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Weak and Strong Convergence Theorems for an α-Nonexpansive Mapping and a Generalized Nonexpansive Mapping in Hilbert Spaces

Kriengsak Wattanawitoon and Yaowalux Khamlae[1]

Department of Mathematics and Statistics
Faculty of Science and Agricultural Technology
Rajamangala University of Technology
Lanna Tak, Tak 63000, Thailand
e-mail : kriengsak.wat@rmutl.ac.th (K. Wattanawitoon)
y.khamlae@rmutl.ac.th (Y. Khamlae)

Abstract : In this paper, we investigate iterative scheme for approximating common solution of fixed point problems involving an α-nonexpansive mapping and a generalized nonexpansive mapping in the framework of Hilbert spaces via Takahashi and Tamura's scheme. We obtain the weak convergence theorem under appropriate conditions and strong convergence theorem by adding some necessary condition in the same scheme. Our results extend and improve some recent results in the literature.

Keywords : generalized nonexpansive mapping; α-nonexpansive mapping; fixed point problem; Hilbert space.
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1 Introduction

Throughout this paper, we study the convergence theorems in the framework of a Hilbert space H. Let C be a nonempty closed and convex subset of H . We

[1]Corresponding author.

recall some definitions of mapping as shown in the following:

- $T : C \to C$ is said to be *nonexpansive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$.
- $T : C \to C$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed point of T.

In 2008, Suzuki [3] proposed the condition which is called condition (C). He found that it is the weaker condition when comparing with the nonexpansive mapping and it is stronger than a quasi-nonexpansive mapping. Moreover, he also proved the fixed point theorem and convergence theorem for this mapping.

Then, Dhompongsa et al. [6] improved and extended the main results of Suzuki's theorems. He proposed the fixed point theorem for mappings satisfying condition (C) in Banach spaces under some appropriate conditions.

Later, Kohsaka and Takahashi [7] studied the existence of fixed points of firmly nonexpansive type mappings and also introduced the class of nonspreading mappings in Banach spaces.

Afterward, Takahashi and Tamura [9] studied about the weak convergence of two nonexpansive mappings. They proposed this following scheme:

$$\begin{cases} x_1 = x \in E, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1\{\beta_n T_2 x_n + (1 - \beta_n)x_n\}. \end{cases} \tag{1.1}$$

Recently, Dhompongsa et al. [2] proposed the algorithm for proving the weak convergence theorem of nonspreading mapping and mapping satisfying condition (C) by using the Takahashi and Tamura's scheme in the framework of Hilbert spaces.

Remark 1.1. *From [8], it can be concluded that*

- *T is called* nonexpansive mapping *if T is 0-nonexpansive;*
- *T is called* nonspreading mapping *if T is $\frac{1}{2}$- nonexpansive;*
- *T is called* hybrid mapping *if T is $\frac{1}{3}$-nonexpansive;*
- *If T is an α-nonexpansive mapping and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.*

In the same period, Aoyama and Kohsaka [8] proposed the new class of α-nonexpansive mappings in a Banach space E. This class can be reduced to nonexpansive, nonspreading and hybrid mappings.

Motivated and inspired by the above results, we proposed the iterative scheme for proving the convergence theorem for an α-nonexpansive mapping and mapping satisfying the condition (C) by using the iterative scheme of Takahashi and Tamura [9] as shown in the following:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n S\{\beta_n T x_n + (1 - \beta_n)x_n\} + (1 - \alpha_n)x_n, \end{cases} \tag{1.2}$$

for all $n \in \mathbb{N}$ where $\{\alpha_n\} \subset (0, 1]$ and $\{\beta_n\} \subset [0, 1]$ satisfy the appropriate conditions. Then we prove weak and strong convergence theorems.

2 Preliminaries

In this section, we give some necessary notations, definitions, lemmas and theorems which will be needed for the proof of our main results.

First, we recall some basic notations in Hilbert spaces. Note that $\mathbb{N}$ and $\mathbb{R}$ stand for the set of integers and the set of real numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

In a Hilbert space H, it is known that

$$\|x+y\|^2 = \|x\|^2 + \|y^2\| + 2\langle x, y\rangle.$$

The nearest projection P_C from H to C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

We know that a Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\}$ in H such that $x_n \rightharpoonup x$ implies that

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{i\to\infty} \inf \|x_n - y\|, \ \forall y \in H \ \text{ and } \ y \neq x. \tag{2.1}$$

A mapping $T : C \to C$ is said to satisfy the condition (A) if there exists a nondecreasing $f : [0, \infty] \to [0, \infty]$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$f(d(x, F(T))) \leq \|x - Tx\|, \tag{2.2}$$

for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.
Let $\mathcal{F} := F(S) \cap F(T)$, the mappings $S, T : C \to C$ are said to satisfy the condition (A') if there exists a nondecreasing $f : [0, \infty] \to [0, \infty]$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$f(d(x, \mathcal{F})) \leq \frac{1}{2}(\|x - Tx\| + \|x - Sx\|), \tag{2.3}$$

for all $x \in C$, where $d(x, \mathcal{F}) = \inf\{\|x - x^*\| : x^* \in \mathcal{F}\}$.

Definition 2.1. Let E be a Banach space, let C be a nonempty closed convex subset of E, and let α be a real number such that $\alpha < 1$. A mapping $T : C \to E$ is said to be α-nonexpansive if

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2 \tag{2.4}$$

for all $x, y \in C$.

Definition 2.2 ([3])**.** Let T be a mapping on a subset C of a Banach space E. Then T is said to satisfy the condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \ \text{ which implies } \|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

Theorem 2.3 ([3, Theorem 4]). *Let T be a mapping on a convex subset C of a Banach space E. Assume that T satisfies condition (C) and either of the following holds:*

- *C is compact.*
- *C is weakly compact and C has the Opial's property.*

Then T has a fixed point.

Remark 2.4. *From [3], we also obtain the following facts.*

- *Every nonexpansive mappings satisfy condition (C) but the converse is not true.*
- *If a mapping T satisfies condition (C) and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.*
- *Let T be a mapping on a closed subset C of a Banach space E. Assume that T satisfies condition (C). Then $F(T)$ is closed. Moreover, if E is strictly convex and C is convex, then $F(T)$ is also convex; see also Itoh and Takahashi [1].*
- *Let T be a mapping on a subset C of a Banach space E with the Opial's property. Assume that T satisfies condition (C). If $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tz = z$.*

Theorem 2.5 ([6]). *Let C be a nonempty bounded closed convex subset of a Banach space E. Let $T : C \to C$ be a mapping satisfying condition (C). Suppose that the asymptotic center in C of each bounded sequence of E is nonempty and compact. Then T has a fixed point.*

Recently, C.Mongkolkeha et.al prove the demiclosed lemma for T and the closed and convex property of $F(T)$ where T is an α-nonexpansive mapping as shown in the following.

Lemma 2.6 ([10]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be an α -nonexpansive mapping. If a sequence $\{x_n\}$ in C with $x_n \rightharpoonup x^*$ and $\|x_n - Tx_n\| \to 0$, then $x^* = Tx^*$.*

Theorem 2.7 ([10]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be an α-nonexpansive mapping such that $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex.*

Lemma 2.8 ([4]). *Suppose that $\{s_n\}$ and $\{e_n\}$ are sequences of nonnegative real numbers such that $s_{n+1} \leq s_n + e_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} e_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.*

Lemma 2.9. *Let C be a nonempty closed convex subset of H and let $\{x_n\}$ be a sequence on H. Suppose that, for all $y \in C$,*

$$\|x_{n+1} - y\| \leq \|x_n - y\|$$

for all $n \in \mathbb{N}$. Then $\{P_C x_n\}$ converges strongly to some $z_0 \in C$, where P_C is a metric projection of H onto C.

Lemma 2.10. *Let C be a nonempty closed convex subset of a Hilbert space H. Then, for $x \in H$ and $y \in C$, $y = P_C x$ if and only if $\langle x - y, y - z \rangle \geq 0$ for all $z \in C$.*

3 Weak Convergence Theorem

In this section, we prove the weak convergence theorem for approximating a common fixed point for mapping which satisfies the condition (C) and α-nonexpansive mapping.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $S : C \to C$ be an α-nonexpansive mapping and let $T : C \to C$ be a mapping satisfying the condition (C) such that $F(S) \cap F(T) \neq \emptyset$. We generate the sequence $\{x_n\}$ recursively by the following scheme:*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n S\{\beta_n T x_n + (1 - \beta_n) x_n\} + (1 - \alpha_n) x_n, \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$ where $0 < a \leq \alpha \leq 1$ and $\{\beta_n\} \subset [0, 1]$ satisfy the following conditions:

(C1) $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$,

(C2) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$.

Then $\{x_n\}$ converges weakly to $w \in F(S) \cap F(T)$, where $w = \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Proof. **Step 1.** We first prove that the sequence $\{x_n\}$ generated by the above scheme is bounded for all $n \in \mathbb{N}$.
Let $p \in F(S) \cap F(T)$. By Definition 2.1 and Definition 2.2, we consider

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|S\beta_n T x_n + (1 - \beta_n) x_n - p\| + (1 - \alpha_n)\|x_n - p\| \\ &\leq \alpha_n \beta_n \|T x_n - p\| + \alpha_n (1 - \beta_n)\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.2}$$

So, $\lim_{n\to\infty} \|x_n - p\|$ exists and hence $\{x_n\}$ is bounded.
Step 2. In this step, we will show that $\lim_{n\to\infty} \|Su_n - x_n\| = 0$. Let $u_n = \beta_n T x_n + (1 - \beta_n) x_n$ for all $n \in \mathbb{N}$. From $p \in F(S) \cap F(T)$, then we have

$$\begin{aligned} \|Su_n - p\| &\leq \|u_n - p\| \\ &= \|\beta_n T x_n + (1 - \beta_n) x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.3}$$

Next, we consider

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n Su_n+(1-\alpha_n)x_n-p\|^2\\ &= \alpha_n\|Su_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\alpha_n(1-\alpha_n)\|Su_n-x_n\|^2.\end{aligned}$$

We reorder the above equation, then we obtain that

$$\begin{aligned}\alpha_n(1-\alpha_n)\|Su_n-x_n\|^2 &= \alpha_n\|Su_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &= \|x_n-p\|^2-\|x_{n+1}-p\|^2.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n(1-\alpha_n)>0$ and $\lim_{n\to\infty}\|x_n-p\|$ exists, we can conclude that

$$\lim_{n\to\infty}\|Su_n-x_n\|=0. \tag{3.4}$$

Step 3. We will show that $\lim_{n\to\infty}\|Tx_n-x_n\|$ and $\lim_{n\to\infty}\|Sx_n-x_n\|=0$. Let $p\in F(S)\cap F(T)$ and from the existence of $\lim_{n\to\infty}\|x_n-p\|$, we let $\lim_{n\to\infty}\|x_n-p\|=c$. Then, we obtain

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n Su_n+(1-\alpha_n)x_n-p\|\\ &\le \alpha_n\|Su_n-p\|+(1-\alpha_n)\|x_n-p\|\\ &\le \|u_n-p\|+(1-\alpha_n)\|x_n-p\|.\end{aligned}$$

We rewritten the above inequality, therefore we have

$$\frac{\|x_{n+1}-p\|-\|x_n-p\|}{\alpha_n}+\|x_n-p\|\le\|u_n-p\|. \tag{3.5}$$

By (3.3), (3.5) and the existence of $\lim_{n\to\infty}\|x_n-p\|$, we obtain

$$c\le\liminf_{n\to\infty}\|u_n-p\|\le\limsup_{n\to\infty}\|u_n-p\|\le\limsup_{n\to\infty}\|x_n-p\|=c$$

and then $\lim_{n\to\infty}\|u_n-p\|=c$.
Next, we consider

$$\begin{aligned}\beta_n(1-\beta_n)\|Tx_n-x_n\|^2 &= \beta_n\|Tx_n-p\|^2+(1-\beta_n)\|x_n-p\|^2\\ &\quad -\|\beta_n(Tx_n-p)+(1-\beta_n)(x_n-p)\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|x_n-p\|^2-\|u_n-p\|^2\\ &= \|x_n-p\|^2-\|u_n-p\|^2.\end{aligned}$$

Since $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$ and $\lim_{n\to\infty}\|u_n-p\|=\lim_{n\to\infty}\|x_n-p\|$, then we can conclude that

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0. \tag{3.6}$$

Next, we observe that

$$\begin{aligned}\|x_n - u_n\|^2 &= \|x_n - \beta_n Tx_n - (1-\beta_n)x_n\|^2 \\ &= \beta_n^2\|x_n - Tx_n\|^2.\end{aligned}$$

Since $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, we have that $\|x_n - u_n\|$ tends to zero and next we consider

$$\begin{aligned}\|Sx_n - u_n\|^2 &= \|Sx_n - x_n\|^2 + \|x_n - u_n\|^2 + 2\langle Sx_n - x_n, x_n - u_n\rangle \\ &\le \|Sx_n - x_n\|^2 + \|x_n - u_n\|^2 + 2\|Sx_n - x_n\|\|x_n - u_n\|.\end{aligned}$$

By the above inequality and definition of α-nonexpansive, then we obtain the following:

$$\begin{aligned}\|Sx_n - x_n\|^2 &= \|Sx_n - Su_n + Su_n - x_n\|^2 \\ &= \|Sx_n - Su_n\|^2 + \|Su_n - x_n\|^2 + 2\langle Sx_n - Su_n, Su_n - x_n\rangle \\ &\le \|Sx_n - Su_n\|^2 + \|Su_n - x_n\|^2 + 2\|Sx_n - Su_n\|\|Su_n - x_n\| \\ &\le \alpha\|Sx_n - u_n\|^2 + \alpha\|Su_n - x_n\|^2 + (1-2\alpha)\|x_n - u_n\|^2 \\ &\quad +\|Su_n - x_n\|^2 + 2\|Sx_n - Su_n\|\|Su_n - x_n\| \\ &\le \alpha\|Sx_n - x_n\|^2 + \alpha\|x_n - u\|^2 + 2\alpha\|Sx_n - x_n\|\|x_n - u_n\| \\ &\quad +\alpha\|Su_n - x_n\|^2 + (1-2\alpha)\|x_n - u_n\|^2 + \|Su_n - x_n\|^2 \\ &\quad +2\|Sx_n - Su_n\|\|Su_n - x_n\|.\end{aligned}$$

Then we reorder the above inequality, we obtain

$$\begin{aligned}&(1-\alpha)\|Sx_n - x_n\|^2 \\ &\quad \le \alpha\|x_n - u_n\|^2 + 2\alpha\|Sx_n - x_n\|\|x_n - u_n\| + \alpha\|Su_n - x_n\|^2 \\ &\qquad + (1-2\alpha)\|x_n - u_n\|^2 + \|Su_n - x_n\|^2 + 2\|Sx_n - Su_n\|\|Su_n - x_n\|.\end{aligned}$$

Since $\alpha-$nonexpansive is quasi-nonexpansive, we obtain that

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0.$$

Step 4. We will show that $\{x_n\}$ converges weakly to some $v \in F(S)\cap F(T)$. Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that it converges weakly to v. From Lemma 2.6, we have that $v \in F(S)$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$. We show that $w = v$. Next, we will show that for any $z \in F(S)$, $\lim_{n\to\infty}\|x_n - z\|$ exists. Obviously,

$$\begin{aligned}\|x_{n+1} - z\| &= \|\alpha_n S\{\beta_n Tx_n + (1-\beta_n)x_n\} + (1-\alpha_n)x_n - z\| \\ &\le \alpha_n\|S\{\beta_n Tx_n + (1-\beta_n)x_n\} - z\| + (1-\alpha_n)\|x_n - z\| \\ &\le \alpha_n\beta_n\|Tx_n - z\| + \alpha_n(1-\beta_n)\|x_n - z\| + (1-\alpha_n)\|x_n - z\| \\ &= \alpha_n\beta_n\|Tx_n - z\| + (1-\alpha_n\beta_n)\|x_n - z\| \\ &\le \beta_n\|Tx_n - z\| + \|x_n - z\|.\end{aligned}$$

By condition (C2) and from Lemma 2.8, we obtain that $\lim_{n\to\infty}\|x_n - z\|$ exists. Suppose that $w \neq v$, from Opial's theorem [5] and $v, w \in F(S)$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|x_n - w\| &= \lim_{i\to\infty}\|x_{n_i} - w\| < \lim_{i\to\infty}\|x_{n_i} - v\| \\ &= \lim_{n\to\infty}\|x_n - v\| < \lim_{j\to\infty}\|x_{n_j} - v\| \\ &< \lim_{j\to\infty}\|x_{n_j} - w\| = \lim_{n\to\infty}\|x_n - w\|.\end{aligned}$$

This is a contradiction and we can conclude that $v = w$. So, $x_n \rightharpoonup w \in F(S)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to some $w \in F(S) \cap F(T)$. As the same of the above proof, we can conclude that $x_n \rightharpoonup w$.

Finally, we will show that $w = \lim_{n\to\infty} P_{F(S)\cap F(T)}x_n$. Let $u_n = P_{F(S)\cap F(T)}x_n$ and from $w \in F(S) \cap F(T)$ with Lemma 2.10, we have that $\langle u - u_n, u_n - x_n\rangle \geq 0$ for all $n \in \mathbb{N}$. From (3.2), we obtain that for each $p \in F(S) \cap F(T)$, $\|x_{n+1} - p\| \leq \|x_n - p\|$. Then by Lemma 2.9, we have $u_n \to z_0$ for some $z_0 \in F(S) \cap F(T)$ and hence $\langle w - z_0, z - 0 - w\rangle \geq 0$ as $n \to \infty$. This implies that $w = z_0$. This completes the proof. □

From Theorem 3.1, we can deduce to the following results.

Corollary 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $S : C \to C$ be an α-nonexpansive mapping and let $T : C \to C$ be a mapping satisfying the condition (C) such that $F(S) \cap F(T) \neq \emptyset$. We generate the sequence $\{x_n\}$ recursively by the following scheme:*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n S\{\beta_n T x_n + (1 - \beta_n)x_n\} + (1 - \alpha_n)x_n, \end{cases} \tag{3.7}$$

for all $n \in \mathbb{N}$ where $0 < a \leq \alpha \leq 1$ and $\{\beta_n\} \subset [0, 1]$ satisfy the following conditions:

(C1) $\liminf_{n\to\infty}\alpha_n(1 - \alpha_n) > 0$,

(C2) $\sum_{n=1}^{\infty}\beta_n < \infty$.

Then $\{x_n\}$ converges weakly to $w \in F(S)$.

Next, we put $\beta_n \equiv 0$ then we have the following corollary.

Corollary 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $S : C \to C$ be an α-nonexpansive mapping such that $F(S) \neq \emptyset$. We generate the sequence $\{x_n\}$ recursively by the following scheme:*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n S x_n + (1 - \alpha_n)x_n, \end{cases} \tag{3.8}$$

for all $n \in \mathbb{N}$ where $0 < a \leq \alpha \leq 1$ and $\liminf_{n\to\infty}\alpha_n(1 - \alpha_n) > 0$. Then $\{x_n\}$ converges weakly to $w \in F(S)$.

4 Strong Convergence Theorem

Based on the scheme in the Main Theorem, we propose the strong convergence theorem by adjusting some conditions in our algorithm.

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $S : C \to C$ be an α-nonexpansive mapping and $T : C \to C$ be a mapping satisfying condition (C). Assume that both mappings also satisfy condition (A') and $\mathcal{F} := F(S) \cap F(T) \neq 0$. Let the sequence x_n is generated iteratively by the following scheme:*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n S\{\beta_n T x_n + (1-\beta_n)x_n\} + (1-\alpha_n)x_n, \end{cases} \tag{4.1}$$

for all $n \in \mathbb{N}$ where $0 < a \leq \alpha \leq 1$ and $\beta \subset [0,1]$ satisfy the following conditions:

(C1) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$,

(C2) $\sum_{n=1}^{\infty} \beta_n < \infty$.

Then, the sequence x_n converges strongly to $w^* \in \mathcal{F}$ where $w^* = P_{\mathcal{F}} x_n$.

Proof. By the mentioned proof in the Main Theorem, we have

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{4.2}$$

Moreover, we also already confirm that

$$\|x_{n+1} - p\| \leq \|x_n - p\| \tag{4.3}$$

for any $p \in F(S) \cap F(T)$. Taking infemum in both sides of the inequality 4.3, we can conclude that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists.

Next, we will claim that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Suppose that $\lim_{n\to\infty} d(x_n, \mathcal{F}) \neq 0$, then we can choose $t_1 \in \mathbb{N}$ such that $0 < \frac{k}{2} \leq d(x_n, \mathcal{F})$ for all $n \geq t_1$.
Since f is nondecreasing and $f(0) = 0$, therefore

$$0 < f(\frac{k}{2}) \neq f(d(x_n, \mathcal{F})). \tag{4.4}$$

By condition (A') of S and T, we have that

$$f(d(x_n, \mathcal{F})) \leq \frac{1}{2}\big\{\|x_n - Sx_n\| + \|x_n - Tx_n\|\big\}, \tag{4.5}$$

and from (4.2), it leads to the conclusion that

$$0 < f(\frac{k}{2}) \leq f(d(x_n, \mathcal{F})) \leq \frac{1}{2}\big\{\|x_n - Sx_n\| + \|x_n - Tx_n\|\big\} \to 0, \tag{4.6}$$

as $n \to \infty$. This is the contradiction. Therefore, we can conclude that

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$$

and hence, there exists $t_2 \in \mathbb{N}$ such that

$$d(x_n, \mathcal{F}) \leq \frac{\epsilon}{2} \tag{4.7}$$

for all $n \geq t_2$. Let $m, n \geq t_2$ and $p \in \mathcal{F}$, it follows that

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - p\| + \|x_n - p\| \leq \|x_{t_2} - p\| + \|x_{t_2} - p\| \\ &\leq 2d(x_{t_2}, \mathcal{F}) \\ &< \epsilon \end{aligned}$$

for all $m, n \geq t_2$. Thus, the sequence $\{x_n\}$ is a Cauchy sequence. Finally, we suppose that $\lim_{n\to\infty} x_n = w^*$ for some $w^* \in H$. Since $\mathcal{F}$ is closed, so we have $w^* \in \mathcal{F}$ Therefore, the sequence $\{x_n\}$ converges strongly to $w^* \in \mathcal{F}$. This completes the proof. □

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