



Some Sequence Spaces Defined by $|A|$ Summability and a Modulus Function in Seminormed Space

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Abstract : The object of this paper is to introduce a new sequence space which arises from the notions of $|A|$ summability and a modulus function in seminormed complex linear space. We examine various algebraic and topological properties of this space and also investigate some inclusion relations. Our results generalize and unify the corresponding earlier results of Altin et al. [1], Bhardwaj and Singh [2].

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1 Introduction

Given an infinite series $\sum_{n=0}^{\infty} a_n$, let

$$s_n = a_0 + a_1 + \cdots + a_n. \quad (1.1)$$

Denote the sequence (a_n) by a and the sequence (s_n) by s . We will suppose throughout that a, s are related by (1.1).

Let A be a lower triangular matrix. Set $A_n = \sum_{\nu=0}^n a_{n\nu} s_\nu$.

It is well known (see e.g. [3, 4]) that a series $\sum a_n$ is summable $|A|_k$, $k \geq 1$ if $\sum_{n=1}^{\infty} n^{k-1} |A_n - A_{n-1}|^k < \infty$. In particular, $\sum a_n$ is $|A|$ summable if $\sum_{n=1}^{\infty} |A_n - A_{n-1}| < \infty$.

We may associate with A two lower triangular matrices \bar{A} and \hat{A} defined as follows:

$$\bar{a}_{n\nu} = \sum_{r=\nu}^n a_{nr}, \quad n, \nu = 0, 1, 2, \dots,$$

and

$$\hat{a}_{n\nu} = \bar{a}_{n\nu} - \bar{a}_{n-1,\nu}, \quad n = 1, 2, \dots$$

Also we define

$$y_n(a) = \sum_{i=0}^n a_{ni} s_i = \sum_{i=0}^n a_{ni} \sum_{\nu=0}^i a_\nu = \sum_{\nu=0}^n a_\nu \sum_{i=\nu}^n a_{ni} = \sum_{\nu=0}^n \bar{a}_{n\nu} a_\nu$$

and

$$\phi_n(a) = y_n(a) - y_{n-1}(a) = \sum_{\nu=0}^n (\bar{a}_{n\nu} - \bar{a}_{n-1,\nu}) a_\nu = \sum_{\nu=0}^n \hat{a}_{n\nu} a_\nu. \quad (1.2)$$

Note that for any sequences a, b and scalar λ , we have

$$\phi_n(a + b) = \phi_n(a) + \phi_n(b) \text{ and } \phi_n(\lambda a) = \lambda \phi_n(a).$$

The idea of modulus was structured in 1953 by Nakano [5]. Following Ruckle [6] and Maddox [7] we recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

- (i) $f(x) = 0$ if and only if $x = 0$,
- (ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0, y \geq 0$,
- (iii) f is increasing,
- (iv) f is continuous from the right at 0.

Because of (ii), $|f(x) - f(y)| \leq f(|x - y|)$ so that in view of (iv), f is continuous everywhere on $[0, \infty)$. A modulus may be unbounded (for example, $f(x) = x^p, 0 < p \leq 1$) or bounded (for example, $f(x) = \frac{x}{(1+x)}$).

It is easy to see that $f_1 + f_2$ is a modulus function when f_1 and f_2 are modulus functions, and that the function f^v (v is a positive integer), the composition of a modulus function f with itself v times, is also a modulus function.

Ruckle [6] used the idea of a modulus function f to construct a class of FK spaces

$$L(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

The space $L(f)$ is closely related to the space ℓ_1 which is an $L(f)$ space with $f(x) = x$ for all real $x \geq 0$.

Recently many researchers have studied sequence spaces of fuzzy numbers using the concept of a modulus function. For a detailed account, one may refer to [8–11] where many more references can be found.

By w we shall denote the space of all scalar sequences. $\ell_{\infty, c}$ and c_0 denote the spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_{\infty} = \sup_k |x_k|$.

Let q_1 and q_2 be seminorms on a linear space X . Then q_1 is stronger than q_2 [12] if there exists a constant L such that $q_2(x) \leq Lq_1(x)$ for all $x \in X$. If each is stronger than the other, q_1 and q_2 are said to be equivalent.

The following inequalities (see e.g. [13, p. 190]) are needed throughout the paper.

Let $r = (r_k)$ be a bounded sequence of strictly positive real numbers. If $H = \sup_k r_k$, then for any complex a_k and b_k ,

$$|a_k + b_k|^{r_k} \leq C(|a_k|^{r_k} + |b_k|^{r_k}), \tag{1.3}$$

where $C = \max(1, 2^{H-1})$. Also for any complex λ ,

$$|\lambda|^{r_k} \leq \max(1, |\lambda|^H). \tag{1.4}$$

Let X be a seminormed space with seminorm q , f be a modulus function, $s \geq 0$ be a real number and $r = (r_k)$ be a bounded sequence of strictly positive real numbers. The symbol $w(X)$ denotes the space of all X -valued sequences.

We now introduce the following generalized X -valued sequence space using modulus function f .

$$|A|(f, r, q, s) = \{a \in w(X) : \sum_{n=1}^{\infty} n^{-s} [f(q(\phi_n(a)))]^{r_n} < \infty\}.$$

$|A|(f, r, q, s)$ is the generalization of several known sequence spaces, for instance, the following classes arise from $|A|(f, r, q, s)$ as the special cases:

- (i) If $X = \mathbb{C}, q(x) = |x|, f(x) = x, s = 0, (a_{nk})$ is $a_{nk} = \frac{p_k}{P_n}$ if $k \leq n$ and $a_{nk} = 0$ if $k > n$, then $|A|(f, r, q, s) = |\bar{N}_p|(r)$ (Bhardwaj and Singh [14]).

- (ii) If $X = \mathbb{C}$, $q(x) = |x|$, $s = 0$, (a_{nk}) is $a_{nk} = \frac{p^k}{P_n}$ if $k \leq n$ and $a_{nk} = 0$ if $k > n$, then $|A|(f, r, q, s) = |\bar{N}_p|(f, r)$ (Bhardwaj and Singh [2]).
- (iii) If (a_{nk}) is $a_{nk} = \frac{p^k}{P_n}$ if $k \leq n$ and $a_{nk} = 0$ if $k > n$, then $|A|(f, r, q, s) = |\bar{N}_p|(f, r, q, s)$ (Altin et al. [1]).
- (iv) If $X = \mathbb{C}$, $q(x) = |x|$, $f(x) = x$ and $s = 0$, then $|A|(f, r, q, s) = |A|(r)$ (Savas et al. [15]).
- (v) If $X = \mathbb{C}$, $q(x) = |x|$, $f(x) = x$, $s = 0$, $r_k = 1$ for all k , (a_{nk}) is $a_{nk} = \frac{p^k}{P_n}$ if $k \leq n$ and $a_{nk} = 0$ if $k > n$, then $|A|(f, r, q, s) = |\bar{N}_p|$.

We denote $|A|(f, r, q, s)$ by $|A|(r, q, s)$ when $f(x) = x$ and by $|A|(f, r, q)$ when $s = 0$.

2 Linear Topological Structure of $|A|(f, r, q, s)$ Space and Inclusion Theorems

In this section we examine various algebraic and topological properties of the space $|A|(f, r, q, s)$ and investigate some inclusion relations.

Theorem 2.1. *For any modulus f , $|A|(f, r, q, s)$ is a linear space over the complex field \mathbb{C} .*

The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 2.2. *$|A|(f, r, q, s)$ is a topological linear space, paranormed by*

$$g(a) = \left(\sum_{n=1}^{\infty} n^{-s} [f(q(\phi_n(a)))]^{r_n} \right)^{\frac{1}{M}}$$

where $M = \max(1, \sup_k r_k)$.

The proof uses ideas similar to those used (e.g.) in [2, p. 1793] and the fact that every paranormed space is a topological linear space [16, p. 37].

Remark 2.3. *It is clear from the properties of f and q that g is not total.*

Lemma 2.4 ([17]). *Let f be a modulus function and let $0 < \delta < 1$. Then for each $x > \delta$ we have $f(x) \leq 2f(1)\delta^{-1}x$.*

Theorem 2.5. *Let f, f_1, f_2 be modulus functions, then*

- (i) if $s > 1$, then $|A|(f_1, r, q, s) \subseteq |A|(f \circ f_1, r, q, s)$,
- (ii) $|A|(f_1, r, q, s) \cap |A|(f_2, r, q, s) \subseteq |A|(f_1 + f_2, r, q, s)$,
- (iii) if $s > 1$ and $\limsup_{t \rightarrow \infty} \frac{f_1(t)}{f_2(t)} < \infty$, then $|A|(f_2, r, q, s) \subseteq |A|(f_1, r, q, s)$.

Proof. (i) Let $a \in |A|(f_1, r, q, s)$. Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $u_n = f_1(q(\phi_n(a)))$ and consider

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-s} [f(u_n)]^{r_n} &= \sum_{u_n \leq \delta} n^{-s} [f(u_n)]^{r_n} + \sum_{u_n > \delta} n^{-s} [f(u_n)]^{r_n} \\ &< \max(1, \epsilon^H) \sum_{n=1}^{\infty} n^{-s} + \max(1, (2f(1)\delta^{-1})^H) \sum_{n=1}^{\infty} n^{-s} [u_n]^{r_n} \\ &< \infty, \end{aligned}$$

by inequality (1.4) and Lemma 2.4 and hence $a \in |A|(fof_1, r, q, s)$.

(ii) The proof is easy in view of inequality (1.3).

(iii) Let $a \in |A|(f_2, r, q, s)$ and $\limsup_{t \rightarrow \infty} \frac{f_1(t)}{f_2(t)} = L < \infty$. Then for a given $\epsilon > 0$ there is a positive integer N such that for all t with $t > N$ we have $f_1(t) < (L + \epsilon)f_2(t)$. Let $u_n = q(\phi_n(a))$, then $\sum_{n=1}^{\infty} n^{-s} [f_1(u_n)]^{r_n} = \sum_1 + \sum_2$, where the first summation is over $u_n \leq N$ and the second over $u_n > N$. Then, using (1.4), we have

$$\sum_1 n^{-s} [f_1(u_n)]^{r_n} \leq [Nf_1(1)]^H \sum_{n=1}^{\infty} n^{-s}$$

and

$$\sum_2 n^{-s} [f_1(u_n)]^{r_n} \leq \max(1, (L + \epsilon)^H) \sum_{n=1}^{\infty} n^{-s} [f_2(u_n)]^{r_n}$$

and so $a \in |A|(f_1, r, q, s)$. □

Proposition 2.6. *For any modulus f and $s > 1$, $|A|(r, q, s) \subseteq |A|(f, r, q, s)$.*

The proof follows by taking $f_1(x) = x$ in Theorem 2.5(i).

Maddox [18, Proposition 1] proved that for any modulus f there exists $\lim_{t \rightarrow \infty} \frac{f(t)}{t}$. Using this result we give a sufficient condition for the inclusion $|A|(f, r, q, s) \subseteq |A|(r, q, s)$.

Theorem 2.7. *For any modulus f , if $\lim_{t \rightarrow \infty} \frac{f(t)}{t} = \beta > 0$, then $|A|(f, r, q, s) \subseteq |A|(r, q, s)$.*

Proof. Following the proof of Proposition 1 of Maddox [18], we have $\beta = \lim_{t \rightarrow \infty} \frac{f(t)}{t} = \inf\{\frac{f(t)}{t} : t > 0\}$, so that $0 \leq \beta \leq f(1)$. Let $\beta > 0$. By definition of β we have $\beta t \leq f(t)$ for all $t \geq 0$. Since $\beta > 0$ we have $t \leq \beta^{-1}f(t)$ for all $t \geq 0$. Now $a \in |A|(f, r, q, s)$ implies

$$\sum_{n=1}^{\infty} n^{-s} [q(\phi_n(a))]^{r_n} \leq \max(1, \beta^{-H}) \sum_{n=1}^{\infty} n^{-s} [f(q(\phi_n(a)))]^{r_n}$$

by (1.4) whence $a \in |A|(r, q, s)$ and the proof is complete. □

Theorem 2.8. Let f be a modulus function, q, q_1, q_2 be seminorms and s, s_1, s_2 be non-negative real numbers. Then

- (i) $|A|(f, r, q_1, s) \cap |A|(f, r, q_2, s) \subseteq |A|(f, r, q_1 + q_2, s)$,
- (ii) if q_1 is stronger than q_2 , then $|A|(f, r, q_1, s) \subseteq |A|(f, r, q_2, s)$,
- (iii) if q_1 is equivalent to q_2 , then $|A|(f, r, q_1, s) = |A|(f, r, q_2, s)$,
- (iv) if $s_1 \leq s_2$, then $|A|(f, r, q, s_1) \subseteq |A|(f, r, q, s_2)$.

Proof. The proof of (i) is straightforward using (1.3).

(ii) Let $a \in |A|(f, r, q_1, s)$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-s} [f(q_2(\phi_n(a)))]^{r_n} &\leq \sum_{n=1}^{\infty} n^{-s} [f(Lq_1(\phi_n(a)))]^{r_n} \\ &\leq (1 + [L])^H \sum_{n=1}^{\infty} n^{-s} [f(q_1(\phi_n(a)))]^{r_n} \end{aligned}$$

by (1.4) whence $a \in |A|(f, r, q_2, s)$.

The proofs of (iii) and (iv) are trivial. \square

Theorem 2.9. If $t = (t_k)$ and $r = (r_k)$ are bounded sequences of positive real numbers with $0 < t_k \leq r_k < \infty$ for each k , then for any modulus f ,

- (i) $|A|(f, t, q) \subseteq |A|(f, r, q)$,
- (ii) $|A|(f, r, q) \subseteq |A|(f, r, q, s)$.

Proof. (i) Let $a \in |A|(f, t, q)$. This implies that

$$f(q(\phi_i(a))) \leq 1$$

for sufficiently large values of i , say $i \geq k_0$ for some fixed $k_0 \in \mathbb{N}$. Since f is increasing, we have

$$\sum_{n \geq n_0}^{\infty} [f(q(\phi_n(a)))]^{r_n} \leq \sum_{n \geq n_0}^{\infty} [f(q(\phi_n(a)))]^{t_n} < \infty.$$

This shows that $a \in |A|(f, r, q)$ and completes the proof.

The proof of (ii) is trivial. \square

3 Composite Space $|A|(f^v, r, q, s)$ Using Composite Modulus Function f^v

Taking modulus function f^v instead of f in the space $|A|(f, r, q, s)$, we can define the composite space $|A|(f^v, r, q, s)$ as follows.

Definition 3.1. For a fixed natural number v , we define

$$|A|(f^v, r, q, s) = \left\{ a \in w(X) : \sum_{n=1}^{\infty} n^{-s} [f^v(q(\phi_n(a)))]^{r_n} < \infty \right\}.$$

Theorem 3.2. For any modulus function f and $v \in \mathbb{N}$,

- (i) $|A|(f^v, r, q, s) \subseteq |A|(r, q, s)$ if $\lim_{t \rightarrow \infty} \frac{f(t)}{t} = \beta > 0$,
- (ii) $|A|(r, q, s) \subseteq |A|(f^v, r, q, s)$ if there exists a positive constant α such that $f(t) \leq \alpha t$ for all $t \geq 0$.

Proof. Since proof of (i) is similar to that of Theorem 2.7, hence is omitted.

(ii) Let $a \in |A|(r, q, s)$. Since $f(t) \leq \alpha t$ for all $t \geq 0$ and f is increasing, we have $f^v(t) \leq \alpha^v t$ for each $v \in \mathbb{N}$. Again, using (1.4), we have

$$\sum_{n=1}^{\infty} n^{-s} [f^v(q(\phi_n(a)))]^{r_n} \leq \max(1, \alpha^{vH}) \sum_{n=1}^{\infty} n^{-s} [q(\phi_n(a))]^{r_n}$$

Hence, $a \in |A|(f^v, r, q, s)$ and this completes the proof. □

Example 3.3. $f_1(t) = t + t^{1/2}$ and $f_2(t) = \log(1 + t)$ for all $t \geq 0$ satisfy the conditions given in Theorem 3.2(i), (ii) respectively.

Theorem 3.4. Let $i, v \in \mathbb{N}$ and $i < v$. If f is a modulus such that $f(t) \leq \alpha t$ for all $t \geq 0$, where α is a positive constant, then

$$|A|(r, q, s) \subseteq |A|(f^i, r, q, s) \subseteq |A|(f^v, r, q, s).$$

Proof. Let $j = v - i$. Since $f(t) \leq \alpha t$, we have $f^v(t) < M^j f^i(t) < M^v t$, where $M = 1 + [\alpha]$. Let $a \in |A|(r, q, s)$. By the above inequality and using (1.4), we get

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-s} [f^v(q(\phi_n(a)))]^{r_n} &< M^{jH} \sum_{n=1}^{\infty} n^{-s} [f^i(q(\phi_n(a)))]^{r_n} \\ &< M^{vH} \sum_{n=1}^{\infty} n^{-s} [q(\phi_n(a))]^{r_n}. \end{aligned}$$

Hence, the required inclusion follows. □

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References

- [1] Y. Altin, M. Et, H. Altinok, The sequence space $|\bar{N}_p|(f, r, q, s)$ on seminormed spaces, *Internat. J. Appl. Math.* 12 (2) (2003) 125–132.
- [2] V.K. Bhardwaj, N. Singh, Some sequence spaces defined by $|\bar{N}, p_n|$ summability and a modulus function, *Indian J. Pure Appl. Math.* 32 (12) (2001) 1789–1801.
- [3] B. E. Rhoades, Inclusion theorems for absolute matrix summability methods, *J. Math. Anal. Appl.*, 238(1)(1999), 82-90.
- [4] E. Savas, A study on absolute summability factors for a triangular matrix, *Math. Inequal. Appl.* 12 (1) (2009) 141–146.
- [5] H. Nakano, Concave modulars, *J. Math. Soc. Japan* 5 (1953) 29–49.
- [6] W.H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.* 25 (1973) 973–978.
- [7] I.J. Maddox, Sequence spaces defined by a modulus, *Math. Proc. Camb. Philos. Soc.* 100 (1986) 161–166.
- [8] Y. Altin, M. Et, Generalized difference sequence spaces defined by a modulus function in a locally convex space, *Soochow J. Math.* 31 (2) (2005) 233–243.
- [9] Y. Altin, H. Altinok, R. Çolak, On some seminormed sequence spaces defined by a modulus function, *Kragujevac J. Math.* 29 (2006) 121–132.
- [10] Y. Altin, M. Isik, R. Çolak, A new sequence space defined by a modulus, *Stud. Univ. Babeş-Bolyai Math.* 53 (2) (2008) 3–13.
- [11] Y. Altin, Properties of some sets of sequences defined by a modulus function, *Acta Math. Sci. Ser. B Engl. Ed.* 29 (2) (2009) 427–434.
- [12] A. Wilansky, *Functional analysis*, Blaisdell Publishing Company, New York, 1964.
- [13] I.J. Maddox, *Elements of Functional Analysis*, Cambridge Univ. Press, 1970 (first edition).
- [14] V.K. Bhardwaj, N. Singh, Some sequence spaces defined by $|\bar{N}, p_n|$ summability, *Demonstratio Math.* 32 (3) (1999) 539–546.
- [15] E. Savas, V. Karakaya, N. Simsek, Some $\ell(p)$ -type new sequence spaces and their geometric properties, *Abstract and Appl. Anal.*, Vol. 2009 (2009), Article ID 696971, 12 pages.
- [16] A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, 1978.
- [17] S. Pehlivan, B. Fisher, On some sequence spaces, *Indian J. Pure Appl. Math.* 25 (10) (1994) 1067–1071.

- [18] I.J. Maddox, Inclusion between FK spaces and Kuttner's theorem, Math. Proc. Camb. Philos. Soc. 101 (1987) 523–527.

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