sigma}$ we denote the extension of the hypersubstitution σ to a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X).$$

The term $\hat{\sigma}[t]$ is defined inductively by

(i)　$\hat{\sigma}[x] = x$ for any variable x in the alphabet X, and

(ii)　$\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

Here $\sigma(f_i)^{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra with the universe $W_\tau(X)$.

Graph algebras have been invented in [3] to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra* $\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and with two basic operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if} \quad (u, v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

In [4], graph varieties had been investigated for finite undirected graphs in order to get graph theoretic results (structure theorems) from universal algebra via graph algebras. In [5], these investigations are extended to arbitrary (finite) directed graphs where the authors ask for a graph theoretic characterization of graph varieties, i.e., of classes of graphs which can be defined by identities for

their corresponding graph algebras. The answer is a theorem of **Birkhoff-type**, which uses graph theoretic closure operations. *A class of finite directed graphs is equational* (i.e., a graph variety) *if and only if it is closed with respect to finite restricted pointed subproducts and isomorphic copies.*

In [6–8], Ananpinitwatna and Poomsa-ard characterized special M-hyperidentity in all biregular leftmost, in all (x(yz))z with loop and in all triregular leftmost without loop and reverse arc graph varieties respectively. In [9], Krapeedang and Poomsa-ard characterized the class of (x(yz))z with opposite loop and reverse arc graph varieties. In [10], Ananpinitwatna and Poomsa-ard characterized identities in (x(yz))z with opposite loop and reverse arc graph varieties. In [11], Tongmoon and Poomsa-ard characterized hyperidentities in (x(yz))z with opposite loop and reverse arc graph varieties.

In this paper we characterize special M-hyperidentities in each (x(yz))z with opposite loop and reverse arc graph variety.

2   Terms, Identities and Graph Varieties

Dealing with terms for graph algebras, the underlying formal language has to contain a binary operation symbol (juxtaposition) and a symbol for the constant ∞ (denoted by ∞, too).

Definition 2.1. The set $T(X)$ of all terms over the alphabet

$$X = \{x_1, x_2, x_3, \ldots\}$$

is defined inductively as follows:

   (i) every variable $x_i, i = 1, 2, 3, \ldots$, and ∞ are terms;

  (ii) if t_1 and t_2 are terms, then $t_1 t_2$ is a term;

 (iii) $T(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $T(X_2)$. The leftmost variable of a term t is denoted by $L(t)$ and rightmost variable of a term t is denoted by $R(t)$. A term, in which the symbol ∞ occurs is called a *trivial term*.

Definition 2.2. For each non-trivial term t of type $\tau = (2, 0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1 t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$$

where $t = t_1 t_2$ is a compound term.

$L(t)$ is called the *root* of the graph $G(t)$, and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, we assign the empty graph ϕ to every trivial term t.

Definition 2.3. A non-trivial term t of type $\tau = (2,0)$ is called $(x(yz))z$ with opposite loop and reverse arc term if and only if $G(t)$ is a graph with $V(t) = \{x, y, z\}$ and $E(t) = E \cup (\cup_{X \in E'} X)$, where $E = \{(x,y),(x,z),(y,z)\}$, $E' \subset \{U, V, W\}$, $E' \neq \phi$ and $U = \{(y,x),(z,z)\}$, $V = \{(z,x),(y,y)\}$, $W = \{(z,y),(x,x)\}$. A term equation $s \approx t$ is called $(x(yz)))z$ with opposite loop and reverse arc equation if s and t are $(x(yz))z$ with opposite loop and reverse arc terms

Definition 2.4. We say that a graph $G = (V, E)$ *satisfies an identity* $s \approx t$ if the corresponding graph algebra $A(G)$ satisfies $s \approx t$ (i.e. we have $s = t$ for every assignment $V(s) \cup V(t) \to V \cup \{\infty\})$, and in this case, we write $G \models s \approx t$. Given a class $\mathcal{G}$ of graphs and a set Σ of identities (i.e., $\Sigma \subset T(X) \times T(X)$) we introduce the following notation:

$G \models \Sigma$ if $G \models s \approx t$ for all $s \approx t \in \Sigma$, $\quad$ $\mathcal{G} \models s \approx t$ if $G \models s \approx t$ for all $G \in \mathcal{G}$,

$\mathcal{G} \models \Sigma$ if $G \models \Sigma$ for all $G \in \mathcal{G}$, $\quad$ $Id\mathcal{G} = \{s \approx t \mid s, t \in T(X),\ \mathcal{G} \models s \approx t\}$,

$Mod_g \Sigma = \{G \mid G \text{ is a graph and } G \models \Sigma\}$, $\quad$ $\mathcal{V}_g(\mathcal{G}) = Mod_g Id\mathcal{G}$.

$\mathcal{V}_g(\mathcal{G})$ is called the *graph variety generated by* $\mathcal{G}$ and $\mathcal{G}$ is called *graph variety* if $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$. $\mathcal{G}$ is called *equational* if there exists a set Σ' of identities such that $\mathcal{G} = \text{Mod}_g \Sigma'$. Obviously $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$ if and only if $\mathcal{G}$ is an equational class.

3　(x(yz))z with Opposite Loop and Reverse Arc Graph Varieties and Identities

All (x(yz))z with opposite loop and reverse arc graph varieties were characterized in [9] which found that $\mathcal{K} = \{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_3, \dots, \mathcal{K}_8\}$, where

$\mathcal{K}_0 = Mod_g\{(x(yz))z \approx (x(yz))z\}$,

$\mathcal{K}_1 = Mod_g\{((xx)(y(zy)))z \approx (x((yy)(zx)))z\}$,

$\mathcal{K}_2 = Mod_g\{((xx)(y(zy)))z \approx (x((yx)(zz)))z\}$,

$\mathcal{K}_3 = Mod_g\{(x((yy)(zx)))z \approx (x((yx)(zz)))z\}$,

$\mathcal{K}_4 = Mod_g\{((xx)(y(zy)))z \approx ((xx)((yy)((zx)y)))z\}$,

$\mathcal{K}_5 = Mod_g\{(x((yy)(zx)))z \approx ((xx)((yy)((zx)y)))z\}$,

$\mathcal{K}_6 = Mod_g\{(x((yx)(zz)))z \approx ((xx)((yx)((zy)z)))z\}$,

$\mathcal{K}_7 = Mod_g\{((xx)((yy)((zx)y)))z \approx ((xx)(((yx)y)(((zx)y)z)))z\}$,

$\mathcal{K}_8 = Mod_g\{((xx)(y(zy)))z \approx (x((yy)(zx)))z, ((xx)(y(zy)))z \approx (x(yx)(zz)))z\}$

are all (x(yz))z with opposite loop and reverse arc graph varieties.

In [10], Ananpinitwatna and Poomsa-ard characterized identities in all (x(yz))z with opposite loop and reverse arc graph varieties which found that if $s \approx t$ is a trivial equation (i.e. s and t are both trivial or $L(s) = L(t)$, $G(s) = G(t)$), then $s \approx t$ is an identity in every (x(yz))z with opposite loop and reverse arc graph varieties. Otherwise the common properties are $L(s) = L(t)$ and $V(s) = V(t)$. For other properties we need some notation about terms. For any non-trivial term t and $x \in V(t)$, let

$A_x(t) = \{x' \in V(t) \mid x' \text{ is an in-neighbor of } x \text{ in } G(t)\},$

$A'_x(t) = \{x' \in V(t) \mid x' \text{ is an out-neighbor of } x \text{ in } G(t)\},$

$A''_x(t) = \{x' \in V(t) \mid x' \text{ is an in-neighbor and an out-neighbor of } x \text{ in } G(t)\},$

$A^0_x(t) = \{x\}$, $A^1_x(t) = \{x' \in V(t) \mid x' \in A''_x(t) \text{ or } x' \in A'_x(t) \text{ which has } z$ such that $(x,z),(z,x),(x',z) \in E(t)$ or $x' \in A_x(t)$ which has z',z'' such that $(x,z'),(z',x),(x',z') \in E(t)$ or $(x,z''),(z'',x),(z'',x') \in E(t)\},$

$A^2_x(t) = \bigcup_{y \in A^1_x(t)} A^1_y(t),\ldots, A^n_x(t) = \bigcup_{y \in A^{n-1}_x(t)} A^1_y(t)$, $A^*_x(t) = \bigcup_{i=0}^{\infty} A^i_x(t),$

$B^0_x(t) = \{x\}$, $B^1_x(t) = \{x' \in V(t) \mid x' \in A''_x(t) \text{ or } x' \in A_x(t) \text{ which has }$ z such that $(x,z),(z,x),(x',z) \in E(t)$ or $x' \in A'_x(t)$ which has z' such that $(x,z'),(z',x),(z',x') \in E(t)\},$

$B^2_x(t) = \bigcup_{y \in B^1_x(t)} B^1_y(t),\ldots, B^n_x(t) = \bigcup_{y \in B^{n-1}_x(t)} B^1_y(t)$, $B^*_x(t) = \bigcup_{i=0}^{\infty} B^i_x(t),$

$C^0_x(t) = \{x\}$, $C^1_x(t) = \{x' \in V(t) \mid x' \in A''_x(t) \text{ or } x' \in A'_x(t) \text{ which has } z,z'$ such that $(x,z),(z,x),(x',z) \in E(t)$ or $(x,z'),(z',x),(z',x') \in E(t)$ or $x' \in A_x(t)$ which has z'' such that $(x,z''),(z'',x),(z'',x') \in E(t)\},$

$C^2_x(t) = \bigcup_{y \in C^1_x(t)} C^1_y(t),\ldots, C^n_x(t) = \bigcup_{y \in C^{n-1}_x(t)} C^1_y(t)$, $C^*_x(t) = \bigcup_{i=0}^{\infty} C^i_x(t),$

$D^0_x(t) = \{x\}$, $D^1_x(t) = \{x' \in V(t) \mid x' \in A''_x(t) \text{ or } x' \in A_x(t) \text{ which has } z \text{ such}$ that $(x,z),(z,x),(x',z) \in E(t)\}$,

$D^2_x(t) = \bigcup_{y \in D^1_x(t)} D^1_y(t),\ldots, D^n_x(t) = \bigcup_{y \in D^{n-1}_x(t)} D^1_y(t)$, $D^*_x(t) = \bigcup_{i=0}^{\infty} D^i_x(t),$

$F^0_x(t) = \{x\}$, $F^1_x(t) = \{x' \in V(t) \mid x' \in A''_x(t) \text{ or } x' \in A_x(t) \text{ which has }$ z such that $(x,z),(z,x),(z,x') \in E(t)$ or $x' \in A'_x(t)$ which has z' such that $(x,z'),(z',x),(x',z') \in E(t)\},$

$F^2_x(t) = \bigcup_{y \in F^1_x(t)} F^1_y(t),\ldots, F^n_x(t) = \bigcup_{y \in F^{n-1}_x(t)} F^1_y(t)$, $F^*_x(t) = \bigcup_{i=0}^{\infty} F^i_x(t),$

$H^0_x(t) = \{x\}$, $H^1_x(t) = \{x' \in V(t) \mid x' \in A''_x(t) \text{ or } x' \in A'_x(t) \text{ which has } z \text{ such}$ that $(x,z),(z,x),(z,x') \in E(t)\},$

$H^2_x(t) = \bigcup_{y \in H^1_x(t)} H^1_y(t),\ldots, H^n_x(t) = \bigcup_{y \in H^{n-1}_x(t)} H^1_y(t)$, $H^*_x(t) = \bigcup_{i=0}^{\infty} H^i_x(t),$

$I^0_x(t) = \{x\}$, $I^1_x(t) = \{x' \in V(t) \mid x' \in A''_x(t)\},$

$I^2_x(t) = \bigcup_{y \in I^1_x(t)} I^1_y(t),\ldots, I^n_x(t) = \bigcup_{y \in I^{n-1}_x(t)} I^1_y(t)$, $I^*_x(t) = \bigcup_{i=0}^{\infty} I^i_x(t)$. Then all identities in each (x(yz))z with opposite loop and reverse arc graph variety are characterized in the following table:

Table 1. The properties of identities in each graph variety.

Variety	Properties of s and t.
$\mathcal{K}_1$	(i) for any $x \in V(s)$, there exists $y \in A_x^*(s)$ such that $(y,y) \in E(s)$ iff there exists $z \in A_x^*(t)$ such that $(z,z) \in E(t)$, (ii) for any $x,y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y' \in A_y^*(s)$, $x' \in A_x^*(s)$ such that $(y',y') \in E(s)$ and $(x',x') \in E(s)$ iff $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y'' \in A_y^*(t)$, $x'' \in A_x^*(t)$ such that $(y'',y'') \in E(t)$ and $(x'',x'') \in E(t)$
$\mathcal{K}_2$	(i) for any $x \in V(s)$, there exists $y \in B_x^*(s)$ such that $(y,y) \in E(s)$ iff there exists $z \in B_x^*(t)$ such that $(z,z) \in E(t)$, (ii) for any $x,y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y' \in B_y^*(s)$, $x' \in B_x^*(s)$ such that $(y',y') \in E(s)$ and $(x',x') \in E(s)$ iff $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y'' \in B_y^*(t)$, $x'' \in B_x^*(t)$ such that $(y'',y'') \in E(t)$ and $(x'',x'') \in E(t)$
$\mathcal{K}_3$	(i) for any $x \in V(s)$, there exists $y \in C_x^*(s)$ such that $(y,y) \in E(s)$ iff there exists $z \in C_x^*(t)$ such that $(z,z) \in E(t)$, (ii) for any $x,y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y' \in C_y^*(s)$, $x' \in C_x^*(s)$ such that $(y',y') \in E(s)$ and $(x',x') \in E(s)$ iff $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y'' \in C_y^*(t)$, $x'' \in C_x^*(t)$ such that $(y'',y'') \in E(t)$ and $(x'',x'') \in E(t)$
$\mathcal{K}_4$	(i) for any $x \in V(s)$, there exists $y \in D_x^*(s)$ such that $(y,y) \in E(s)$ iff there exists $z \in D_x^*(t)$ such that $(z,z) \in E(t)$, (ii) for any $x,y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y' \in D_y^*(s)$, $x' \in D_x^*(s)$ such that $(y',y') \in E(s)$ and $(x',x') \in E(s)$ iff $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y'' \in D_y^*(t)$, $x'' \in D_x^*(t)$ such that $(y'',y'') \in E(t)$ and $(x'',x'') \in E(t)$
$\mathcal{K}_5$	(i) for any $x \in V(s)$, there exists $y \in F_x^*(s)$ such that $(y,y) \in E(s)$ iff there exists $z \in F_x^*(t)$ such that $(z,z) \in E(t)$, (ii) for any $x,y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y' \in F_y^*(s)$, $x' \in F_x^*(s)$ such that $(y',y') \in E(s)$ and $(x',x') \in E(s)$ iff $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y'' \in F_y^*(t)$, $x'' \in F_x^*(t)$ such that $(y'',y'') \in E(t)$ and $(x'',x'') \in E(t)$
$\mathcal{K}_6$	(i) for any $x \in V(s)$, there exists $y \in H_x^*(s)$ such that $(y,y) \in E(s)$ iff there exists $z \in H_x^*(t)$ such that $(z,z) \in E(t)$, (ii) for any $x,y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y' \in H_y^*(s)$, $x' \in H_x^*(s)$ such that $(y',y') \in E(s)$ and $(x',x') \in E(s)$ iff $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y'' \in H_y^*(t)$, $x'' \in H_x^*(t)$ such that $(y'',y'') \in E(t)$ and $(x'',x'') \in E(t)$

Table 1. (Continue).

Variety	Properties of s and t.
$\mathcal{K}_7$	(i) for any $x \in V(s)$, there exists $y \in I_x(s)$ such that $(y,y) \in E(s)$ iff there exists $z \in I_x(t)$ such that $(z,z) \in E(t)$, (ii) for any $x, y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $S \subset E(s)$ such that if $G \in K_7$ and h is a homomorphism from $G(s)$ into G, then $(h(x), h(y)) \in E$ iff $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $S' \subset E(t)$ such that if $G' \in K_7$ and h is a homomorphism from $G(s)$ into G', then $(h'(x), h'(y)) \in E'$
$\mathcal{K}_8$	(i) for any $x \in V(s)$, there exists $y \in A_x^*(s) \cup B_x^*(s)$ such that $(y,y) \in E(s)$ iff there exists $z \in A_x^*(t) \cup B_x^*(t)$ such that $(z,z) \in E(t)$, (ii) for any $x, y \in V(s)$ with $x \neq y$, $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y' \in A_y^*(s) \cup B_y^*(s)$, $x' \in A_x^*(s) \cup B_x^*(s)$ such that $(y',y') \in E(s)$ and $(x',x') \in E(s)$ iff $(x,y) \in E(s)$ or $(y,x) \in E(s)$ and, there exists $y'' \in A_y^*(t) \cup B_y^*(t)$, $x'' \in A_x^*(t) \cup B_x^*(t)$ such that $(y'', y'') \in E(t)$ and $(x'', x'') \in E(t)$

4 Hypersubstitution and Proper Hypersubstitution

Let $\mathcal{K}$ be a graph variety. Now we want to formulate precisely the concept of a graph hypersubstitution for graph algebras.

Definition 4.1. A mapping $\sigma : \{f, \infty\} \to T(X_2)$, where $X_2 = \{x_1, x_2\}$ and f is the operation symbol corresponding to the binary operation of a graph algebra is called *graph hypersubstitution* if $\sigma(\infty) = \infty$ and $\sigma(f) = s \in T(X_2)$. The graph hypersubstitution with $\sigma(f) = s$ is denoted by σ_s.

Definition 4.2. An identity $s \approx t$ is a $\mathcal{K}$ *graph hyperidentity* iff for all graph hypersubstitutions σ, the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in $\mathcal{K}$.

If we want to check that an identity $s \approx t$ is a hyperidentity in $\mathcal{K}$ we can restrict our consideration to a (small) subset of $Hyp\mathcal{G}$ - the set of all graph hypersubstitutions.

In [12], the following relation between hypersubstitutions was defined:

Definition 4.3. Two graph hypersubstitutions σ_1, σ_2 are called $\mathcal{K}$-*equivalent* iff $\sigma_1(f) \approx \sigma_2(f)$ is an idetity in $\mathcal{K}$. In this case we write $\sigma_1 \sim_\mathcal{K} \sigma_2$.

The following lemma was proved in [13].

Lemma 4.4. *If $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}$ and $\sigma_1 \sim_\mathcal{K} \sigma_2$ then, $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}$.*

Therefore, it is enough to consider the quotient set $Hyp\mathcal{G}/\sim_\mathcal{K}$.

In [14], it was shown that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Without difficulties, we can show that $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

The following definition was given in [15].

Definition 4.5. The graph hypersubstitution $\sigma_{NF(t)}$, is called *normal form graph hypersubstitution*. Here $NF(t)$ is the normal form of the binary term t.

Since for any binary term t the rooted graphs of t and $NF(t)$ are the same, we have $t \approx NF(t) \in Id\mathcal{K}$. Then for any graph hypersubstitution σ_t with $\sigma_t(f) = t \in T(X_2)$, one obtains $\sigma_t \sim_\mathcal{K} \sigma_{NF(t)}$.

In [15], all rooted graphs with at most two vertices were considered. Then we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given in the following table.

Table 2. Normal form terms.

normal form term	graph hypers	normal form term	graph hypers
x_1x_2	σ_0	x_1	σ_1
x_2	σ_2	x_1x_1	σ_3
x_2x_2	σ_4	x_2x_1	σ_5
$(x_1x_1)x_2$	σ_6	$(x_2x_1)x_2$	σ_7
$x_1(x_2x_2)$	σ_8	$x_2(x_1x_1)$	σ_9
$(x_1x_1)(x_2x_2)$	σ_{10}	$(x_2(x_1x_1))x_2$	σ_{11}
$x_1(x_2x_1)$	σ_{12}	$x_2(x_1x_2)$	σ_{13}
$(x_1x_1)(x_2x_1)$	σ_{14}	$(x_2(x_1x_2))x_2$	σ_{15}
$x_1((x_2x_1)x_2)$	σ_{16}	$x_2((x_1x_1)x_2)$	σ_{17}
$(x_1x_1)((x_2x_1)x_2)$	σ_{18}	$(x_2((x_1x_1)x_2))x_2$	σ_{19}

Let $M_\mathcal{G}$ be the set of all normal form graph hypersubstitutions. Then we get,

$$M_\mathcal{G} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}.$$

We define the product of two normal form graph hypersubstitutions in $M_\mathcal{G}$ as follows.

Definition 4.6. The *product $\sigma_{1N} \circ_N \sigma_{2N}$ of two normal form graph hypersubstitutions* is defined by $(\sigma_{1N} \circ_N \sigma_{2N})(f) = NF(\hat{\sigma}_{1N}[\sigma_{2N}(f)])$.

The concept of a proper hypersubstitution of a class of algebras was introduced in [13].

Definition 4.7. A hypersubstitution σ is called *proper with respect to a class $\mathcal{K}$ of algebras* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}$ for all $s \approx t \in Id\mathcal{K}$.

The following lemma was proved in [15].

Lemma 4.8. *For each non-trivial term $s, (s \neq x \in X)$ and for all $u, v \in X$, we have*

$$E(\hat{\sigma}_6[s]) = E(s) \cup \{(u, u)|(u, v) \in E(s)\},$$

$$E(\hat{\sigma}_8[s]) = E(s) \cup \{(v, v)|(u, v) \in E(s)\},$$

and

$$E(\hat{\sigma}_{12}[s]) = E(s) \cup \{(v, u)|(u, v) \in E(s)\}.$$

By the similar way we prove that,

$$E(\hat{\sigma}_{10}[s]) = E(s) \cup \{(u, u), (v, v)|(u, v) \in E(s)\}.$$

Let $PM_{\mathcal{K}}$ be the set of all proper graph hypersubstitutions with respect to the class $\mathcal{K}$. In [11] it was found out that,
$$PM_{\mathcal{K}_1} = PM_{\mathcal{K}_2} = PM_{\mathcal{K}_3} = PM_{\mathcal{K}_4} = PM_{\mathcal{K}_6} = PM_{\mathcal{K}_8} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$$
and $PM_{\mathcal{K}_5} = PM_{\mathcal{K}_7} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}\}$.

5 Special M-hyperidentities

We know that a graph identity $s \approx t$ is a graph hyperidentity, if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is a graph identity for all $\sigma \in M_{\mathcal{G}}$. Let M be a subgroupoid of $M_{\mathcal{G}}$. Then, a graph identity $s \approx t$ is an M-graph hyperidentity (M-hyperidentity), if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is a graph identity for all $\sigma \in M$. In [1], Denecke and Wismath defined special subgroupoid of $M_{\mathcal{G}}$ as the following.

Definition 5.1.

(i) A hypersubstitution $\sigma \in Hyp(\tau)$ is said to be leftmost if for every $i \in I$, the first variable in $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]$ is x_1. Let $Left(\tau)$ be the set of all leftmost hypersubstitutions of type τ.

(ii) A hypersubstitution $\sigma \in Hyp(\tau)$ is said to be outermost if for every $i \in I$, the first variable in $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]$ is x_1 and the last variable is x_{n_i}. Let $Out(\tau)$ be the set of all outermost hypersubstitutions of type τ.

(iii) A hypersubstitution $\sigma \in Hyp(\tau)$ is said to be rightmost if for every $i \in I$, the last variable in $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]$ is x_{n_i}. Let $Right(\tau)$ be the set of all rightmost hypersubstitutions of type τ. Note that $Out(\tau) = Right(\tau) \cap Left(\tau)$.

(iv) A hypersubstitution $\sigma \in Hyp(\tau)$ is called regular if for every $i \in I$, each of the variables $x_1, \ldots, x_{n_i}$ occurs in $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]$. Let $Reg(\tau)$ be the set of all regular hypersubstitutions of type τ.

(v) A hypersubstitution $\sigma \in Hyp(\tau)$ is called symmetrical if for every $i \in I$, there is a permutation s_i on the set $\{1, \ldots, n_i\}$ such that $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})] = f_i(x_{s_i(1)}, \ldots, x_{s_i(n_i)})$. Let $D(\tau)$ be the set of all symmetrical hypersubstitutions of type τ.

(vi) We will call a hypersubstitution σ of type τ a pre-hypersubstitution if for every $i \in I$, the term $\sigma(f_i)$ is not a variable. Let $Pre(\tau)$ be the set of all pre-hypersubstitutions of type τ.

From Definition 5.1, we have:

$M_{Left} = \{\sigma_0, \sigma_1, \sigma_3, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$.

$M_{Right} = \{\sigma_0, \sigma_2, \sigma_4, \sigma_6, \sigma_7, \sigma_8, \sigma_{10}, \sigma_{11}, \sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$.

$M_{Out} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{16}, \sigma_{18}\}$.

$M_{Reg} = \{\sigma_0, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$.

$M_D = \{\sigma_0, \sigma_5\}$.

$M_{Pre} = \{\sigma_0, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$.

Definition 5.2. Let $\mathcal{V}$ be a graph variety of type τ, and let $s \approx t$ be an identity of $\mathcal{V}$. Let M be a subgroupoid of $Hyp(\tau)$. Then $s \approx t$ is called an M-hyperidentity with respect to $\mathcal{V}$, if for every $\sigma \in M$, $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity of $\mathcal{V}$.

For any (x(yz))z with opposite loop and reverse arc graph variety $\mathcal{K}$ and for any $s \approx t \in Id\mathcal{K}$. We want to characterize the property of s and t such that $s \approx t$ is an M_{Left}-hyperidentity, M_{Right}-hyperidentity, M_{Out}-hyperidentity, M_{Reg}-hyperidentity, M_D-hyperidentity and M_{Pre}-hyperidentity with respect to $\mathcal{K}$ for all (x(yz))z with opposite loop and reverse arc graph varieties $\mathcal{K}$. For any term s we see that $s \approx s$ is an M-hyperidentity with respect to $\mathcal{K}$ for all (x(yz))z with opposite loop and reverse arc graph varieties $\mathcal{K}$ and for all special subgroupoid M of $M_{\mathcal{G}}$.

At first we consider the M_D-hyperidentity. Let $\mathcal{K}$ be any (x(yz))z with opposite loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. Since $M_D = \{\sigma_0, \sigma_5\}$, we see that if s and t are trivial terms, then $s \approx t$ is an M_D-hyperidentity with respect to $\mathcal{K}$. For the case s and t are non-trivial terms, we have $s \approx t$ is an M_D-hyperidentity with respect to $\mathcal{K}$ if and only if $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}$.

For M_{Left}-hyperidentity. Let $\mathcal{K}$ be any (x(yz))z with opposite loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. Since $M_{Left} = \{\sigma_0, \sigma_1, \sigma_3, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$, we see that if s and t are trivial terms, then $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}$ if and only if $L(s) = L(t)$. Now we consider the case s and t are non-trivial and different terms. We characterize M_{Left}-hyperidentity with respect to all (x(yz))z with opposite loop and reverse arc graph varieties as the following theorem:

Theorem 5.3. *Let s and t be non-trivial and different terms and let $\mathcal{K}_i$, $i \in \{1, 2, 3, \ldots, 8\}$ be (x(yz))z with opposite loop and reverse arc graph varieties. If $s \approx t \in Id\mathcal{K}_i$, then $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}_i$.*

Proof. Consider for $\mathcal{K}_1$. If $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$, then σ is a proper hypersubstitution. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$. Since $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$ and $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$, we have $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}_1$ and $\hat{\sigma}_3[s] \approx \hat{\sigma}_3[t] \in Id\mathcal{K}_1$. By Table 2, we have $\sigma_{10} \sim_{\mathcal{K}_1} \sigma_{14} \sim_{\mathcal{K}_1} \sigma_{16} \sim_{\mathcal{K}_1} \sigma_{18}$. We get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$ for all $\sigma \in \{\sigma_{14}, \sigma_{16}, \sigma_{18}\}$. Hence, $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}_1$.

The proof of $\mathcal{K}_i$, $i \in \{2, 3, \ldots, 8\}$ are similar to the proof of $\mathcal{K}_1$. $\qquad\square$

For M_{Out}-hyperidentity. Let $\mathcal{K}$ be any (x(yz))z with opposite loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. Since $M_{Out} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{16}, \sigma_{18}\}$, we see that if s and t are trivial terms, then $s \approx t$ is an M_{Out}-hyperidentity with respect to $\mathcal{K}$. For the case s and t are non-trivial and different terms, since $M_{Out} \subset M_{Left}$, so we can check that it has the same results as M_{Left}-hyperidentity.

For M_{Reg}-hyperidentity. Let $\mathcal{K}$ be any (x(yz))z with opposite loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. Since $M_{Reg} = \{\sigma_0, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$, we see that if s and t are trivial terms, then $s \approx t$ is an M_{Reg}-hyperidentity with respect to $\mathcal{K}$. For the case s and t are non-trivial and different terms, we have the following theorem:

Theorem 5.4. *Let s and t be non-trivial and different terms and let $\mathcal{K}_i$, $i \in \{1, 2, 3, \ldots, 8\}$ be (x(yz))z with opposite loop and reverse arc graph varieties. If $s \approx t \in Id\mathcal{K}_i$, then $s \approx t$ is an M_{Reg}-hyperidentity with respect to $\mathcal{K}_i$ if and only if $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_i$.*

Proof. Consider for $\mathcal{K}_1$. If $s \approx t$ is an M_{Reg}-hyperidentity with respect to $\mathcal{K}_1$, then $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in $\mathcal{K}_1$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_1$ and that $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in $\mathcal{K}_1$. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Reg}.

If σ is a proper, then we get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$. Hence, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$ for all $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$. By assumption, $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is an identity in $\mathcal{K}_1$. Since $\sigma_{10} \sim_{\mathcal{K}_1} \sigma_{14} \sim_{\mathcal{K}_1} \sigma_{16} \sim_{\mathcal{K}_1} \sigma_{18}$, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$ for all $\sigma \in \{\sigma_{14}, \sigma_{16}, \sigma_{18}\}$.

Since $\sigma_6 \circ_N \sigma_5 = \sigma_7$, $\sigma_8 \circ_N \sigma_5 = \sigma_9$, $\sigma_{10} \circ_N \sigma_5 = \sigma_{11}$, $\sigma_{12} \circ_N \sigma_5 = \sigma_{13}$, $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_1$ and σ_6, σ_8, σ_{10}, σ_{12} are proper, we have that $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t]$, $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t]$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ are identities in $\mathcal{K}_1$. Since $\sigma_{11} \sim_{\mathcal{K}_1} \sigma_{15} \sim_{\mathcal{K}_1} \sigma_{17} \sim_{\mathcal{K}_1} \sigma_{19}$, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$ for all $\sigma \in \{\sigma_{15}, \sigma_{17}, \sigma_{19}\}$.

The proof of $\mathcal{K} \in \{\mathcal{K}_2, \mathcal{K}_3, \ldots, \mathcal{K}_8\}$ are similar to the proof of $\mathcal{K}_1$. $\qquad\square$

For M_{Pre}-hyperidentity. Let $\mathcal{K}$ be any (x(yz))z with opposite loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. Since $M_{Pre} = \{\sigma_0, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$, we see that if s and t are trivial terms, then $s \approx t$ is an M_{Pre}-hyperidentity with respect to $\mathcal{K}$ if and only if they

have the same leftmost and the same rightmost. For the case s and t are non-trivial and different terms, since $M_{Reg} = M_{Pre} - \{\sigma_3, \sigma_4\}$, we have the same results as M_{Reg}-hyperidentity.

For M_{Right}-hyperidentity. Let $\mathcal{K}$ be any (x(yz))z with opposite loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. Since $M_{Right} = \{\sigma_0, \sigma_2, \sigma_4, \sigma_6, \sigma_7, \sigma_8, \sigma_{10}, \sigma_{11}, \sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$, we see that if s and t are trivial terms, then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}$ if and only if they have the same rightmost variables. For the case s and t are non-trivial and different terms, we characterize M_{Right}-hyperidentity with respect to each (x(yz))z with opposite loop and reverse arc graph variety as the following theorem:

Theorem 5.5. *Let s and t be non-trivial and different terms and let $\mathcal{K}_i$, $i \in \{1, 2, 3, \ldots, 8\}$ be (x(yz))z with opposite loop and reverse arc graph varieties. If $s \approx t \in Id\mathcal{K}_i$, then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_i$ if and only if $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_i$ and $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_i$.*

Proof. Consider for $\mathcal{K}_1$, let $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_1$. We have $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_1$ and $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_1$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_1$, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_1$ and $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_1$. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}. If $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}\}$, then σ is a proper graph hypersubstitution. Hence, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$. By assumption, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_1$ and $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_1$. Hence, $R(s) = L(\hat{\sigma}_7[s]) = L(\hat{\sigma}_7[t]) = R(t)$. Since $\hat{\sigma}_2[s] = R(s) = R(t) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = R(s)R(s) = R(t)R(t) = \hat{\sigma}_4[t]$, we have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_1$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \in Id\mathcal{K}_1$. Since $\sigma_{10} \circ \sigma_7 = \sigma_{11}$ and σ_{10} is a proper graph hypersubstitution, we have $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t] \in Id\mathcal{K}_1$. Since $\sigma_{10} \sim_{\mathcal{K}_1} \sigma_{16} \sim_{\mathcal{K}_1} \sigma_{18}$ and $\sigma_{11} \sim_{\mathcal{K}_1} \sigma_{15} \sim_{\mathcal{K}_1} \sigma_{17} \sim_{\mathcal{K}_1} \sigma_{19}$, we get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$ for all $\sigma \in \{\sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$. Hence, $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_1$. The proof of other $\mathcal{K}_i$ graph variety are similar to the proof of $\mathcal{K}_1$. $\square$

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