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On Estimates for the Bessel Transform in the Space $L_{p,\alpha}(\mathbb{R}_+)$

R. Daher and M. El Hamma[1]

Department of Mathematics, Faculty of Science Aïn Chock
Casablanca, Morocco
e-mail : ra_daher@yahoo.fr (R. Daher)
m_elhamma@yahoo.fr (M. El Hamma)

Abstract : In this paper, we prove the estimates for the Bessel transform in $L_{p,\alpha}(\mathbb{R}_+)$ as applied to some classes of functions characterized by a generalized modulus of continuity.

Keywords : Bessel operator; Bessel transform; Bessel generalized translation; generalized modulus of continuity.
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1 Introduction and Preliminaries

Integral transforms and their inverses the Bessel transform are widely used to solve various problems in calculus, mechanics, mathematical physics, and computational mathematics (see [1–6]). The main aim of this paper is to generalise the Theorem 1 in [7]. $L_{p,\alpha}(\mathbb{R}_+)$, $1 < p \leq 2$, is the Banach space of measurable functions $f(t)$ on $\mathbb{R}_+$ with the finite norm

$$\|f\| = \|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{1/p}.$$

Everywhere α is a real number, $\alpha > -\frac{1}{2}$, let

$$\mathrm{B} = \frac{d^2}{dx^2} + \frac{(2\alpha+1)}{x}\frac{d}{dx}.$$

[1]Corresponding author.

be the Bessel differential operator. By $j_\alpha(x)$ we denote the Bessel normed function of the first kind, i.e,

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(x)}{x^\alpha},$$

where $J_\alpha(x)$ is the Bessel function of the first kind and $\Gamma(x)$ is the gamma-function (see [8]). The function $y = j_\alpha(x)$ satisfies the differential equation $\mathrm{B}y + y = 0$ with the condition initial $y(0) = 1$ and $y'(0) = 0$. The function $j_\alpha(x)$ is the infinitely differentiable and even. In $L_{p,\alpha}(\mathbb{R}_+)$, consider the Bessel generalized translation T_h (see [8])

$$\mathrm{T}_h f(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})} \int_0^\pi f(\sqrt{x^2+h^2-2xh\cos x}) \sin^{2\alpha} x dx.$$

The Bessel transform is defined by formula [8–10]

$$\widehat{f}(t) = \int_0^\infty f(x) j_\alpha(tx) x^{2\alpha+1} dx, \quad t \in \mathbb{R}^+$$

The inverse Bessel transform is given by the formula

$$f(x) = (2^\alpha \Gamma(\alpha+1))^{-2} \int_0^\infty \widehat{f}(t) j_\alpha(tx) t^{2\alpha+1} dt$$

We have the Young inequality

$$\|\widehat{f}\|_{q,\alpha} \leq K \|f\|_{p,\alpha}, \tag{1.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and K is positive constant.

We note the important property of the Bessel transform: If $f \in L_{p,\alpha}(\mathbb{R}_+)$

$$\widehat{\mathrm{B}f}(t) = (-t^2)\widehat{f}(t). \tag{1.2}$$

The following relation connect the Bessel generalized translation and the Bessel transform:

$$\widehat{\mathrm{T}_h f}(t) = j_\alpha(th)\widehat{f}(t). \tag{1.3}$$

The finite differences of the first and higher orders are defined as follows:

$$\Delta_h f(x) = \mathrm{T}_h f(x) - f(x) = (\mathrm{T}_h - \mathrm{I}) f(x),$$

where I is the identity operator in $L_{p,\alpha}(\mathbb{R}_+)$, and

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (\mathrm{T}_h - \mathrm{I})^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \mathrm{T}_h^i f(x), \tag{1.4}$$

where $\mathrm{T}_h^0 f(x) = f(x)$, $\mathrm{T}_h^i f(x) = \mathrm{T}_h(\mathrm{T}_h^{i-1} f(x))$, $(i = 1, 2, ..., k$ and $k = 1, 2, ...)$.

The k^{th} order generalized modulus of continuity of a function $f \in L_{p,\alpha}(\mathbb{R}_+)$ is defined by

$$\Omega_k(f,\delta) = \sup_{0<h\leq\delta} \|\Delta_h^k f(x)\|_{p,\alpha}.$$

Let $\mathrm{W}_{p,\phi}^{r,k}(\mathrm{B})$ denote the class of functions $f \in L_{p,\alpha}(\mathbb{R}_+)$ that have generalized derivatives in the sense of Levi (see [11]) satisfying the estimate

$$\Omega_k(\mathrm{B}^r f,\delta) = O(\phi(\delta^k)),\ \delta \longrightarrow 0$$

i.e

$$\mathrm{W}_{p,\phi}^{r,k}(\mathrm{B}) = \{f \in L_{p,\alpha}(\mathbb{R}_+)/\ \mathrm{B}^r f \in L_{p,\alpha}(\mathbb{R}_+)\ \mathit{and}\ \Omega_k(\mathrm{B}^r f,\delta) = O(\phi(\delta^k)),\ \delta \longrightarrow 0\}$$

where $\phi(t)$ is any nonnegative function given on $[0,\infty)$, and $\mathrm{B}^0 f = f$, $\mathrm{B}^r f = \mathrm{B}(\mathrm{B}^{r-1}f)$; $r = 1, 2, ...$

2 Main Results

In this section, we can prove an estimate for the integral

$$\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1} dt,$$

which are useful in applications.

Lemma 2.1. *For $f \in L_{p,\alpha}(\mathbb{R}_+)$*

$$\left(\int_0^\infty t^{2qr+2\alpha+1} |j_\alpha(th) - 1|^{qk} |\widehat{f}(t)|^q dt\right)^{1/q} \leq K\|\Delta_h^k \mathrm{B}^r f(x)\|_{p,\alpha}.$$

where K is positive constant.

Proof. From formula (1.2), we obtain

$$\widehat{\mathrm{B}^r f}(t) = (-1)^r t^{2r} \widehat{f}(t);\ r = 0, 1, ... \tag{2.1}$$

We use the formulas (1.3) and (2.1), we conclude

$$\widehat{\mathrm{T}_h^i \mathrm{B}^r f}(t) = (-1)^r j_\alpha^i(th) t^{2r} \widehat{f}(t),\ 1 \leq i \leq k. \tag{2.2}$$

From the definition of finite difference (1.4) and formula (2.2) the image $\Delta_h^k \mathrm{B}^r f(x)$ under the Bessel transform has the form

$$\widehat{\Delta_h^k \mathrm{B}^r f}(t) = (-1)^r (1 - j_\alpha(th))^k t^{2r} \widehat{f}(t).$$

By the inequality (1.1), we have the result. □

Theorem 2.2. *For functions $f(x) \in L_{p,\alpha}(\mathbb{R}_+)$ in the class $\mathrm{W}^{r,k}_{p,\phi}(\mathrm{B})$,*

$$\sup_{\mathrm{W}^{r,k}_{p,\phi}(\mathrm{B})}\left(\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt\right) = O\left(N^{-2rq}\phi^q\left(\left(\frac{C}{N}\right)^k\right)\right);$$

where $r = 0, 1, 2, ...; k = 1, 2, ..., C > 0$ is a fixed constant, and $\phi(t)$ is any nonnegative function defined on the interval $[0, \infty)$.

Proof. In the terms of $j_\alpha(x)$, we have (see [12])

$$1 - j_p(x) = O(1),\ x \geq 1, \tag{2.3}$$

$$1 - j_p(x) = O(x^2),\ 0 \leq x \leq 1, \tag{2.4}$$

$$\sqrt{hx}J_p(hx) = O(1),\ hx \geq 0. \tag{2.5}$$

Let $f \in \mathrm{W}^{r,k}_{p,\phi}(\mathrm{B})$. By the Hölder inequality, we have

$$\begin{aligned}
&\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt - \int_N^\infty j_\alpha(th)|\widehat{f}(t)|^q t^{2\alpha+1}dt = \int_N^\infty (1 - j_\alpha(th))|\widehat{f}(t)|^q t^{2\alpha+1}dt\\
&= \int_N^\infty (1 - j_\alpha(th))(|\widehat{f}(t)|t^{\frac{2\alpha+1}{q}})^q dt\\
&= \int_N^\infty (1 - j_\alpha(th))(|\widehat{f}(t)|t^{\frac{2\alpha+1}{q}})^{q-1/k}(|\widehat{f}(t)|t^{\frac{2\alpha+1}{q}})^{1/k}dt\\
&\leq \left(\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt\right)^{\frac{qk-1}{qk}}\left(\int_N^\infty |1 - j_\alpha(th)|^{qk}|\widehat{f}(t)|^q t^{2\alpha+1}dt\right)^{\frac{1}{qk}}\\
&= \left(\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt\right)^{\frac{qk-1}{qk}}\left(\int_N^\infty t^{-2qr}|1 - j_\alpha(th)|^{qk}|\widehat{f}(t)|^q t^{2qr+2\alpha+1}dt\right)^{\frac{1}{qk}}\\
&\leq N^{-2r/k}\left(\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt\right)^{\frac{qk-1}{qk}}\left(\int_N^\infty |1 - j_\alpha(th)|^{qk}|\widehat{f}(t)|^q t^{2qr+2\alpha+1}dt\right)^{\frac{1}{qk}}.
\end{aligned}$$

In view of Lemma 2.1, we conclude that

$$\int_N^\infty |1 - j_\alpha(th)|^{qk}|\widehat{f}(t)|^q t^{2qr+2\alpha+1}dt \leq K^q\|\Delta_h^k \mathrm{B}^r f(x)\|^q_{p,\alpha}.$$

Therefore,

$$\begin{aligned}
\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt &\leq \int_N^\infty j_\alpha(th)|\widehat{f}(t)|^q t^{2\alpha+1}dt\\
&\quad + K^q N^{-2r/k}\left(\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt\right)^{\frac{qk-1}{qk}}\|\Delta_h^k \mathrm{B}^r f(x)\|^{1/k}_{p,\alpha}.
\end{aligned}$$

From formula (2.5) and definition of $j_\alpha(x)$, we have

$$j_\alpha(th) = O((th)^{-\alpha-\frac{1}{2}}).$$

Then

$$\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt$$
$$= O\left(\int_N^\infty (th)^{-\alpha-\frac{1}{2}}|\widehat{f}(t)|^q t^{2\alpha+1}dt + K^q N^{-2r/k}\left(\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt\right)^{\frac{qk-1}{qk}} \|\Delta_h^k \mathrm{B}^r f(x)\|_{p,\alpha}^{1/k}\right)$$
$$= O\left((Nh)^{-\alpha-\frac{1}{2}}\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt + K^q N^{-2r/k}\left(\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt\right)^{\frac{qk-1}{qk}} \|\Delta_h^k \mathrm{B}^r f(x)\|_{p,\alpha}^{1/k}\right)$$

or

$$\left(1 - O(Nh)^{-\alpha-\frac{1}{2}}\right)\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt = O(N^{-2r/k})\left(\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt\right)^{\frac{qk-1}{qk}} \|\Delta_h^k \mathrm{B}^r f(x)\|_{p,\alpha}^{1/k}.$$

Setting $h = C/N$ in the last inequality and choosing $C > 0$ such that $1 - O(C^{-\alpha-\frac{1}{2}}) \geq \frac{1}{2}$, we obtain

$$\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt = O(N^{-2r/k})\left(\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt\right)^{\frac{qk-1}{qk}} \phi^{1/k}\left[\left(\frac{C}{N}\right)^k\right].$$

Then

$$\int_N^\infty |\widehat{f}(t)|^q t^{2\alpha+1}dt = O\left(N^{-2rq}\phi^q\left[\left(\frac{C}{N}\right)^k\right]\right),$$

which proves the theorem. □

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