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On Some Properties of Fractional Calculus Operators Associated with Generalized Mittag-Leffler Function

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Abstract : This paper deals with the study of an entire function of the form

$$E^{\gamma,q}_{\alpha,\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\delta)_n},$$

where

$$\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0 \text{ and } \mathrm{q} \in (0,1) \bigcup \mathrm{N}.$$

Various properties that exist between $E^{\gamma,q}_{\alpha,\beta,\delta}(z)$ and Rieman Liouville fractional integrals and derivatives are investigated. It has been shown that the fractional integration and differentiation operators transform such functions with power multipliers into the function of the same form. Some of the results given earlier by Khan and Ahmed follow as special cases.

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1 Introduction

In 1903, the Swedish mathematician Mittag-Leffler [1] introduced the function

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \tag{1.1}$$

where z is a complex variable and Γ is a Gamma function $\alpha\geq 0$. The Mittag-Leffler function is a direct generalisation of exponential function to which it reduces for $\alpha=1$. For $0<\alpha<1$ it interpolates between the pure exponential and hypergeometric function $\frac{1}{1-z}$. Its importance is realized during the last two decades due to its involvement in the problems of physics, chemistry, biology, engineering and applied sciences. Mittag-Leffler function naturally occurs as the solution of fractional order differential or fractional order integral equation. The generalization of $E_\alpha(z)$ was studied by Wiman [2] in 1905 and he defined the function as

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ \ (\alpha,\beta\in\mathbb{C};\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0)\,. \tag{1.2}$$

In 1971, Prabhakar [3] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form of

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!},\ \ (\alpha,\beta,\gamma,\in\mathbb{C};\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0)\,, \tag{1.3}$$

where $(\gamma)_n$ is the Pochhammer symbol (Rainville [4]).

$$(\gamma)_0=1\ ,(\gamma)_n=\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1).$$

In 2007, Shukla and Prajapati [5] introduced the function $E^{\gamma,q}_{\alpha,\beta}(z)$ which is defined for $\alpha,\beta,\gamma\in\mathbb{C}$; $\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0$ and $q\in(0,1)\bigcup N$ as

$$E^{\gamma,q}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)n!},\ \ (\alpha,\beta,\gamma,\in\mathbb{C};\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0) \tag{1.4}$$

In 2009, Salim [6] introduced the function the function $E^{\gamma,\delta}_{\alpha,\beta}(z)$ which is defined for $\alpha,\beta,\gamma,\delta\in\mathbb{C};\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0,\mathrm{Re}(\delta)>0$ as

$$E^{\gamma,\delta}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)(\delta)_n}, \tag{1.5}$$

Recently, Khan and Ahmed [7] defined and studied an entire function of the form

$$E^{\gamma,q}_{\alpha,\beta,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\delta)_n}, \tag{1.6}$$

where

$$\min\left(\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\gamma),\mathrm{Re}(\delta)>0\right) \text{ and } \mathrm{q}\in(0,1)\bigcup \mathrm{N}. \tag{1.7}$$

It is an entire function of order $[\mathrm{Re}(\alpha)]^{-1}$. The eq.(1.6) is a generalization of all above functions defined by the eqs. (1.1)-(1.5). Note that

$$E^{\gamma,1}_{\alpha,\beta,\delta}(z)=E^{\gamma,\delta}_{\alpha,\beta}(z),E^{\gamma,q}_{\alpha,\beta,1}(z)=E^{\gamma,q}_{\alpha,\beta,}(z),E^{\gamma,1}_{\alpha,\beta,1}(z)=E^{\gamma}_{\alpha,\beta,}(z),$$

$$E^{1,1}_{\alpha,\beta,1}(z)=E_{\alpha,\beta}(z),E^{1,1}_{\alpha,1,1}(z)=E_{\alpha}(z),E^{1,1}_{1,1,1}(z)=\exp(z).$$

2 Preliminaries

This section contain some basic results and formulas which will be used in our main results. The fractional calculus operators are defined by (see Samko, Kilbas and Marichev [8, Sect. 2]) for $\nu>0$:

$$I^{\nu}_{0+}f=\frac{1}{\Gamma(\nu)}\int_0^x\frac{f(t)}{(x-t)^{1-\nu}}dt, \tag{2.1}$$

$$I^{\nu}_{0-}f=\frac{1}{\Gamma(\nu)}\int_x^{\infty}\frac{f(t)}{(x-t)^{1-\nu}}dt, \tag{2.2}$$

$$\begin{aligned}\left(D^{\nu}_{0+}f\right)(x)&=\left(\frac{d}{dx}\right)^{[\nu]+}\left(I^{1-\nu}_{0+}f\right)(x)\\&=\frac{1}{\Gamma(1-\{\nu\})}\left(\frac{d}{dx}\right)^{[\nu]+1}\int_0^x\frac{f(t)}{(x-t)^{\{\nu\}}}dt,\end{aligned} \tag{2.3}$$

$$\begin{aligned}\left(D^{\nu}_{-}f\right)(x)&=\left(\frac{d}{dx}\right)^{[\nu]+1}\left(I^{1-\nu}_{-}f\right)(x)\\&=\frac{1}{\Gamma(1-\{\nu\})}\left(\frac{d}{dx}\right)^{[\nu]+1}\int_0^x\frac{f(t)}{(x-t)^{\{\nu\}}}dt,\end{aligned} \tag{2.4}$$

where $[\nu]$ means the maximal integer not exceeding ν and $\{\nu\}$ is the fractional part of ν. The object of this paper is to derive the relations that exist between the generalized Mittag-Leffler function defined by (1.6) and the left- and right-sided operators of Riemann-Liouville fractional calculus operators defined by (2.1)-(2.4). The results derived in this papers are believed to be new.

Lemma 2.1. *If* $\alpha,\beta,\gamma,\delta\in\mathbb{C};\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0,\mathrm{Re}(\delta)>0$ and $\mathrm{q}\in(0,1)\bigcup\mathrm{N}$*, then*

$$E^{\gamma,q}_{\alpha,\beta,\delta}(ax^{\alpha})=\beta E^{\gamma,q}_{\alpha,\beta+1,\delta}(ax^{\alpha})+x\frac{d}{dx}E^{\gamma,q}_{\alpha,\beta+1,\delta}(ax^{\alpha}) \tag{2.5}$$

3 Main Results

In this section we derive several interesting properties of the generalized Mittag-Leffler functions $E^{\gamma,q}_{\alpha,\beta,\delta}(z)$ defined by (1.6) associated Riemann-Liouville fractional integrals and derivatives.

Theorem 3.1. *Let* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0$ and $\mathrm{q} \in (0,1) \bigcup \mathrm{N}$ and $\mathrm{a} \in \mathrm{R}$. *Let* I^{ν}_{0+} *be the left-sided operator of Rimann-Liouville fractional integral (2.1). Then there holds the formula*

$$\left(I^{\nu}_{0+}[t^{\beta-1}E^{\gamma,q}_{\alpha,\beta,\delta}(at^{\alpha})]\right)(x) = x^{\beta+\nu-1}E^{\gamma,q}_{\alpha,\beta+\nu,\delta}(ax^{\alpha}), \tag{3.1}$$

Proof. By virtue of (1.6) and (2.1) we have

$$\begin{aligned} K &\equiv \left(I^{\nu}_{0+}[t^{\beta-1}E^{\gamma,q}_{\alpha,\beta,\delta}(at^{\alpha})]\right)(x) \\ &= \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\delta)_n}a^n t^{\alpha n+\beta-1}dt. \end{aligned}$$

Interchanging the order of integration and summation and evaluating the inner integral by beta function formula, we have

$$K = x^{\nu+\beta-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\nu+\alpha n+\beta)(\delta)_n}(ax^{\alpha})^n$$

or

$$K = x^{\nu+\beta-1}E^{\gamma,q}_{\alpha,\beta+\nu,\delta}(ax^{\alpha}).$$

□

Corollary 3.2. *For* $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\nu) > 0$ and $\mathrm{a} \in \mathrm{R}$*, then there holds the formula*

$$\left(I^{\nu}_{0+}[t^{\beta-1}E^{\gamma,q}_{\alpha,\beta}(at^{\alpha})]\right)(x) = x^{\nu+\beta-1}E^{\gamma,q}_{\alpha,\beta+\nu}(ax^{\alpha}) \tag{3.2}$$

Corollary 3.3. *For* $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0, \mathrm{Re}(\nu) > 0$ *and* $a \in R$*, then there holds the formula*

$$\left(I^{\nu}_{0+}[t^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta}(at^{\alpha})]\right)(x) = x^{\nu+\beta-1}E^{\gamma,\delta}_{\alpha,\beta+\nu}(ax^{\alpha}) \tag{3.3}$$

Remark 3.4. *The formula (3.3) is a known result of Khan and Ahmed [9, (2.1)]*

Theorem 3.1 and Lemma 2.1 imply

Theorem 3.5. *Let* $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0$ and $\mathrm{q} \in (0,1) \bigcup \mathrm{N}$ and $\mathrm{a} \in \mathrm{R}$. *Let* I_{0+}^{ν} *be the left-sided operator of Rimann-Liouville fractional integral (2.1). Then there holds the formula*

$$\left(I_{0+}^{\nu}[t^{\beta-1}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{\alpha})]\right)(x) = x^{\nu+\beta-1}[(\beta+\nu))E_{\alpha,\beta+\nu+1,\delta}^{\gamma,q}(ax^{\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta+\nu+1,\delta}^{\gamma,q}(ax^{\alpha})]. \tag{3.4}$$

Corollary 3.6. *For* $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\nu) > 0, \mathrm{q} \in \mathrm{N}$ and $\mathrm{a} \in \mathrm{R}$. *Then there holds the formula*

$$\left(I_{0+}^{\nu}[t^{\beta-1}E_{\alpha,\beta}^{\gamma,q}(at^{\alpha})]\right)(x) = x^{\nu+\beta-1}[(\beta+\nu))E_{\alpha,\beta+\nu+1}^{\gamma,q}(ax^{\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta+\nu+1}^{\gamma,q}(ax^{\alpha})]. \tag{3.5}$$

Corollary 3.7. *For* $\mathrm{Re}(\alpha) > 0$ $\mathrm{Re}(\beta) > 0$ $\mathrm{Re}(\gamma) > 0$ $\mathrm{Re}(\gamma) > 0$ $\mathrm{Re}(\nu) > 0$ and $\mathrm{a} \in \mathrm{R}$. *Then there holds the formula*

$$\left(I_{0+}^{\nu}[t^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta}(at^{\alpha})]\right)(x) = x^{\nu+\beta-1}[(\beta+\nu))E_{\alpha,\beta+\nu+1}^{\gamma,\delta}(ax^{\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta+\nu+1}^{\gamma,\delta}(ax^{\alpha})]. \tag{3.6}$$

Remark 3.8. *The formula (3.6) is a known result of Khan and Ahmed [9, (2.5)].*

Theorem 3.9. *Let* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0, \mathrm{q} \in (0,1) \bigcup \mathrm{N}$, and $\mathrm{a} \in \mathrm{R}$. *Let* I_{-}^{ν} *be the right-sided operator of Rimann-Liouville fractional integral (2.2). Then there holds the formula*

$$\left(I_{-}^{\nu}[t^{-\nu-\beta}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{-\alpha})]\right)(x) = x^{-\beta}E_{\alpha,\beta+\nu,\delta}^{\gamma,q}(ax^{-\alpha}). \tag{3.7}$$

Proof. By virtue of (1.6) and (2.2) we have

$$\begin{aligned} K &\equiv \left(I_{-}^{\nu}[t^{-\nu-\beta}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{-\alpha})]\right)(x) \\ &= \frac{1}{\Gamma(\nu)}\int_{x}^{\infty}(t-x)^{\nu-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\delta)_n}a^n t^{-\alpha n}dt. \end{aligned}$$

Interchanging the order of integration and summation and evaluating the inner integral by beta function formula, we have

$$\begin{aligned} K &= x^{-\beta}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\nu+\alpha n+\beta)(\delta)_n}(ax^{-\alpha})^n \\ &= x^{-\beta}E_{\alpha,\beta+\nu,\delta}^{\gamma,q}(ax^{-\alpha}). \end{aligned}$$

$\square$

Corollary 3.10. *For* $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\nu) > 0, \mathrm{a} \in \mathrm{R}$ and $\mathrm{q} \in (0,1) \bigcup \mathrm{N}$. *Then there holds the formula*

$$\left(I_{-}^{\nu}[t^{-\nu-\beta}E_{\alpha,\beta}^{\gamma,q}(at^{-\alpha})]\right)(x) = x^{-\beta}E_{\alpha,\beta+\nu}^{\gamma,q}(ax^{-\alpha}). \tag{3.8}$$

Corollary 3.11. *For* $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\nu) > 0$ and $\mathrm{a} \in \mathrm{R}$. *Then there holds the formula*

$$\left(I_{-}^{\nu}[t^{-\nu-\beta}E_{\alpha,\beta}^{\gamma,\delta}(at^{-\alpha})]\right)(x) = x^{-\beta}E_{\alpha,\beta+\nu}^{\gamma,\delta}(ax^{-\alpha}). \tag{3.9}$$

Remark 3.12. *The formula (3.9) is a known result of Khan and Ahmed [9, (2.8)].*

Theorem 3.13. *Let* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0, \mathrm{q} \in (0,1) \bigcup \mathrm{N}$ and $\mathrm{a} \in \mathrm{R}$. *Let* I_{-}^{ν} *be the right-sided operator of Rimann-Liouville fractional integral (2.2). Then there holds the formula*

$$\begin{aligned}&\left(I_{-}^{\nu}[t^{-\nu-\beta}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{-\alpha})]\right)(x)\\ &\qquad = x^{-\nu-\beta}[(\beta+\nu)E_{\alpha,\beta+\nu+1,\delta}^{\gamma,q}(ax^{\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta+\nu+1,\delta}^{\gamma,q}(ax^{-\alpha})].\end{aligned} \tag{3.10}$$

Corollary 3.14. *For* $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\nu) > 0, \mathrm{q} \in (0,1) \bigcup \mathrm{N}$ and $\mathrm{a} \in \mathrm{R}$. *Then there holds the formula*

$$\begin{aligned}&\left(I_{-}^{\nu}[t^{-\nu-\beta}E_{\alpha,\beta}^{\gamma,q}(at^{-\alpha})]\right)(x)\\ &\qquad = x^{-\nu-\beta}[(\beta+\nu))E_{\alpha,\beta+\nu+1}^{\gamma,q}(ax^{-\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta+\nu+1}^{\gamma,q}(ax^{-\alpha})].\end{aligned} \tag{3.11}$$

Corollary 3.15. *For* $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\nu) > 0$ and $\mathrm{a} \in \mathrm{R}$. *Then there holds the formula*

$$\begin{aligned}&\left(I_{-}^{\nu}[t^{-\nu-\beta}E_{\alpha,\beta}^{\gamma,\delta}(at^{-\alpha})]\right)(x)\\ &\qquad = x^{-\nu-\beta}[(\beta+\nu))E_{\alpha,\beta+\nu+1}^{\gamma,\delta}(ax^{-\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta+\nu+1}^{\gamma,\delta}(ax^{-\alpha})].\end{aligned} \tag{3.12}$$

Remark 3.16. *The formula (3.12) is a known result of Khan and Ahmed [9, (2.11)].*

We now proceed to derive certain other properties of $E_{\alpha,\beta,\delta}^{\gamma,q}(z)$ associated with Rimann-Liouville fractional derivative operators D_{0+}^{ν} *and* D_{-}^{ν} defined by (2.3) and (2.4) respectively.

Theorem 3.17. *Let* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0, \mathrm{q} \in (0,1) \bigcup \mathrm{N}$ and $\mathrm{a} \in \mathrm{R}$. *Let* D_{0+}^{ν} *be the left-sided operator of Rimann-Liouville fractional derivative (2.3). Then there holds the formula*

$$\left(D_{0+}^{\nu}[t^{\beta-1}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{\alpha})]\right)(x) = x^{\beta-\nu-1}E_{\alpha,\beta-\nu,\delta}^{\gamma,q}(ax^{\alpha}) \tag{3.13}$$

Proof. By virtue of (1.6) and (2.3), we have

$$\begin{aligned}
K &\equiv \left(D_{0+}^{\nu}[t^{\beta-1}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{\alpha})]\right)(x) \\
&= \left(\frac{d}{dx}\right)^{[\nu]+1}\left(I_{0+}^{1-\{\nu\}}[t^{\beta-1}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{\alpha})]\right)(x) \\
&= \sum_{n=0}^{\infty}\frac{a^n(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\delta)_n}\left(\frac{d}{dx}\right)^{[\nu]+1}\int_0^x t^{\alpha n+\beta-1}(x-t)^{-\{\nu\}}dt.
\end{aligned}$$

Interchanging the order of integration and summation and evaluating the inner integral by beta function formula, we have

$$\begin{aligned}
K &\equiv \sum_{n=0}^{\infty}\frac{a^n(\gamma)_{qn}}{\Gamma(\alpha n+\beta+1-\{\nu\})(\delta)_n}\left(\frac{d}{dx}\right)^{[\nu]+1}x^{\alpha n+\beta-\{\nu\}} \\
&= \sum_{n=0}^{\infty}\frac{a^n(\gamma)_{qn}}{\Gamma(\alpha n+\beta-\nu)(\delta)_n}x^{\alpha n+\beta-\nu-1} \\
&= x^{\beta-\nu-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta-\nu)(\delta)_n}(ax^{\alpha})^n \\
&= x^{\beta-\nu-1}E_{\alpha,\beta-\nu,\delta}^{\gamma,q}(ax^{\alpha}).
\end{aligned}$$

□

Corollary 3.18. *For* $\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\nu)>0, \mathrm{q}\in(0,1)\bigcup \mathrm{N}$ and $\mathrm{a}\in\mathrm{R}$. *Then there holds the formula*

$$\left(D_{0+}^{\nu}[t^{\beta-1}E_{\alpha,\beta}^{\gamma,q}(at^{\alpha})]\right)(x) = x^{\beta-\nu-1}E_{\alpha,\beta-\nu}^{\gamma,q}(ax^{\alpha}). \tag{3.14}$$

Corollary 3.19. *For* $\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\gamma)>0, \mathrm{Re}(\delta)>0, \mathrm{Re}(\nu)>0$ and $\mathrm{a}\in\mathrm{R}$. *Then there holds the formula*

$$\left(D_{0+}^{\nu}[t^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta}(at^{\alpha})]\right)(x) = x^{\beta-\nu-1}[E_{\alpha,\beta-\nu}^{\gamma,\delta}(ax^{\alpha})]. \tag{3.15}$$

Remark 3.20. *The formula (3.15) is a known result of Khan and Ahmed [9, (2.14)].*

Theorem 3.21. *Let* $\mathrm{Re}(\nu)>0, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\gamma)>0, \mathrm{Re}(\delta)>0, \mathrm{q}\in(0,1)\bigcup\mathrm{N}, \mathrm{a}\in\mathrm{R}$. *Let* D_{0+}^{ν} *be the left-sided operator of Rimann-Liouville fractional integral (2.3). Then there holds the formula*

$$\begin{aligned}
&\left(D_{0+}^{\nu}[t^{\beta-1}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{\alpha})]\right)(x) \\
&\quad = x^{\beta-\nu-1}[(\beta-\nu))E_{\alpha,\beta-\nu+1,\delta}^{\gamma,q}(ax^{\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta-\nu+1,\delta}^{\gamma,q}(ax^{\alpha})]. \qquad (3.16)
\end{aligned}$$

Corollary 3.22. *For* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0,$ and $\mathrm{q} \in (0,1) \bigcup \mathrm{N}, \mathrm{a} \in \mathrm{R}.$ *Then there holds the formula*

$$\left(D_{0+}^{\nu}[t^{\beta-1}E_{\alpha,\beta}^{\gamma,q}(at^{\alpha})]\right)(x)$$
$$= x^{\beta-\nu-1}[(\beta-\nu))E_{\alpha,\beta-\nu+1}^{\gamma,q}(ax^{\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta-\nu+1}^{\gamma,q}(ax^{\alpha})]. \tag{3.17}$$

Corollary 3.23. *For* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{q} \in (0,1) \bigcup \mathrm{N}$ *and* $a \in R$*. Then there holds the formula*

$$\left(D_{0+}^{\nu}[t^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta}(at^{\alpha})]\right)(x)$$
$$= x^{\beta-\nu-1}[(\beta-\nu))E_{\alpha,\beta-\nu+1}^{\gamma,\delta}(ax^{\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta-\nu+1}^{\gamma,\delta}(ax^{\alpha})]. \tag{3.18}$$

Theorem 3.24. *Let* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0$ with $\beta - \alpha + \{\alpha\} > 1$ and $\mathrm{q} \in (0,1) \bigcup \mathrm{N}, \mathrm{a} \in \mathrm{R}.$ *Let* D_{-}^{ν} *be the right-sided operator of Rimann-Liouville fractional derivative (2.4). Then there holds the formula*

$$\left(D_{-}^{\nu}[t^{\nu-\beta}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{-\alpha})]\right)(x) = x^{-\beta}E_{\alpha,\beta-\nu,\delta}^{\gamma,q}(ax^{-\alpha}). \tag{3.19}$$

Proof. By virtue of (1.6) and (2.4) we have

$$\begin{aligned} K &\equiv \left(D_{-}^{\nu}[t^{\nu-\beta}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{-\alpha})]\right)(x) \\ &= \left(-\frac{d}{dx}\right)^{[\nu]+1}\left(I_{-}^{1-\{\nu\}}[t^{\nu-\beta}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{-\alpha})]\right)(x) \\ &= \sum_{n=0}^{\infty}\frac{a^n(\gamma)_{qn}}{\Gamma(\alpha n+\beta)\Gamma(1-\{\nu\})(\delta)_n}\left(-\frac{d}{dx}\right)^{[\nu]+1}\int_x^{\infty} t^{-\alpha n+\nu-\beta}(t-x)^{-\{\nu\}}dt. \end{aligned}$$

If we set $t = \frac{x}{z}$, then the above expression transforms into the form

$$\begin{aligned} K \equiv &\sum_{n=0}^{\infty}\frac{a^n(\gamma)_{qn}}{\Gamma(\alpha n+\beta)\Gamma(1-\{\nu\})(\delta)_n} \\ &\times\int_0^1 z^{\alpha n-\nu+\beta+\{\nu\}-2}(1-z)^{\{-\nu\}}dz\left(-\frac{d}{dx}\right)^{[\nu]+1}x^{\nu-\alpha n-\beta-\{\nu\}+1}. \end{aligned}$$

Using beta integral, we have

$$\begin{aligned} K &\equiv \sum_{n=0}^{\infty}\frac{a^n(\gamma)_{qn}}{\Gamma(\alpha n+\beta-\nu)(\delta)_n}x^{-\alpha n-\beta} \\ &= x^{-\beta}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta-\nu)(\delta)_n}(ax^{-\alpha})^n \\ &= x^{-\beta}E_{\alpha,\beta-\nu,\delta}^{\gamma,q}(ax^{-\alpha}). \end{aligned}$$

□

Corollary 3.25. *For* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ with $\beta - \alpha + \{\alpha\} > 1$. *Let* D_{-}^{ν} *be the right-sided operator of Rimann-Liouville fractional derivative (2.4). Then there holds the formula*

$$\left(D_{-}^{\nu}[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,q}(at^{-\alpha})]\right)(x) = x^{-\beta}E_{\alpha,\beta-\nu}^{\gamma,q}(ax^{-\alpha}). \tag{3.20}$$

Corollary 3.26. *For* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0$ with $\beta - \alpha + \{\alpha\} > 1$. *Let* D_{-}^{ν} *be the right-sided operator of Rimann-Liouville fractional derivative (2.4). Then there holds the formula*

$$\left(D_{-}^{\nu}[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,\delta}(at^{-\alpha})]\right)(x) = x^{-\beta}E_{\alpha,\beta-\nu}^{\gamma,\delta}(ax^{-\alpha}). \tag{3.21}$$

Remark 3.27. *The formula (3.21) is a known result of Khan and Ahmed [9, (2.20)].*

Theorem 3.28. *Let* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0$ q $\in (0,1)\bigcup \mathrm{N}$ and a $\in$ R. *Let* D_{-}^{ν} *be the right-sided operator of Rimann-Liouville fractional derivative (2.4). Then there holds the formula*

$$\begin{aligned}&\left(D_{-}^{\nu}[t^{\nu-\beta}E_{\alpha,\beta,\delta}^{\gamma,q}(at^{-\alpha})]\right)(x)\\ &\qquad = x^{-\beta}[(\beta-\nu)E_{\alpha,\beta-\nu+1,\delta}^{\gamma,q}(ax^{-\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta-\nu+1,\delta}^{\gamma,q}(ax^{-\alpha})].\end{aligned} \tag{3.22}$$

Corollary 3.29. *For* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0$ and a $\in$ R. *Then there holds the formula*

$$\begin{aligned}&\left(D_{-}^{\nu}[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,\delta}(at^{-\alpha})]\right)(x)\\ &\qquad = x^{-\beta}[(\beta-\nu)E_{\alpha,\beta-\nu+1}^{\gamma,\delta}(ax^{-\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta-\nu+1}^{\gamma,\delta}(ax^{-\alpha})].\end{aligned} \tag{3.23}$$

Corollary 3.30. *For* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ q $\in (0,1)\bigcup \mathrm{N}$ and a $\in$ R. *Then there holds the formula*

$$\begin{aligned}&\left(D_{-}^{\nu}[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,q}(at^{-\alpha})]\right)(x)\\ &\qquad = x^{-\beta}[(\beta-\nu)E_{\alpha,\beta-\nu+1}^{\gamma,q}(ax^{-\alpha}) + x\frac{d}{dx}[E_{\alpha,\beta-\nu+1}^{\gamma,q}(ax^{-\alpha})].\end{aligned} \tag{3.24}$$

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