



# Rate of Convergence of S-iteration and Ishikawa Iteration for Continuous Functions on Closed Intervals

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**Abstract :** In this paper, we first give a necessary and sufficient condition for convergence of S-iteration and then prove equivalence between S-iteration and Ishikawa iteration. Finally, we compare the rate of convergence between S-iteration and Ishikawa iteration. Some numerical examples for comparing the rate of convergence between these two methods are also given.

**Keywords :** rate of convergence; S-iteration; Ishikawa iteration; continuous functions; closed interval.

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## 1 Introduction

Fixed point iteration methods play very important role in approximation for solutions of nonlinear equations. Let  $E$  be a closed interval on the real line and  $f : E \rightarrow E$  be a continuous function. A point  $p \in E$  is a *fixed point* of  $f$  if  $f(p) = p$ . We denote by  $F(f)$  the set of fixed points of  $f$ . It is known that if  $E$  also bounded, then  $F(f)$  is nonempty. The *Mann iteration* (see [1]) is defined by  $u_1 \in E$  and

$$u_{n+1} = (1 - \alpha_n) u_n + \alpha_n f(u_n) \quad (1.1)$$

for all  $n \geq 1$ , where  $\{\alpha_n\}_{n=1}^{\infty}$  is sequences in  $[0, 1]$ , and will be denoted by  $M(u_1, \alpha_n, f)$ . The *Ishikawa iteration* (see [3]) is defined by  $s_1 \in E$  and

$$\begin{cases} t_n = (1 - \beta_n) s_n + \beta_n f(s_n) \\ s_{n+1} = (1 - \alpha_n) s_n + \alpha_n f(t_n) \end{cases} \quad (1.2)$$

for all  $n \geq 1$ , where  $\{\alpha_n\}_{n=1}^{\infty}$ ,  $\{\beta_n\}_{n=1}^{\infty}$  are sequences in  $[0, 1]$ , and will be denoted by  $I(s_1, \alpha_n, \beta_n, f)$ . The *S-iteration* (see [4]) is defined by  $x_1 \in E$  and

$$\begin{cases} y_n = (1 - \beta_n) x_n + \beta_n f(x_n) \\ x_{n+1} = (1 - \alpha_n) f(x_n) + \alpha_n f(y_n) \end{cases} \quad (1.3)$$

for all  $n \geq 1$ , where  $\{\alpha_n\}_{n=1}^{\infty}$  and  $\{\beta_n\}_{n=1}^{\infty}$  are sequences in  $[0, 1]$ , and will be denoted by  $S(x_1, \alpha_n, \beta_n, f)$ . The S-iteration was first introduced by Agarwal, O'Regan and Sahu [2]. Clearly Mann iterations is a special case of Ishikawa iteration.

In 1976, Rhoades [5] proved the convergence of the Mann and Ishikawa iterations for the class of continuous and nondecreasing functions on unit closed interval. After that in 1991, Borwein and Borwein [6] proved the convergence of the Mann iteration of continuous functions on a bounded closed interval. Recently, Qing and Qihou [8] extended their results to an arbitrary interval and to the Ishikawa iteration and gave some control conditions for the convergence of Ishikawa iteration on an arbitrary interval.

It was shown in [9] that the Mann and Ishikawa iterations are equivalent for the class of Zamfirescu operators. In 2006, Babu and Prasad [10] showed that the Mann iteration converges faster than the Ishikawa iteration for these class of operators. Two year later, Qing and Rhoades [11] provided an example to show that the claim of Babu and Prasad [10] is false.

In this paper, we give a necessary and sufficient condition for the convergence of the S-iteration of continuous nondecreasing functions on an arbitrary interval. We also prove that if the Ishikawa iterations converges, then the S-iteration converges and converges faster than the Ishikawa iteration for the class of continuous and nondecreasing functions. Moreover, we present the numerical examples for the S-iteration to compare with the Mann and Ishikawa iterations.

## 2 Main Results

We first give some useful facts for our main results.

**Lemma 2.1.** *Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  a continuous and non-decreasing function. Let  $\{\alpha_n\}, \{\beta_n\}$  be sequences in  $[0, 1]$ . For  $x_1 \in E$ , let  $\{x_n\}$  be the sequence defined by (1.3).*

1. (i) *If  $f(x_1) < x_1$ , then  $f(x_n) \leq x_n \quad \forall n$  and  $\{x_n\}$  is nonincreasing.*
2. (ii) *If  $f(x_1) > x_1$ , then  $f(x_n) \geq x_n \quad \forall n$  and  $\{x_n\}$  is nondecreasing.*

*Proof.* (i) Let  $f(x_1) < x_1$ . Assume that  $f(x_k) \leq x_k$  for  $k > 1$ . Then  $f(x_k) \leq y_k \leq x_k$ . Since  $f$  is non-decreasing, we have  $f(y_k) \leq f(x_k) \leq y_k$ . By (1.4), we have  $f(y_k) \leq f(x_{k+1}) \leq f(x_k)$ . This implies  $x_{k+1} \leq f(x_k) \leq y_k$ . Since  $f$  is nondecreasing, we have  $f(x_{k+1}) \leq f(y_k)$ . Thus  $f(x_{k+1}) \leq x_{k+1}$ . By induction, we can conclude that  $f(x_n) \leq x_n$  for all  $n \geq 1$ . This together with (1.4), we have  $y_n \leq x_n$  for all  $n \geq 1$ , since  $f$  is nondecreasing, we have  $f(x_n) \leq f(x_n) \leq x_n$  for all  $n \geq 1$ . It follows that  $x_{n+1} = (1 - \alpha_n)f(x_n) + \alpha_n f(y_n) \leq x_n$  for all  $n \geq 1$ . Thus  $\{x_n\}$  is nonincreasing.

(ii) By using the same argument as in (i), we obtain the desired result.  $\square$

**Theorem 2.2.** *Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  a continuous and nondecreasing function. For  $x_1 \in E$ , let the S-iteration  $\{x_n\}_{n=1}^{\infty}$  be defined by 1.3, where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  are sequences in  $[0, 1]$  and  $\lim_{n \rightarrow \infty} \beta_n = 0$ . Then  $\{x_n\}$  is bounded if and only if  $\{x_n\}$  converges to a fixed point of  $f$ .*

*Proof.* If  $\{x_n\}$  is convergent, then it is bounded. Now, assume that  $\{x_n\}$  is bounded. we will show that  $\{x_n\}$  is convergent. If  $f(x_1) = x_1$ , by (1.3) we can show by induction that  $x_n = x_1$  for all  $n \geq 1$ . Thus  $\{x_n\}$  is convergent. Suppose that  $f(x_1) \neq x_1$ . By Lemma 2.1, we obtain that  $\{x_n\}$  is non-decreasing or non-increasing. Since  $\{x_n\}$  is bounded, it implies that it is convergent. Next, we prove that  $\{x_n\}$  converges to a fixed point of  $f$ . Let  $\lim_{n \rightarrow \infty} x_n = p$  for some  $p \in E$ . By continuity of  $f$ , we have  $\{f(x_n)\}$  is bounded. By (1.4),  $y_n - x_n = \beta_n(f(x_n) - x_n)$ . Since  $\beta_n \rightarrow 0$ , we obtain that  $y_n - x_n \rightarrow 0$ . This implies  $y_n \rightarrow p$ . By continuity of  $f$ , we have

$$\lim_{n \rightarrow \infty} (f(y_n) - f(x_n)) = f(p) - f(p) = 0. \quad (2.1)$$

By (1.3), we have  $x_n = f(x_n) + \alpha_n(f(y_n) - f(x_n))$ .

By continuity of  $f$ , we have

$$\begin{aligned} p &= \lim_{n \rightarrow \infty} x_n \\ &= \lim_{n \rightarrow \infty} f(x_n) + \lim_{n \rightarrow \infty} \alpha_n(f(y_n) - f(x_n)) \\ &= f(p) \end{aligned}$$

Hence  $\{x_n\}$  converge to a fixed point of  $f$ .  $\square$

**Lemma 2.3.** *Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  a continuous and nondecreasing function. Let  $\{x_n\}$  be the S-iteration defined by (1.3), where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $[0, 1]$ . Then we have the following:*

(i) *If  $p \in F(f)$  with  $x_1 > p$ , then  $x_n \geq p \quad \forall n$ .*

(ii) *If  $p \in F(f)$  with  $x_1 < p$ , then  $x_n \leq p \quad \forall n$ .*

*Proof.* (i) Suppose that  $p \in F(f)$  and  $x_1 > p$ . Since  $f$  is nondecreasing, we have  $f(x_1) \geq f(p) = p$ . By (1.3), we have  $y_1 \geq p$ . thus  $f(y_1) \geq f(p) = p$ . Then

$$\begin{aligned} x_n &= (1 - \alpha_1)f(x_1) + \alpha_1 f(y_1) \\ &\geq (1 - \alpha_1)p + \alpha_1 p = p. \end{aligned}$$

Assume that  $x_k \geq p$ . Thus  $f(x_k) \geq f(p) = p$ . Then

$$y_k = (1 - \beta_k)x_k + \beta_k f(x_k) \geq (1 - \beta_k)p + \beta_k p = p.$$

Hence  $f(y_k) \geq f(p) = p$ . It follows that

$$x_{k+1} = (1 - \alpha_k)f(x_k) + \alpha_k f(y_k) \geq (1 - \alpha_k)p + \alpha_k p = p.$$

By induction, we can conclude that  $x_n \geq p$  for all  $n \geq 1$ .

(ii) By using the same argument as in (i), we can show that  $x_n \leq p$  for all  $n \geq 1$ .  $\square$

**Lemma 2.4.** *Let  $E$  be a closed interval on real line and  $f : E \rightarrow E$  a continuous and nondecreasing function. Let  $\{\alpha_n\}, \{\beta_n\}$  be sequences in  $[0, 1]$ . For  $u_1 = s_1 = x_1 \in E$ , let  $\{u_n\}, \{s_n\}$  and  $\{x_n\}$  be the sequences defined by (1.1)-(1.3), respectively. Then we have the following:*

(i) *If  $f(u_1) < u_1$ , then  $x_n \leq s_n \leq u_n$  for all  $n \geq 1$ .*

(ii) *If  $f(u_1) > u_1$ , then  $x_n \geq s_n \geq u_n$  for all  $n \geq 1$ .*

*Proof.* Let  $f(u_1) < u_1$ . Since  $u_1 = s_1 = x_1$ , we set  $f(s_1) < s_1$ ,  $f(x_1) < x_1$ . First, we show that  $x_n \leq s_n$  for all  $n \geq 1$ . By (1.2) and (1.3),

$$y_1 - t_1 = (1 - \beta_1)(x_1 - s_1) + \beta_1(f(x_1) - f(s_1)) = 0,$$

so  $y_1 = t_1$ . This implies

$$x_2 - s_2 = (1 - \alpha_1)(f(x_1) - s_1) + \alpha_1(f(y_1) - f(t_1)) \leq 0.$$

So  $x_2 \leq s_2$ . Assume that  $x_k \leq s_k$ . Thus  $f(x_k) \leq f(s_k)$ . By Lemma 2.3,  $f(s_k) \leq s_k$  and  $f(x_k) \leq x_k \leq s_k$ . This implies  $f(x_k) \leq y_k \leq x_k$  and  $f(y_k) \leq f(x_k) \leq y_k$ . By (1.2) and (1.3), we have

$$y_k - t_k = (1 - \beta_k)(x_k - s_k) + \beta_k(f(x_k) - f(s_k)) \leq 0.$$

Thus  $y_k \leq t_k$ . Since  $f$  is nondecreasing, we have  $f(y_k) \leq f(t_k)$ . This implies

$$x_{k+1} - s_{k+1} = (1 - \alpha_k)(f(x_k) - s_k) + \alpha_k(f(y_k) - f(t_k)) \leq 0.$$

Hence  $x_{k+1} \leq s_{k+1}$ . By mathematical induction, we obtain  $x_n \leq s_n$  for all  $n \geq 1$ .

(ii) By using the same argument as in (i), we obtain the desired result.  $\square$

The next two propositions show that convergence of S-iteration depends on how far the initial point from the fixed point set.

**Proposition 2.5.** *Let  $E$  be a closed interval on the real line and  $f : E \rightarrow E$  be a continuous and nondecreasing function such that  $F(f)$  is nonempty and bounded with  $x_1 > \sup\{p \in E : p = f(p)\}$ . Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $[0, 1]$ . If  $f(x_1) > x_1$ , then the sequences  $\{x_n\}$  defined by S-iteration does not converge to a fixed point of  $f$ .*

*Proof.* By Lemma 2.1 (ii), we have  $\{x_n\}$  is non-decreasing. Since the initial point  $x_1 > \sup\{p \in E : p = f(p)\}$ , it follows that  $\{x_n\}$  does not converge to a fixed point of  $f$ .  $\square$

**Proposition 2.6.** *Let  $E$  be a closed interval on the real line and  $f : E \rightarrow E$  be a continuous and nondecreasing function such that  $F(f)$  is nonempty and bounded with  $x_1 < \inf\{p \in E : p = f(p)\}$ . Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $[0, 1]$ . If  $f(x_1) < x_1$ , then the sequences  $\{x_n\}$  defined by S-iteration does not converge to a fixed point of  $f$ .*

*Proof.* By Lemma 2.1 (i), we have that  $\{x_n\}$  is nonincreasing. Since the initial point  $x_1 < \inf\{p \in E : p = f(p)\}$ , it follows that  $\{x_n\}$  does not converge to a fixed point of  $f$ .  $\square$

**Theorem 2.7.** *Let  $E$  be a closed interval on the real line and  $f : E \rightarrow E$  be a continuous and nondecreasing function such that  $F(f)$  is nonempty and bounded. For  $u_1 = s_1 = x_1 \in E$ , let  $\{u_n\}$ ,  $\{s_n\}$  and  $\{x_n\}$  be the sequences defined by (1.1)-(1.3), respectively. If the Ishikawa iteration  $\{s_n\}$  converges to  $p \in F(f)$ , then the S-iteration  $\{x_n\}$  converges to  $p$ . Moreover, the S-iteration converges faster than the Ishikawa iteration.*

*Proof.* Suppose the Ishikawa iteration  $\{x_n\}$  converges to  $p \in F(f)$ . Put  $l = \inf\{p \in E : p = f(p)\}$  and  $u = \sup\{p \in E : p = f(p)\}$ . We divide our proof into the following three cases:

Case 1:  $s_1 = x_1 > u$ . By [7], Proposition 3.5, we get  $f(s_1) < s_1$  and  $f(x_1) < x_1$ . By lemma 2.4 (i), we have  $x_n \leq s_n$  for all  $n \geq 1$ . By continuity of  $f$ , we have  $f(u) = u$ , so  $u = f(u) \leq f(x_1) < x_1$ . This implies by (1.3) that  $f(x_1) \leq y_1 \leq x_1$ . So  $u \leq y_1 \leq x_1$ . Since  $f$  is nonincreasing,  $u = f(u) \leq f(x_1) < f(x_1)$ . It follows by (1.3),  $u \leq f(y_1) \leq x_2 \leq f(x_1)$ . By mathematical induction, we can show that  $u \leq x_n$  for all  $n \geq 1$ . Hence, we have  $p \leq x_n \leq s_n$  for all  $n \geq 1$ . Then  $|x_n - p| \leq |s_n - p|$  for all  $n \geq 1$ . It follows that  $x_n \rightarrow p$  and the S-iteration  $\{x_n\}$  converges to  $p \in F(f)$  faster than the Ishikawa iteration  $\{s_n\}$ .

Case 2:  $s_1 = x_1 < l$ . By Proposition 2.6, we get  $f(x_1) > x_1$  and  $f(s_1) > s_1$ . This implies by Lemma 2.4 (ii),  $x_n \geq s_n$  for all  $n \geq 1$ . We note that  $x_1 < l$  and by using (1.3) and mathematical induction, we can show that  $x_n < l$  for all  $n \geq 1$ . So we have  $s_n \geq x_n \geq p$  for all  $n \in \mathbb{N}$ . Then  $|x_n - p| \leq |s_n - p|$  for all  $n \geq 1$ . It follows that  $\{x_n\}$  converges to  $p$  and the S-iteration  $\{x_n\}$  converges to  $p \in F(f)$  faster than the Ishikawa iteration  $\{s_n\}$ .

Case 3:  $l \leq s_1 \leq x_1 \leq u$ . suppose that  $f(x_1) \neq x_1$ . If  $f(x_1) < x_1$ , we have by Lemma 2.1 (i) that  $\{x_n\}$  is nonincreasing with limit  $p$ . By Lemma 2.3 (i) and Lemma 2.4 (i), we have  $p \leq x_n \leq s_n$  for all  $n \geq 1$ . It follows that  $|x_n - p| \leq |s_n - p|$  for all  $n \geq 1$ . Hence  $\{x_n\}$  converges to  $p$  and the S-iteration  $\{x_n\}$  converges to  $p$  faster than the Ishikawa iteration.

If  $f(x_1) > x_1$ , we have by Lemma 2.1(ii) that  $\{x_n\}$  is nondecreasing with limit  $p$ . By Lemma 2.3 (ii) and Lemma 2.4 (ii), we have  $p \geq x_n \geq s_n$  for all  $n \geq 1$ . It

follows that  $|x_n - p| \leq |s_n - p|$  for all  $n \geq 1$ . Hence, we have  $\{x_n\}$  converges to  $p$  and the S-iteration  $\{x_n\}$  converges to  $p$  faster than the Ishikawa iteration.  $\square$

**Example 2.8.** Let  $f : [0, 8] \rightarrow [0, 8]$  be defined by  $f(x) = \frac{x^2+9}{10}$ . Then  $f$  is a continuous and non-decreasing function. The comparisons of the convergence of the Mann, Ishikawa and the S-iteration to the exact fixed point  $p = 1$  are given in the following table with the initial point  $x_1 = s_1 = u_1 = 4$  and  $\alpha_n = \beta_n = \frac{1}{n}$ .

$n$	Mann	Ishikawa	S-iteration	
	$u_n$	$s_n$	$x_n$	$ f(x_n) - x_n $
3	1.776671875	1.217094392	1.020004999	0.082558017
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
8	1.424154537	1.110061637	1.000028652	0.000116500
9	1.383987922	1.098907130	1.000005659	2.29215E-05
10	1.351493959	1.090012877	1.000001121	4.52707E-06

Table 1:

From Table 1, we see that the S-iteration converges faster than Mann, and Ishikawa iterations.

**Example 2.9.** Let  $f : [-6, \infty) \rightarrow [-6, \infty)$  be defined by  $f(x) = \sqrt{x+6}$ . Then  $f$  is a continuous and nondecreasing function. The comparisons of the convergence for S-iteration, Mann and Ishikawa iterations where the fixed point  $p = 3$  are given in the following table with initial point  $u_1 = s_1 = x_1 = 9$  and  $\alpha_n = \beta_n = \frac{1}{n}$ .

$n$	Mann	Ishikawa	S-iteration	
	$u_n$	$s_n$	$x_n$	$ f(x_n) - x_n $
3	3.507556865	3.07795462	3.018687231	0.118534778
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
8	3.183039411	3.026875838	3.000001918	9.75403E-06
9	3.163953612	3.024017604	3.000000315	1.59802E-06
10	3.148759019	3.021752320	3.000000052	2.62869E-07

Table 2:

From Table 2, we see that the S-iteration converges faster than Mann, and Ishikawa iterations.

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