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Coupled Coincidence Point Theorems for (φ, ψ)-Contractive Mixed Monotone Mapping in Partially Ordered Metric Spaces[1]

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Abstract : In this paper, we proved some coupled coincidence and coupled common fixed point theorem for such nonlinear contractive mappings in partially ordered complete metric spaces. The main results of this paper are generalization of the main results of Alotaibi and Alsulami (Coupled coincidence points for monotone operators in partially ordered metric spaces, Fixed Point Theory and Applications 2011, 2011:44).

Keywords : coupled coincidence point; coupled common fixed point; partially ordered metric spaces; mixed monotone.
2000 Mathematics Subject Classification : 47H10; 54H25.

1 Introduction

The existence of a fixed point for contraction type of mappings in partially ordered metric spaces has been considered recently by Ran and Reurings [1], Bhaskar and Lakshmikantham [2], Nieto and lopez [3, 4], Agarwal et al. [5], Lakshmikantham and Ćirić [6], Luong and Thuan [7], Berinde [8] and Alotaibi and Alsulami

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[9].

Bhaskar and Lakshmikantham [2] introduced notions of a mixed monotone mapping and a coupled fixed point and proved some coupled fixed point theorems for mixed monotone mapping. Later, Luong and Thuan , [7] proved some coupled fixed point theorems for mappings having a mixed monotone property in partially ordered metric spaces and generalization of the main results of Bhaskar and Lakshmikantham [2]. In 2012 Berinde [8] extended the coupled fixed point theorems for mixed monotone operator $F : X \times X \to X$ obtain by Bhaskar and Lakshmikantham [2] and Luong and Thuan [7].

Lakshmikantham and Ćirić [6] defined a mixed g-monotone mapping and prove coupled coincidence and coupled common fixed point theorems for such nonlinear contractive mappings in partially ordered complete metric spaces. In 2011 Alotaibi and Alsulami [9] proved the existence and uniqueness of coupled coincidence point involving a (φ, ψ)-contractive condition for a mappings having the mixed g-monotone property.

The purpose of this paper is to present some coupled coincidence point theorems for a mixed g-monotone mapping in a partially ordered metric space which are generalizations of the results of Alotaibi and Alsulami [9] and Berinde [8].

2 Preliminaries

Let $(X, \leqslant)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Consider on the product space $X \times X$ the following partial order :

$$for (x, y), (u, v) \in X \times X,\ (u, v) \leqslant (x, y) \Longleftrightarrow x \geqslant u, y \leqslant v. \tag{2.1}$$

Definition 2.1 ([2]). Let$(X, \leqslant)$ be a partially ordered set and $F : X \times X \to X$. We say F has the mixed monotone property if for any $x, y \in X$

$$x_1, x_2 \in X,\ x_1 \leqslant x_2 \quad implies \quad F(x_1, y) \leqslant F(x_2, y) \tag{2.2}$$

and

$$y_1, y_2 \in X,\ y_1 \leqslant y_2 \quad implies \quad F(x, y_1) \geqslant F(x, y_2). \tag{2.3}$$

Definition 2.2 ([2]). An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$. if $x = F(x, y)$ and $y = F(y, x)$.

Let Φ denote the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying

(i_φ) φ is continuous and non-decreasing,

(ii_φ) $\varphi(t) = 0$ if and only if $t = 0$ and,

(iii_φ) $\varphi(t + s) \leqslant \varphi(t) + \varphi(s)$ for all $t, s \in [0, \infty)$

and Ψ denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy

(i_ψ) $\lim\limits_{t\to r} \psi(t) > 0$ for all $r > 0$ and (ii_ψ) $\lim\limits_{t\to 0^+} \psi(t) = 0$.

Luong and Thuan [7] proved the following coupled fixed point theorems.

Theorem 2.3 ([7])**.** *Let $(X, \leqslant)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi(d(F(x, y), F(u, v))) \leqslant \frac{1}{2}\varphi(d(x, u) + d(y, v)) - \psi(\frac{d(x, u) + d(y, v)}{2}) \tag{2.4}$$

for all $x, y, z, u, v \in X$ for which $x \geqslant u$ and $y \leqslant v$.
Suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leqslant x$ for all n.

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \leqslant y_n$ for all n.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leqslant F(x_0, y_0) \quad \textit{and} \quad y_0 \geqslant F(y_0, x_0),$$

Then there exist $x, y \in X$ such that

$$x = F(x, y) \quad \textit{and} \quad y = F(y, x),$$

that is, F has a coupled fixed point in X.

The main result in Berinde [8] is the following coupled fixed point theorem which generalizes Theorem 2.3 in [7].

Theorem 2.4 ([8])**.** *Let $(X, \leqslant)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} &\varphi\left(\frac{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))}{2}\right) \\ &\quad \leqslant \varphi\left(\frac{d(x, u) + d(y, v)}{2}\right) - \psi\left(\frac{d(x, u) + d(y, v)}{2}\right) \end{aligned} \tag{2.5}$$

for all $x, y, z, u, v \in X$ for which $x \geqslant u$ and $y \leqslant v$.
suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leqslant x$ for all n.

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \leqslant y_n$ for all n.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leqslant F(x_0, y_0) \quad and \quad y_0 \geqslant F(y_0, x_0),$$

Then there exist $x, y \in X$ such that

$$x = F(x, y) \quad and \quad y = F(y, x),$$

that is, F has a coupled fixed point in X.

Lakshmikantham and Ćirić [6] introduced a mixed g-monotone mapping.

Definition 2.5 ([6])**.** Let$(X, \leqslant)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the g-mixed monotone property if for any $x, y \in X$

$$x_1, x_2 \in X,\ g(x_1) \leqslant g(x_2) \quad implies \quad F(x_1, y) \leqslant F(x_2, y) \tag{2.6}$$

and

$$y_1, y_2 \in X,\ g(y_1) \leqslant g(y_2) \quad implies \quad F(x, y_1) \geqslant F(x, y_2). \tag{2.7}$$

Definition 2.6 ([6])**.** An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$. and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Definition 2.7 ([6])**.** Let X be a non-empty set and $F : X \times X \to X$. and $g : X \to X$. We say F and g are commutative if $g(F(x, y)) = F(g(x), g(y))$ for all $x, y \in X$.

Theorem 2.8 ([6])**.** *Let$(X, \leqslant)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and $\lim\limits_{r \to t^+} \varphi(t) < t$ for each $t > 0$ and also suppose $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and*

$$d(F(x, y), F(u, v)) \quad \leqslant \quad \varphi\left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\right) \tag{2.8}$$

for all $x, y, u, v \in X$ for which $g(x) \leqslant g(u)$ and $g(y) \geqslant g(v)$.
Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leqslant x$ for all n.

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \leqslant y_n$ for all n.

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leqslant F(x_0, y_0) \quad \text{and} \quad g(y_0) \geqslant F(y_0, x_0),$$

Then there exist $x, y \in X$ such that

$$g(x) = F(x, y) \quad \text{and} \quad g(y) = F(y, x),$$

that is, F and g have a coupled coincidence.

Alotaibi and Alsulami [9] proved the existence and uniqueness of coupled coincidence point involving a (φ, ψ)-contractive condition for a mappings having the mixed g-monotone property.

Theorem 2.9 ([9]). *Let $(X, \leqslant)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\varphi \in \Phi$ and $\psi \in \Psi$ and also suppose $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and*

$$\begin{aligned} \varphi(d(F(x,y), F(u,v))) \quad &\leqslant \quad \frac{1}{2}\varphi(d(g(x), g(u)) + d(g(y), g(v))) \\ &\quad -\psi\left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\right) \end{aligned} \tag{2.9}$$

for all $x, y, u, v \in X$ for which $g(x) \geqslant g(u)$ and $g(y) \leqslant g(v)$.
Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leqslant x$ for all n.

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \leqslant y_n$ for all n.

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leqslant F(x_0, y_0) \quad \text{and} \quad g(y_0) \geqslant F(y_0, x_0),$$

Then there exist $x, y \in X$ such that

$$g(x) = F(x, y) \quad \text{and} \quad g(y) = F(y, x),$$

that is, F and g have a coupled coincidence.

3 Main Results

The first main result in this paper is the following coupled fixed point theorem which generalize Theorem 2.9 [9].

Theorem 3.1. *Let $(X, \leqslant)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\varphi \in \Phi$ and $\psi \in \Psi$ and also suppose $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and*

$$\begin{aligned}\varphi&\left(\frac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2}\right)\\&\leqslant \varphi\left(\frac{d(g(x),g(u))+d(g(y),g(v))}{2}\right)-\psi\left(\frac{d(g(x),g(u))+d(g(y),g(v))}{2}\right)\end{aligned} \tag{3.1}$$

for all $x, y, u, v \in X$ with $g(x) \geqslant g(u)$ and $g(y) \leqslant g(v)$.

Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) F is continuous or

(b) X has the following property:

$$(i)\ \ if\ a\ non-decreasing\ sequence\ \{x_n\} \to x,\ then\ x_n \leqslant x\ for\ all\ n. \tag{3.2}$$

$$(ii)\ if\ a\ non-increasing\ sequence\ \{y_n\} \to y,\ then\ y \leqslant y_n\ for\ all\ n. \tag{3.3}$$

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leqslant F(x_0, y_0) \quad and \quad g(y_0) \geqslant F(y_0, x_0) \tag{3.4}$$

or

$$g(x_0) \geqslant F(x_0, y_0) \quad and \quad g(y_0) \leqslant F(y_0, x_0), \tag{3.5}$$

Then there exist $x, y \in X$ such that

$g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \leqslant F(x_0, y_0)$ and $g(y_0) \geqslant F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that

$$g(x_1) = F(x_0, y_0) \quad and \quad g(y_1) = F(y_0, x_0).$$

Again from $F(X \times X) \subseteq g(X)$ we can choose $x_2, y_2 \in X$ such that

$$g(x_2) = F(x_1, y_1) \quad and \quad g(y_2) = F(y_1, x_1).$$

Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \quad and \quad g(y_{n+1}) = F(y_n, x_n) \ for\, all\ n \geqslant 0. \tag{3.6}$$

Since F is g-mixed monotone, we have

$$(g(x_0), g(y_0)) \leqslant (F(x_0, y_0), F(y_0, x_0)) = (g(x_1), g(y_1)).$$

Assume

$$(g(x_{n-1}), g(y_{n-1})) \leqslant (g(x_n), g(y_n)).$$

by the mathematical induction, we get

$$g(x_n) \leqslant g(x_{n+1}) \quad and \quad g(y_n) \geqslant g(y_{n+1}) \ for\, all\ n \geqslant 0.$$

Consider the sequence of nonnegative real number $\{\delta_n\}_{n=1}^{\infty}$ given by

$$\delta_{n+1} = \frac{d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n))}{2}, \ n \geqslant 0.$$

By taking $x := x_n, y := y_n, u := x_{n-1}, v := y_{n-1}$ in (3.1), we have

$$\begin{aligned} & \frac{d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) + d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))}{2} \\ = \ & \frac{d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n))}{2} \\ = \ & \delta_{n+1}. \end{aligned}$$

While the right hand side of (3.1) will be equal to

$$\begin{aligned} & \varphi\left(\frac{d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1}))}{2}\right) \\ & \quad - \psi\left(\frac{d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1}))}{2}\right) \\ = \ & \varphi(\delta_n) - \psi(\delta_n). \end{aligned}$$

Therefore, the sequence $\{\delta_n\}_{n=1}^{\infty}$ satisfies

$$\varphi(\delta_{n+1}) \leqslant \varphi(\delta_n) - \psi(\delta_n) \leqslant \varphi(\delta_n). \ for\ all\ n \geqslant 0. \tag{3.7}$$

From (3.7) and (i_φ) it follows that the sequence $\{\delta_n\}_{n=1}^{\infty}$ is non-increasing. Therefore, there exists some $\delta \geqslant 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \left[\frac{d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1}))}{2}\right] = \delta. \tag{3.8}$$

We shall prove that $\delta = 0$. Assume, to the contrary, that $\delta > 0$. Then by letting $n \to \infty$ in (3.7), by property of (i_φ) and (i_ψ) we have

$$\varphi(\delta) = \lim_{n\to\infty} \varphi(\delta_{n+1}) \leqslant \lim_{n\to\infty} \varphi(\delta_n) - \lim_{n\to\infty} \psi(\delta_n) = \varphi(\delta) - \lim_{\delta_n\to\delta^+} \psi(\delta_n) < \varphi(\delta),$$

a contradiction. Thus $\delta = 0$ and hence

$$\lim_{n\to\infty} \delta_n = 0. \tag{3.9}$$

We now prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequence in (X, d). Suppose, to the contrary, that at least one of $\{g(x_n)\}$ and $\{g(y_n)\}$ is not Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{g(x_{m(k)})\}$ and $\{g(x_{n(k)})\}$ of $\{g(x_n)\}$, $\{g(y_{m(k)})\}$ and $\{g(y_{n(k)})\}$ of $\{g(y_n)\}$ with $n(k) > m(k) \geqslant K$ such that

$$\frac{1}{2}\left[d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)}))\right] \geqslant \varepsilon, \quad k \in \{1, 2, ...\}. \tag{3.10}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that is the smallest integer with $n(k) > m(k) \geqslant K$ and satisfying (3.10). Then

$$\frac{1}{2}\left[d(g(x_{n(k)-1}), g(x_{m(k)})) + d(g(y_{n(k)-1}), g(y_{m(k)}))\right] < \varepsilon. \tag{3.11}$$

Using (3.10) and (3.11) and the triangle inequality, we have

$$\begin{aligned}
\varepsilon \;&\leqslant\; r_k := \left[\frac{d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)}))}{2}\right] \\
&\leqslant\; \frac{d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(x_{n(k)-1}), g(x_{m(k)}))}{2} \\
&\quad + \frac{d(g(y_{n(k)}), g(y_{n(k)-1})) + d(g(y_{n(k)-1}), g(y_{m(k)}))}{2} \\
&\leqslant\; \frac{d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(y_{n(k)}), g(y_{n(k)-1})}{2} + \varepsilon.
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.9), we get

$$\varepsilon \leqslant \lim_{k\to\infty} r_k \leqslant \lim_{k\to\infty}\left[\frac{d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(y_{n(k)}), g(y_{n(k)-1})}{2} + \varepsilon\right] = \varepsilon,$$

that is,

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty}\left[\frac{d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)}))}{2}\right] = \varepsilon. \tag{3.12}$$

By the triangle inequality, we have

$$\begin{aligned}
d(g(x_{n(k)}), g(x_{m(k)})) \leqslant d(g(x_{n(k)}), g(x_{n(k)+1}) + d(g(x_{n(k)+1}), g(x_{m(k)+1}) \\
+ d(g(x_{m(k)+1}), g(x_{m(k)})),
\end{aligned}$$

similarly

$$d(g(y_{n(k)}), g(y_{m(k)})) \leqslant d(g(y_{n(k)}), g(y_{n(k)+1}) + d(g(y_{n(k)+1}), g(y_{m(k)+1})$$

$$+ d(g(y_{m(k)+1}), g(y_{m(k)})).$$

This show that

$$\begin{aligned} r_k \quad := \quad & \frac{d(g(x_{n(k)}), g(x_{m(k)}) + d(g(y_{n(k)}), g(y_{m(k)}))}{2} \\ \leqslant \quad & \frac{d(g(x_{n(k)}), g(x_{n(k)+1}) + d(g(x_{n(k)+1}), g(x_{m(k)+1}) + d(g(x_{m(k)+1}), g(x_{m(k)}))}{2} \\ & + \frac{d(g(y_{n(k)}), g(y_{n(k)+1}) + d(g(y_{n(k)+1}), g(y_{m(k)+1}) + d(g(y_{m(k)+1}), g(y_{m(k)}))}{2} \\ = \quad & \delta_{n(k)} + \delta_{m(k)} + \frac{d(g(x_{n(k)+1}), g(x_{m(k)+1}) + d(g(y_{n(k)+1}), g(y_{m(k)+1})}{2}. \quad (3.13) \end{aligned}$$

Since $n(k) > m(k)$, we have $g(x_{n(k)}) \geqslant g(x_{m(k)})$ and $g(y_{n(k)}) \leqslant g(y_{m(k)})$ and hence we can use (3.1) with $x := x_{n(k)}$, $y := y_{n(k)}$, $u := x_{m(k)}$, $v := y_{m(k)}$, to obtain

$$\begin{aligned} & \varphi\left(\frac{d(g(x_{n(k)+1}), g(x_{m(k)+1}) + d(g(y_{n(k)+1}), g(y_{m(k)+1})}{2}\right) \\ = \quad & \varphi\left(\frac{d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)})) + d(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))}{2}\right) \\ \leqslant \quad & \varphi\left(\frac{d(g(x_{n(k)}), g(x_{m(k)}) + d(g(y_{n(k)}), g(y_{m(k)})}{2}\right) \\ & -\psi\left(\frac{d(g(x_{n(k)}), g(x_{m(k)}) + d(g(y_{n(k)}), g(y_{m(k)})}{2}\right) \\ = \quad & \varphi(r_k) - \psi(r_k). \qquad (3.14) \end{aligned}$$

On the other hand, by (3.13) and using property (iii_φ), we get

$$\begin{aligned} & \varphi(r_k) \leqslant \varphi(\delta_{n(k)} + \delta_{m(k)}) \\ & \qquad + \varphi\left(\frac{d(g(x_{n(k)+1}), g(x_{m(k)+1}) + d(g(y_{n(k)+1}), g(y_{m(k)+1})}{2}\right). \quad (3.15) \end{aligned}$$

By (3.14), we have

$$\varphi(r_k) \leqslant \varphi(\delta_{n(k)} + \delta_{m(k)}) + \varphi(r_k) - \psi(r_k). \qquad (3.16)$$

Letting $k \to \infty$ in (3.16) and using (3.9) and (3.12) and property of φ and ψ, we have

$$\begin{aligned} \varphi(\varepsilon) = \lim_{k\to\infty} \varphi(r_k) \quad \leqslant \quad & \lim_{k\to\infty} \left[\varphi(\delta_{n(k)} + \delta_{m(k)}) + \varphi(r_k) - \psi(r_k)\right] \\ = \quad & \varphi(\lim_{k\to\infty}(\delta_{n(k)} + \delta_{m(k)})) + \varphi(\lim_{k\to\infty} r_k) - \lim_{k\to\infty} \psi(r_k) \\ = \quad & \varphi(0) + \varphi(\varepsilon) - \lim_{r_k\to\varepsilon^+} \psi(r_k) < \varphi(\varepsilon), \end{aligned}$$

which is a contradiction. This shows that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequence. Since X is a complete matric space, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x \quad and \quad \lim_{n\to\infty} g(y_n) = y. \tag{3.17}$$

From (3.17) and continuity of g,

$$\lim_{n\to\infty} g(g(x_n)) = g(x) \quad and \quad \lim_{n\to\infty} g(g(y_n)) = g(y). \tag{3.18}$$

From (2.6) and commutativity of F and g,

$$g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g(x_n), g(y_n)), \tag{3.19}$$

and

$$g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n)). \tag{3.20}$$

We now show that $F(x, y) = g(x)$ and $F(y, x) = g(y)$.
Suppose that assumption (a) holds.
Taking the limit as $n \to \infty$ in (3.19) and (3.20), by (3.17),(3.18) continuity and commutativity of F and g, we get

$$\begin{aligned} g(x) &= g(\lim_{n\to\infty} g(x_{n+1})) = \lim_{k\to\infty} g(g(x_{n+1})) = \lim_{k\to\infty} g(F(x_n, y_n)) \\ &= \lim_{k\to\infty} F(g(x_n), g(y_n)) = F(\lim_{k\to\infty}(g(x_n), g(y_n))) = F(x, y) \end{aligned}$$

and

$$\begin{aligned} g(y) &= g(\lim_{n\to\infty} g(y_{n+1})) = \lim_{k\to\infty} g(g(y_{n+1})) = \lim_{k\to\infty} g(F(y_n, x_n)) \\ &= \lim_{k\to\infty} F(g(y_n), g(x_n)) = F(\lim_{k\to\infty}(g(y_n), g(x_n))) = F(y, x). \end{aligned}$$

Thus we prove that $F(x, y) = g(x)$ and $F(y, x) = g(y)$. Suppose that the assumption (b) holds. Since $\{g(x_n)\}$ is a non-decreasing and $g(x_n) \to x$, and $\{g(y_n)\}$ is a non-increasing and $g(y_n) \to y$, from (3.2) and (3.3) we have $g(x_n) \leqslant x$ and $g(y_n) \geqslant y$ for all n. Then

$$\begin{aligned} d(g(x), F(x, y)) &\leqslant d(g(x), g(g(x_{n+1}))) + d(g(g(x_{n+1})), F(x, y)) \\ &= d(g(x), g(g(x_{n+1}))) + d(F(g(x_n), g(y_n)), F(x, y)) \end{aligned}$$

and

$$\begin{aligned} d(g(y), F(y, x)) &\leqslant d(g(y), g(g(y_{n+1}))) + d(g(g(y_{n+1})), F(y, x)) \\ &= d(g(y), g(g(y_{n+1}))) + d(F(g(y_n), g(x_n)), F(y, x)). \end{aligned}$$

So

$$d(g(x), F(x, y)) - d(g(x), g(g(x_{n+1}))) \leqslant d(F(g(x_n), g(y_n)), F(x, y))$$

and

$$d(g(y), F(y,x)) - d(g(y), g(g(y_{n+1}))) \leqslant d(F(g(y_n), g(x_n)), F(y,x)).$$

Hence

$$\begin{aligned} &\frac{d(g(x), F(x,y)) - d(g(x), g(g(x_{n+1}))) + d(g(y), F(y,x)) - d(g(y), g(g(y_{n+1})))}{2} \\ \leqslant\ &\frac{d(F(g(x_n), g(y_n)), F(x,y)) + d(F(g(y_n), g(x_n)), F(y,x))}{2}. \end{aligned}$$

By φ is non-decreasing and (3.1), therefore

$$\begin{aligned} &\varphi\left(\frac{d(g(x), F(x,y)) - d(g(x), g(g(x_{n+1}))) + d(g(y), F(y,x))}{2} - \right. \\ &\qquad \left. \frac{d(g(y), g(g(y_{n+1})))}{2}\right) \\ \leqslant\ &\varphi\left(\frac{d(F(g(x_n), g(y_n)), F(x,y)) + d(F(g(y_n), g(x_n)), F(y,x))}{2}\right) \\ \leqslant\ &\varphi\left(\frac{d(g(g(x_n)), g(x)) + d(g(g(y_n)), g(y))}{2}\right) \\ &\quad - \psi\left(\frac{d(g(g(x_n)), g(x)) + d(g(g(y_n)), g(y))}{2}\right). \end{aligned}$$

using the property of ψ, we get

$$\begin{aligned} &\varphi\left(\frac{d(g(x), F(x,y)) - d(g(x), g(g(x_{n+1}))) + d(g(y), F(y,x))}{2} - \right. \\ &\qquad \left. \frac{d(g(y), g(g(y_{n+1})))}{2}\right) \\ \leqslant\ &\varphi\left(\frac{d(g(g(x_n)), g(x)) + d(g(g(y_n)), g(y))}{2}\right). \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain

$$\varphi\left(\frac{d(g(x), F(x,y)) + d(g(y), F(y,x))}{2}\right) \leqslant \varphi(0) = 0$$

which shows, by (ii_φ), that $F(x,y) = g(x)$ *and* $F(y,x) = g(y)$. $\square$

Example 3.2. *Let* $X = \mathbb{R}, d(x,y) = |x-y|$ *and* $F : X \times X \to X$ *be defined by*

$$F(x,y) = \frac{x-2y}{8}, \quad (x,y) \in X^2.$$

Then F *is mixed monotone and satisfies condition* (3.1) *but not satisfy condition* (2.9).

We well prove that (2.9) *is not satisfied when* $g(x) = \frac{x}{2}$. *Assume, to the contrary, that there exist* $\varphi \in \Phi$ *and* $\psi \in \Psi$ *such that* (2.9) *holds. This means that*

$$\begin{aligned}
\varphi\left(\left|\frac{x-2y}{8} - \frac{u-2v}{8}\right|\right) &\leqslant \frac{1}{2}\varphi\left(|g(x)-g(u)|) + |g(y)-g(v)|\right) \\
&\quad -\psi\left(\frac{(|g(x)-g(u)|) + |g(y)-g(v)|)}{2}\right) \\
&= \frac{1}{2}\varphi\left(\frac{|x-u|+|y-v|}{2}\right) \\
&\quad -\psi\left(\frac{|x-u|+|y-v|}{4}\right),
\end{aligned}$$

$g(x) \geqslant g(u)$ *and* $g(y) \leqslant g(v)$.
By which, for $g(x) = g(u)$ *and* $g(y) \neq g(v)$. *in the previous inequality and denote* $r = \left|\frac{y-v}{4}\right|$. *By* (i_φ), *we get*

$$\varphi(r) \leqslant \frac{1}{2}\varphi(2r) - \psi(r) = \varphi(r) - \psi(r),$$

for all $r > 0$, $\psi(r) \leqslant 0$, *this is,* $\psi(r) = 0$, *which contradicts* (i_ψ). *It show that* F *does not satisfy* (2.9).
Now we prove that (3.1) *holds. Indeed, since* $g(x) \geqslant g(u)$ *and* $g(y) \leqslant g(v)$ *we have*

$$\left|\frac{x-2y}{8} - \frac{u-2v}{8}\right| \leqslant \left|\frac{x-u}{8}\right| + \left|\frac{y-v}{4}\right|, \quad g(x) \geqslant g(u), g(y) \leqslant g(v),$$

and

$$\left|\frac{y-2x}{8} - \frac{v-2u}{8}\right| \leqslant \left|\frac{y-v}{8}\right| + \left|\frac{x-u}{4}\right|, \quad g(x) \geqslant g(u), g(y) \leqslant g(v),$$

by summing up the two inequalities above we get exactly (3.1) *with* $\varphi(t) = \frac{t}{2}$ *and* $\psi(t) = \frac{t}{8}$. *Note also that* $x_0 = -4, y_0 = 6$ *satisfy* (3.2). *So by our theorem* 3.1 *we obtain that* F *has a (unique) couple common fixed point* $(0,0)$ *but theorem* 2.9 *in* [9] *does not apply to* F *in this example.*

Corollary 3.3. *Let* $(X, \leqslant)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Assume there is a function* $\varphi \in \Phi$ *and* $\psi \in \Psi$ *and also suppose* $F : X \times X \to X$ *and* $g : X \to X$ *are such that* F *has the mixed* g*-monotone property and*

$$\begin{aligned}
&d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \\
&\leqslant d(g(x), g(u)) + d(g(y), g(v)) - 2\psi\left(\frac{d(g(x),g(u)) + d(g(y),g(v))}{2}\right)
\end{aligned} \tag{3.21}$$

for all $x, y, u, v \in X$ *with* $g(x) \geqslant g(u)$ *and* $g(y) \leqslant g(v)$.

Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leqslant x$ for all n.

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \leqslant y_n$ for all n.

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leqslant F(x_0, y_0) \quad and \quad g(y_0) \geqslant F(y_0, x_0)$$

or

$$g(x_0) \geqslant F(x_0, y_0) \quad and \quad g(y_0) \leqslant F(y_0, x_0),$$

Then there exist $x, y \in X$ such that

$g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof. In theorem 3.1, taking $\varphi(t) = \frac{t}{2}$ and $\psi_1(t) = \frac{\psi}{2}$ and multiple by 4, we get Corollary 3.3 □

Theorem 3.4. *In addition to the hypotheses of theorem 3.1, suppose that for every $(x, y), (x^*, y^*) \in X \times X$ there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Then F and g have a unique couple common fixed point, that is, there exist a unique $(x, y) \in X \times X$ such that*
$x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Proof. From Theorem 3.1, the set of couple coincidence is non-empty. We shall show that if (x, y) and (x^*, y^*) are couple coincidence points, this is,
if $g(x) = F(x, y), g(y) = F(y, x)$ and $g(x^*) = F(x^*, y^*), g(y^*) = F(y^*, x^*)$, then

$$g(x) = g(x^*) \quad and \quad g(y) = g(y^*). \tag{3.22}$$

By assumption there is $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$.
Put $u_0 = u, v_0 = v$ and choose $u_1, u_2 \in X$. So that

$$g(u_1) = F(u_0, v_0) \quad and \quad g(v_1) = g(v_0, u_0).$$

Then similarly as in Theorem 3.1, we can inductively define sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ such that

$$g(u_{n+1}) = F(u_n, v_n) \quad and \quad g(v_{n+1}) = g(v_n, u_n), n \geqslant 0.$$

Since $(F(x,y),F(y,x)) = (g(x),g(y))$ and $(F(u,v),F(v,u)) = (g(u_1),g(v_1))$ are comparable,then $g(x) \leqslant g(u_1)$ and $g(y) \geqslant g(v_1)$. It is easy to show that $(g(x),g(y))$ and $(g(u_n),g(v_n))$ are comparable, that is,

$$g(x) \leqslant g(u_n) \quad and \quad g(y) \geqslant g(v_n) \ for\ all\ n \geqslant 1.$$

Thus from (3.1), we have

$$\begin{aligned}
& \varphi\left(\frac{d(g(x),g(u_{n+1})) + d(g(y),g(v_{n+1}))}{2}\right) \\
= \quad & \varphi\left(\frac{d(F(x,y),F(u_n,v_n)) + d(F(y,x),F(v_n,u_n))}{2}\right) \\
\leqslant \quad & \varphi\left(\frac{d(g(x),g(u_n)) + d(g(y),g(v_n))}{2}\right) - \psi\left(\frac{d(g(x),g(u_n)) + d(g(y),g(v_n))}{2}\right).
\end{aligned} \tag{3.23}$$

Which by the fact that $\psi \geqslant 0$, implies

$$\varphi\left(\frac{d(g(x),g(u_{n+1})) + d(g(y),g(v_{n+1}))}{2}\right) \leqslant \varphi\left(\frac{d(g(x),g(u_n)) + d(g(y),g(v_n))}{2}\right).$$

Thus, by the monotonicity of φ, we obtain that sequence $\{h_n\}$ defined by

$$h_n = \frac{d(g(x),g(u_n)) + d(g(y),g(v_n))}{2}, \qquad n \geqslant 0,$$

is non-increasing. Hence, there exists $\alpha \geqslant 0$ such that

$$\lim_{n\to\infty} h_n = \lim_{n\to\infty}\left[\frac{d(g(x),g(u_n)) + d(g(y),g(v_n))}{2}\right] = \alpha.$$

Suppose, to the contrary, that $\alpha > 0$.
Letting $n \to \infty$ in (3.23), we get

$$\varphi(\alpha) \leqslant \varphi(\alpha) - \lim_{n\to\infty} \psi(h_n) = \varphi(\alpha) - \lim_{h_n\to\alpha^+} \psi(h_n) < \varphi(\alpha).$$

A contradiction. Thus $\alpha = 0$, that is,

$$\lim_{n\to\infty} h_n = \lim_{n\to\infty}\left[\frac{d(g(x),g(u_n)) + d(g(y),g(v_n))}{2}\right] = 0.$$

which implies

$$\lim_{n\to\infty} d(g(x),g(u_n)) = \lim_{n\to\infty} d(g(y),g(v_n)) = 0.$$

Similarly, we obtain

$$\lim_{n\to\infty} d(g(x^*),g(u_n)) = \lim_{n\to\infty} d(g(y^*),g(v_n)) = 0.$$

Hence $g(x) = g(x^*)$ and $g(y) = g(y^*)$.

Since $g(x) = F(x, y)$ and $g(y) = F(y, x)$, by commutativity of F and g, we have

$$g(g(x)) = g(F(x, y)) = F(g(x), g(y)) \quad and \quad g(g(y)) = g(F(y, x)) = F(g(y), g(x)). \tag{3.24}$$

Denote $g(x) = z$ and $g(y) = w$. Then from (3.24),

$$g(z) = F(z, w) \quad and \quad g(w) = F(w, z). \tag{3.25}$$

Thus (z, w) is a coupled coincidence point. Then from (3.22) with $x^* = z$ and $y^* = w$. It follows $g(z) = g(x)$ and $g(w) = g(y)$, that is,

$$g(z) = z \quad and \quad g(w) = w. \tag{3.26}$$

From (3.25) and (3.26),

$$z = g(z) = F(z, w) \quad and \quad w = g(w) = F(w, z).$$

Therefore, (z, w) is a coupled common fixed point of F and g. To prove the uniqueness, assume that (p, q) is another coupled common fixed point. Then by (3.22) we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. $\square$

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