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New Approach for Numerical Solution of the One-Dimensional Bratu Equation

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Abstract : In this paper, parametric cubic spline method is presented for solving Bratu's problem. The convergence analysis of the presented method is discussed. The method is illustrated with two numerical examples and the results show that the method converges rapidly and approximates the exact solution very accurately.

Keywords : numerical method; parametric spline; convergence analysis; Bratu's problem.
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1 Introduction

Consider the Liouville-Bratu-Gelfand equation [1–3]

$$\begin{cases} \Delta u(t) + \lambda e^{u(t)} = 0, & t \in \Omega, \\ u(t) = 0, & t \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$, and Ω is a bounded domain. We consider the Bratu's boundary value problem in one-dimensional planar coordinates [1–5] of the form

$$\begin{aligned} u''(t) + \lambda e^{u(t)} = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0. \end{aligned} \tag{1.2}$$

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Bratu's problem is widely used in science and engineering to describe complicated physical and chemical models. For example, Bratu's problem [2–7] is used in a large variety of applications such as the fuel ignition model of the thermal combustion theory, the model of the thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity concerning the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [8–12].

Several numerical methods for approximating the solution of Bratu's problem are known. Laplace transform decomposition numerical algorithm is used for solving Bratu's problem [13]. The Perturbation-iteration algorithm [14], is applied to Bratu-type equations. Mohsen et al. [15] introduced new smoother to enhance multigrid-based methods for Bratu problem. One-point pseudospectral collocation method has been used for the solution of the one-dimensional Bratu equation [16]. The main purpose of the present paper is to use a parametric cubic spline method [17–19] for the numerical solution of the nonlinear boundary value problem (1.2). The method consists of reducing the problem to a set of nonlinear algebraic equations.

The outline of the paper is as follows. First, in Section 2 we present and describe the parametric cubic spline method and describe the basic formulation of spline approximation required for our subsequent development. Section 3 outlines the convergence analysis of the parametric cubic spline method for the solution of Bratu's problem. Numerical examples are given in Section 4 to illustrate the efficiency of the presented method. Finally, a conclusion is given in Section 5 that briefly summarizes the numerical results.

2 Parametric Cubic Spline Method

We seek a smooth approximate solution of (1.2) using parametric cubic spline functions. Let $n > 1$ and $\Delta := \{a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b\}$ be a partition of a the interval $[a, b]$ with $t_i = t_0 + ih,\ i = 0, 1, \ldots, n$ and $h = \frac{b-a}{n}$. Δ is assumed as mesh of the spline function $S_\Delta(t, \tau)$ of class $C^2[a, b]$. $S_\Delta(t, \tau)$, which interpolates $u(t)$ at the grid points $\{t_i\}_{i=0}^{n}$, depends on a parameter $\tau > 0$ and reduces to a cubic spline function as $\tau \to 0$. For the spline function $S_\Delta(t) := S_\Delta(t, \tau)$ in the interval $[t_i, t_{i+1}]$, we can write:

$$S''_\Delta(t) + \tau S_\Delta(t) = \big[S''_\Delta(t_i) + \tau S_\Delta(t_i)\big]\Big(\frac{t_{i+1} - t}{h}\Big) + \big[S''_\Delta(t_{i+1}) + \tau S_\Delta(t_{i+1})\big]\Big(\frac{t - t_i}{h}\Big). \tag{2.1}$$

We solve (2.1) and determine the constants of integration from the interpolatory conditions, thus we have:

$$\begin{aligned} S_\Delta(t) = \frac{-h^2}{w^2 \sin w}\Big[M_{i+1} \sin\Big(\frac{w(t - t_i)}{h}\Big) + M_i \sin\Big(\frac{w(t_{i+1} - t)}{h}\Big)\Big] \\ + \Big(\frac{h}{w}\Big)^2 \Big[\big(\frac{t - t_i}{h}\big)\Big(M_{i+1} + \big(\frac{w}{h}\big)^2 u_{i+1}\Big) + \big(\frac{t_{i+1} - t}{h}\big)\Big(M_i + \big(\frac{w}{h}\big)^2 u_i\Big)\Big], \end{aligned} \tag{2.2}$$

where $S_\Delta(t_i) = u(t_i) = u_i,\ S''_\Delta(t_i) = M_i$ and $w = h\sqrt{\tau}$. We use the continuity of first derivative of spline function at t_i, and obtain the following result:

$$h^2\big(\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1}\big) = u_{i+1} - 2u_i + u_{i-1},\ \ i = 1, 2, \ldots, n-1, \tag{2.3}$$

where

$$\alpha = \frac{1}{w^2}\big(w \csc w - 1\big), \quad \beta = \frac{1}{w^2}\big(1 - w \cot w\big). \tag{2.4}$$

For a numerical solution of the Bratu's problem (1.2), the interval $[0, 1]$ is divided into a set of grid points with step size h. Setting $t = t_i = t_0 + ih$, in Eq. (1.2), we obtain

$$u''(t_i) = -\lambda e^{u(t_i)}. \tag{2.5}$$

By using the assumption $S''_\Delta(t_i) = M_i$ in (2.5), we have

$$M_i = -\lambda e^{u(t_i)}. \tag{2.6}$$

Replacing M_i as given by Eq. (2.6) into Eq.(2.3), we get

$$\big(\lambda\alpha h^2 e^{u_{i-1}} + u_{i-1}\big) + \big(2\lambda\beta h^2 e^{u_i} - 2u_i\big) + \big(\lambda\alpha h^2 e^{u_{i+1}} + u_{i+1}\big) = 0,\ \ i = 1, 2, \ldots, n-1. \tag{2.7}$$

The above nonlinear system consists of $(n-1)$ equations with $(n-1)$ unknowns $u_i,\ i = 1, \ldots, n-1$. Solving this nonlinear system by $Newton's$ method, we can obtain an approximation to the solution of (1.2).

3 Convergence Analysis

Now we discuss the convergence of the parametric spline method for the Bratu's problem (1.2). We consider the equations in (2.7) and then rewrite them in the matrix form given by the nonlinear system

$$AU + \lambda h^2 BG = 0, \tag{3.1}$$

where $U = \big(u_1, u_2, \ldots, u_{n-1}\big)^T$. Also $A = \big[a_{i,j}\big]$, $B = \big[b_{i,j}\big]$ are tridiagonal matrices of order $(n-1) \times (n-1)$ and define as follows:

$$a_{i,j} = \begin{cases} 2, & i = j, \\ -1, & |i - j| = 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$b_{i,j} = \begin{cases} -2\beta, & i = j, \\ -\alpha, & |i - j| = 1, \\ 0, & \text{otherwise}, \end{cases}$$

and $G = diag\big(e^{u_i}\big),\ i = 1, 2, \ldots, n-1$.

Theorem 3.1. *Let M be a matrix such that $||M|| < 1$, and let I denote the unit matrix. Then $(I+M)^{-1}$ exists, and*

$$||(I+M)^{-1}|| < \frac{1}{1-||M||}.$$

Proof. By applying Theorem 1.7.7 of [20], we can prove the Theorem 3.1. □

Consider Eq.(3.1) and let:

$$C = AU + \lambda h^2 BG. \tag{3.2}$$

We know that the inverse of A exists and bounded as follows [21]:

$$||A^{-1}|| \leq \frac{(b-a)^2}{8h^2}. \tag{3.3}$$

Theorem 3.2. *If $||G||_\infty < \frac{8}{\lambda(b-a)^2}$ then the inverse of C, defined by (3.2), exists.*

Proof. From Eq.(3.2) we can write:

$$C = A(I + \lambda h^2 A^{-1} BG). \tag{3.4}$$

Eq.(3.3) shows that A^{-1} exists. Now, we need to prove the existence of $\big(I + \lambda h^2 A^{-1} BG\big)^{-1}$. According to Theorem 3.1 it is sufficient to show that

$$||\lambda h^2 A^{-1} BE|| < 1.$$

Having used Eq.(3.3) and also $||B|| = 1$, we obtain:

$$\begin{aligned} ||\lambda h^2 A^{-1} BG|| &\leq \lambda h^2 ||A^{-1}||||B||||G|| \\ &\leq \frac{\lambda(b-a)^2}{8}||G||. \end{aligned} \tag{3.5}$$

Considering the assumption $||G|| < \frac{8}{\lambda(b-a)^2}$, we have

$$||\lambda h^2 A^{-1} BG|| < 1. \tag{3.6}$$

Therefore, by using Theorem 3.1 and Eqs.(3.3) and (3.6) we conclude the existence of C^{-1}. □

We can also obtain a bound on the errors $E = U - U_n$ in the maximum norm, where $U = \big(u(x_1), u(x_2), \ldots, u(x_{n-1})\big)$ is the exact solution and $U_n = \big(u_1, u_2, \ldots, u_{n-1}\big)$ is the approximate solution of Bratu's problem (1.2). From Theorem 3.2 we can derive a bound on $||E||$.

Theorem 3.3. *Let T be the vector of local truncation error and $CE = T$, then*

$$||E|| \cong O(h^4), \ \left(when\ \alpha = \frac{1}{12}, \quad \beta = \frac{5}{12} \right),$$
$$||E|| \cong O(h^2), \ \left(when\ \alpha = \frac{1}{6}, \ \beta = \frac{1}{3} \right).$$

Proof. By using Theorem 3.2 and $CE = T$, we can write:

$$E = C^{-1}T = \big(I + \lambda h^2 A^{-1} BG\big)^{-1} A^{-1} T, \tag{3.7}$$

therefore, we get

$$||E|| \leq ||\big(I + \lambda h^2 A^{-1} BG\big)^{-1}|| ||A^{-1}|| ||T||. \tag{3.8}$$

Having used Eq.(3.6) and Theorem 3.1 we obtain

$$||\big(I + \lambda h^2 A^{-1} BG\big)^{-1}|| \leq \frac{1}{1 - ||\lambda h^2 A^{-1} BG||}. \tag{3.9}$$

Now, by applying Eqs.(3.3), (3.8) and (3.9) and also $||B|| = 1$, we have

$$||E|| \leq \frac{(b-a)^2}{h^2\big(8 - \lambda ||G|| (b-a)^2\big)} ||T||. \tag{3.10}$$

For $||T||$ from [18], the following cases arise:

Case(i) when $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ then $||T|| \leq \frac{h^6 M_6}{240}$, where $M_6 = \max |u^{(6)}(x)|$.

Case(ii) when $\alpha = \frac{1}{6}$, $\beta = \frac{1}{3}$ then $||T|| \leq \frac{h^4 M_4}{12}$, where $M_4 = \max |u^{(4)}(x)|$.

Therefore from the above cases and Eq.(3.10), the results can be written as follows:

$$||E|| \leq \frac{(b-a)^2 h^6 M_6}{240 h^2\big(8 - \lambda ||G|| (b-a)^2\big)} \cong O(h^4), \tag{3.11}$$

$$||E|| \leq \frac{(b-a)^2 h^4 M_4}{12 h^2\big(8 - \lambda ||G|| (b-a)^2\big)} \cong O(h^2). \tag{3.12}$$

□

4 Numerical Illustrations

In order to illustrate the performance of the parametric spline method for the Bratu equation (1.2) and justify the accuracy and efficiency of the method, we consider the following examples. The examples have been solved for different values of λ. We take $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$. The errors are reported on the set of uniform grid points

$$S = \{a = t_0, \ldots, t_i, \ldots, t_n = b\},$$

$$t_i = t_0 + ih, \quad i = 0, 1, 2, \ldots, n, \ \ h = \frac{b-a}{n}. \tag{4.1}$$

The absolute error on the uniform grid points S is

$$|u(t_j) - u_n(t_j)|, \quad 1 \leq j \leq n-1, \tag{4.2}$$

where $u(t_j)$ is the exact solution of the given example, and u_j is the computed solution by the parametric cubic spline method. The exact solution of the equation (1.2) is given in [1–3] as:

$$u(t) = -2\ln\left[\frac{\cosh\left((t-\frac{1}{2})\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right], \tag{4.3}$$

where θ satisfies

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right). \tag{4.4}$$

The Bratu problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$ respectively, where the critical value λ_c satisfies the equation

$$1 = \frac{1}{4}\sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right). \tag{4.5}$$

It was evaluated (see [1–5]), that the critical value λ_c is given by $\lambda_c = 3.513830719$. The absolute errors in the computed solutions are tabulated in Tables 1–4.

Example 4.1. *We first consider the Bratu-type model*

$$\begin{aligned} &u''(t) + e^{u(t)} = 0, \quad 0 < t < 1, \\ &u(0) = u(1) = 0. \end{aligned} \tag{4.6}$$

In Eq.(4.6), we have $\lambda = 1$. The numerical results for Example 4.1 are tabulated in Table 1.

For the sake of comparision, we consider the Bratu's problem discussed by Khuri [13], Deeba et al. [22] and Caglar et al. [23]. The authors used the Laplace method, decomposition method and B-spline method to obtain their numerical solution. Table 2 compares the our results for parameters $\lambda = 1$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $n = 10$ with the mentioned methods to same equation.

The exact and approximate solution of Example 4.1 are shown in Figure 1 for $\lambda = 1$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $n = 20$. Figure 1 shows that the approximate solution is indistinguishable (for the given scale) from the exact solution.

Example 4.2. *Consider the second case for Bratu equation as follows when $\lambda = 2$.*

$$u''(t) + 2e^{u(t)} = 0, \quad 0 < t < 1,$$
$$u(0) = u(1) = 0. \tag{4.7}$$

In Table 3, the exact solution for the case $\lambda = 2$ is compared with the numerical solution obtained by the parametric spline method.

Table 4 compares the our results for $\lambda = 2$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $n = 10$ with the methods in [13], [22] and [23].

The exact and approximate solutions of Example 4.2 are shown in Figure 2 for $\lambda = 2$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $n = 20$.

Table 1: Results for Example 4.1: $\lambda = 1$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, $n = 8$.

x	*Exact solution*	*Numerical solution*	*Error*
0.125	0.06068537911	0.06068565922	2.80110×10^{-7}
0.250	0.10478731054	0.10478782429	5.13753×10^{-7}
0.375	0.13156129526	0.13156196444	6.69185×10^{-7}
0.500	0.14053921440	0.14053993814	7.23740×10^{-7}
0.625	0.13156129526	0.13156196444	6.69185×10^{-7}
0.750	0.10478731054	0.10478782429	5.13753×10^{-7}
0.875	0.06068537911	0.06068565922	2.80110×10^{-7}

Table 2: Absolute errors for Example 4.1.

x	Our method	Laplace[13]	Decomposition[22]	B-spline[23]
0.100	9.270×10^{-8}	1.979×10^{-6}	2.685×10^{-3}	2.980×10^{-6}
0.200	1.751×10^{-7}	3.940×10^{-6}	2.022×10^{-3}	5.466×10^{-6}
0.300	2.399×10^{-7}	5.855×10^{-6}	1.523×10^{-4}	7.336×10^{-6}
0.400	2.816×10^{-7}	7.704×10^{-6}	2.202×10^{-3}	8.497×10^{-6}
0.500	2.959×10^{-7}	9.466×10^{-6}	3.02×10^{-3}	8.892×10^{-6}
0.600	2.816×10^{-7}	1.111×10^{-5}	2.202×10^{-3}	8.497×10^{-6}
0.700	2.399×10^{-7}	1.257×10^{-5}	1.523×10^{-4}	7.336×10^{-6}
0.800	1.751×10^{-7}	1.348×10^{-5}	2.022×10^{-3}	5.466×10^{-6}
0.900	9.270×10^{-8}	1.197×10^{-5}	2.685×10^{-3}	2.980×10^{-6}

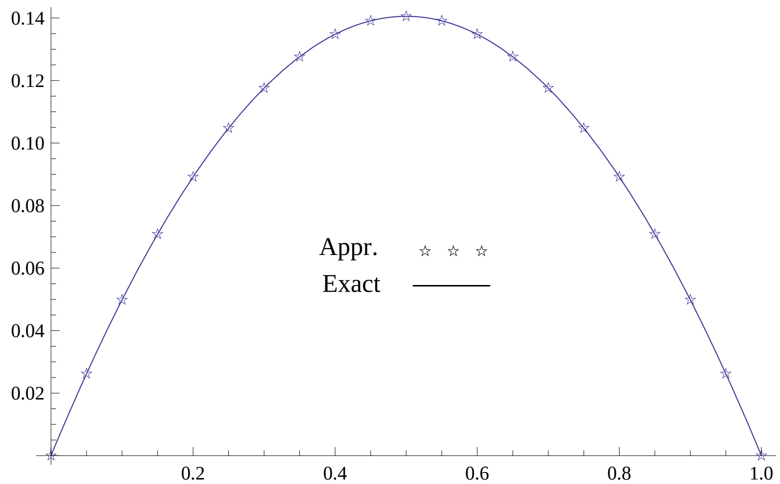

Figure 1: Exact and approximate solutions for Example 1($\lambda = 1$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $n = 20$).

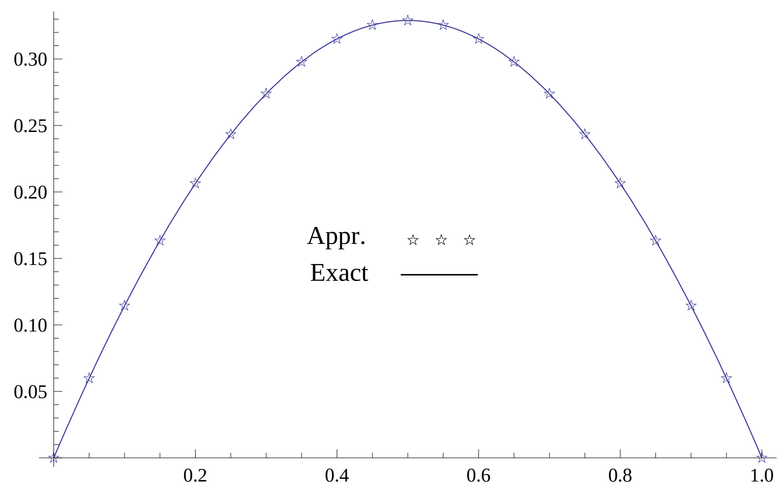

Figure 2: Exact and approximate solutions for Example 2 ($\lambda = 2$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $n = 20$).

Table 3: Results for Example 4.2: $\lambda = 2$, $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, $n = 8$.

x	*Exact solution*	*Numerical solution*	*Error*
0.125	0.13960278219	0.13960597008	3.18789×10^{-6}
0.250	0.24333656779	0.24334289751	6.32972×10^{-6}
0.375	0.30731941062	0.30732808838	8.67776×10^{-6}
0.500	0.32895242134	0.32896197408	9.55274×10^{-6}
0.625	0.30731941062	0.30732808838	8.67776×10^{-6}
0.750	0.24333656779	0.24334289751	6.32972×10^{-6}
0.875	0.13960278219	0.13960597008	3.18789×10^{-6}

Table 4: Absolute errors for Example 4.2.

x	Our method	Laplace[13]	Decomposition[22]	B-spline[23]
0.100	1.034×10^{-6}	2.129×10^{-3}	1.522×10^{-2}	1.718×10^{-5}
0.200	2.092×10^{-6}	4.210×10^{-3}	1.468×10^{-2}	3.260×10^{-5}
0.300	3.023×10^{-6}	6.187×10^{-3}	5.889×10^{-3}	4.490×10^{-5}
0.400	3.667×10^{-6}	8.002×10^{-3}	3.247×10^{-3}	5.286×10^{-5}
0.500	3.898×10^{-6}	9.599×10^{-3}	6.989×10^{-3}	5.561×10^{-5}
0.600	3.667×10^{-6}	1.093×10^{-3}	3.247×10^{-3}	5.286×10^{-5}
0.700	3.023×10^{-6}	1.193×10^{-2}	5.889×10^{-3}	4.490×10^{-5}
0.800	2.092×10^{-6}	1.238×10^{-2}	1.468×10^{-2}	3.260×10^{-5}
0.900	1.034×10^{-6}	1.087×10^{-2}	1.522×10^{-2}	1.718×10^{-5}

5 Conclusion

In this paper a parametric cubic spline method is applied for solving the Bratu equation. The parametric spline method reduce the computation of the Bratu equation to some nonlinear algebraic equations. The analytical results are illustrated with two numerical examples. The proposed scheme is simple and computationally attractive.

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