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On the ϵ-Approximation of the Solution of the Black-Scholes Equation

Amnuay Kananthai

Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200, Thailand
e-mail : malamnka@gmail.com

Abstract : In this paper, we study the well known equation named the Black-Scholes equation. Normally, it is so complicate to find the solution of the Black-Scholes equation which is the option prices directly. But in this work we use the $\epsilon-$approximation to find such option prices and also obtained the interesting kernel related to the interest rate r and the volatility σ of the stock s. Moreover, we obtained the boundedness of the option price in the Sobolev space by giving the suitable initial condition on such option price.

Keywords : the Black-Scholes equation; the Dirac delta function; interest rate and volatility; Fourier transform.
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1 Introduction

In financial mathematics, the famous equation named the Black-Scholes equation plays an important role in solving the option price of stocks. The Black-Scholes equation is given by

$$\frac{\partial u(s,t)}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2 u(s,t)}{\partial s^2} + rs\frac{\partial u(s,t)}{\partial s} - ru(s,t) = 0 \tag{1.1}$$

with the terminal condition

$$u(s,t) = (s-p)^+ \tag{1.2}$$

for $0 \leq t \leq T$ where $u(s,t)$ is the option price at time t, r is the interest rate, s is the price of stock at time t, σ is the volatility of stock and p is the strike price.

They obtain the option price $u(s,t)$ or the solution of (1.1) in the complicated form

$$u(s,t) = s\Phi\frac{\left(ln\frac{s}{p} + \left(r + \frac{\sigma^2}{2}\right)T\right)}{\sigma\sqrt{T}} - pe^{-rT}\Phi\frac{\left(ln\frac{s}{p} - \left(r - \frac{\sigma^2}{2}\right)T\right)}{\sigma\sqrt{T}} \tag{1.3}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy$, see[1, p91].

In this work, we study the solution of (1.1) in the other form. By changing the variable $R = \ln s$, then (1.1) is transformed to

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial R^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial R} - rV = 0 \tag{1.4}$$

where $V = V(R,t) = u(s,t)$. We study the solution of (1.4) for $0 \leq t \leq 1$ with adding the initial condition

$$V(R,0) = f(R) \tag{1.5}$$

where $f \in \mathcal{L}^1(\mathbb{R})$-the space of Lebesgue integrable function.

By applying the Fourier transform to (1.4) we obtain

$$\widehat{V}(\omega,t) = C(\omega)exp\left(\left[\frac{\sigma^2}{2}\omega^2 + \left(r - \frac{\sigma}{2}\right)i\omega + r\right]t\right)$$

and with the condition (1.5)

$$\widehat{V}(\omega,t) = \widehat{f}(\omega)exp\left(\left[\frac{\sigma^2}{2}\omega^2 + \left(r - \frac{\sigma}{2}\right)i\omega + r\right]t\right).$$

Since $\widehat{V}(\omega,t) \notin \mathcal{L}^1(\mathbb{R})$, thus we can not find the inverse Fourier transform $V(R,t)$ of $\widehat{V}(\omega,t)$ we use the $\epsilon-$ approximation $\widehat{V}(\omega,t)$ by defining

$$\widehat{V^\epsilon}(\omega,t) = e^{-\frac{\sigma^2}{2}\omega^2\epsilon}\widehat{V}(\omega,t).$$

Thus

$$\widehat{V^\epsilon}(\omega,t) = e^{rt}\widehat{f}(\omega)e^{-(\epsilon-t)\frac{\sigma^2}{2}\omega^2 + i\left(r-\frac{\sigma}{2}\right)\omega t} \quad \text{for} \quad 0 \leq t < \epsilon < 1.$$

Clearly $\widehat{V^\epsilon}(\omega,t) \to \widehat{V}(\omega,t)$ uniformly as $\epsilon \to 0$, so that $V(R,t)$ will be the limit in the topology of tempered distributions of $V^\epsilon(R,t)$, the inverse Fourier transform of $\widehat{V^\epsilon}(\omega,t)$. Moreover $\widehat{V^\epsilon}(\omega,t) \in \mathcal{L}^1(\mathbb{R})$ for $0 \leq t < \epsilon < 1$. Thus

$$\begin{aligned} V^\epsilon(R,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R}\widehat{V^\epsilon}(\omega,t)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R}e^{\left(\frac{\sigma^2}{2}\omega^2 + \left(r-\frac{\sigma}{2}\right)i\omega + r\right)t - \frac{\sigma^2}{2}\omega^2\epsilon}\widehat{f}(\omega)d\omega \\ &= \frac{e^{rt}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\left[-(\epsilon-t)\left[\omega - \left(\left(r-\frac{\sigma}{2}\right)t + (R-y)\right)\right]i\right]^2} e^{\left(\frac{\left[(R-y)+\left(r-\frac{\sigma}{2}\right)t\right]^2}{2(\epsilon-t)\sigma^2}\right)} f(y)dyd\omega \end{aligned}$$

where $\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega y} f(y)dy$. By direct computation

$$V^{\epsilon}(R,t) = \frac{e^{rt}}{\sqrt{2\pi(\epsilon - t)\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{\left[(R-y)+\left(\frac{\sigma}{2}-r\right)t\right]^2}{2(\epsilon-t)\sigma^2}} f(y)dy \quad \text{for} \quad 0 \leq t < \epsilon < 1$$

or $V^{\epsilon}(R,t) = K^{\epsilon}(R,t) * f(R)$ where

$$K^{\epsilon}(R,t) = \frac{e^{rt}}{\sqrt{2\pi(\epsilon - t)\sigma^2}} e^{-\frac{\left[R+\left(\frac{\sigma}{2}-r\right)t\right]^2}{2(\epsilon-t)\sigma^2}} \quad \text{for} \quad 0 \leq t < \epsilon < 1$$

since $0 \leq t < \epsilon < 1$, thus as $\epsilon \to 0, t = 0$ we have $\lim_{\epsilon \to 0} K^{\epsilon}(R,0) = \delta(R)$ where $\delta(R)$ is the Dirac-delta function. Thus $V(R,0) = \delta(R) * f(R) = f(R)$, as $\epsilon \to 0$ it follow that (1.5) holds.

Moreover, we obtain $V(R,t) = H_k(\mathbb{R}\times[0,1])$ where $H_k(\mathbb{R}\times[0,1])$ is the sobolev space of order k and also $V(R,t) = O\left((\epsilon - t)^{-\frac{1}{2}}\right)$.

Now we have the $\epsilon-$approximation of $V^{\epsilon}(R,t)$ which is the solution (1.4). Thus

$$V^{\epsilon}(\ln s,t) = u^{\epsilon}(s,t) = \frac{e^{rt}}{\sqrt{2\pi(\epsilon - t)\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{\left[(\ln s-y)+\left(\frac{\sigma}{2}-r\right)t\right]^2}{2(\epsilon-t)\sigma^2}} f(y)dy$$

is the solution of (1.1) and

$$V(\ln s,t) = u(s,t) = O\left((\epsilon - t)^{-\frac{1}{2}}\right).$$

2 Preliminaries

In the sequel, we shall need the following definitions

Definition 2.1 Let $f(x) \in \mathcal{L}^1(\mathbb{R})-$the space of integrable function on the set of real $\mathbb{R}$. The Fourier transform $\widehat{f}(\omega)$ of $f(x)$ is defined by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx \tag{2.1}$$

where $\omega, x \in \mathbb{R}$. Also the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \widehat{f}(\omega)d\omega \tag{2.2}$$

Definition 2.2 Let $H_k = H_k(\mathbb{R}^n)$ be the space of the sobolev space of order k on $\mathbb{R}^n$ and is

$$H_k = H_k(\mathbb{R}^n = \left\{f \in \mathcal{L}^2(\mathbb{R}^n) : \partial^{\alpha} f \in \mathcal{L}^2(\mathbb{R}^n)\right\},$$

where k is a nonnegative integer and norm

$$\| f \|_k^2 = \int_{\mathbb{R}^n} | \widehat{f}(\xi) |^2 \left(1 + | \xi^2 |\right)^k d\xi < \infty,$$

$\mathcal{L}^2(\mathbb{R}^n)$ is space of the square integrable in $\mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \alpha_i$ is a non-negative integer, $\partial^\alpha f = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n} f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$.

3 Main Results

Theorem 3.1. *Given the Black-Scholes equation*

$$\frac{\partial u(s,t)}{\partial t} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 u(s,t)}{\partial s^2} + rs \frac{\partial u(s,t)}{\partial s} - ru(s,t) = 0 \tag{3.1}$$

with the initial condition

$$u(s,0) = f(s) \quad \textit{for} \quad 0 \leq t \leq 1 \tag{3.2}$$

where $f(s) \in \mathcal{L}^1(\mathbb{R})-$the space of integrable function of the set $\mathbb{R}$ of real number.

By the method of $\epsilon-$approximation, we obtain

$$u^\epsilon(s,t) = \frac{e^{rt}}{\sqrt{2\pi(\epsilon - t)\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{\left[(\ln s - y) + \left(\frac{\sigma}{2} - r\right)t\right]^2}{2(\epsilon - t)\sigma^2}} f(y) dy \tag{3.3}$$

as the solution of (3.1) for $0 \leq t < \epsilon < 1$ which satisfies (3.2). Moreover,

$$u(s,t) = O\left((\epsilon - t)^{-\frac{1}{2}}\right) \qquad \textit{as } \epsilon \to 0. \tag{3.4}$$

Proof. Let $R = \ln s$, then (3.1) is transformed to

$$\frac{\partial V(R,t)}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V(R,t)}{\partial R^2} + \left(r - \frac{\sigma^2}{2}\right) \frac{\partial V(R,t)}{\partial R} - rV(R,t) = 0 \tag{3.5}$$

where $V(R,t) = V(\ln s, t) = u(s,t$. By applying the Fourier transform in (2.1) we obtain

$$\frac{\partial \widehat{V}(\omega,t)}{\partial t} - \frac{\sigma^2}{2} \omega^2 \widehat{V}(\omega,t) + \left(r - \frac{\sigma}{2}\right)(-i\omega)\widehat{V}(\omega,t) - r\widehat{V}(\omega,t) = 0 \tag{3.6}$$

since $u(s,t) = V(R,t)$ thus the condition(3.2) can be written in the form

$$u(s,0) = V(R,0) = f(R). \tag{3.7}$$

where, $R = \ln s, 1 \leq s$ and $F \in \mathcal{L}^1(\mathbb{R})$. Thus

$$\widehat{V}(\omega,0) = \widehat{f}(\omega) \tag{3.8}$$

we have

$$\widehat{V}(\omega,t) = C(\omega)exp\left(\left[\frac{\sigma^2}{2}\omega^2 + \left(r-\frac{\sigma}{2}\right)i\omega + r\right]t\right)$$

$$\widehat{V}(\omega,t) = \widehat{f}(\omega)exp\left(\left[\frac{\sigma^2}{2}\omega^2 + \left(r-\frac{\sigma}{2}\right)i\omega + r\right]t\right) \tag{3.9}$$

by the condition (3.7).

Since $\widehat{V}(\omega,t) \notin \mathcal{L}^1(\mathbb{R})$, so we can not compute the inverse $V(R,t)$ of the Fourier transform $\widehat{V}(\omega,t)$. Thus we need the $\epsilon-$ approximation to find such inverse. We define

$$\widehat{V^\epsilon}(\omega,t) = e^{-\frac{\sigma^2}{2}\omega^2\epsilon}\widehat{V}(\omega,t) \tag{3.10}$$

where $0 \leq t < \epsilon < 1$. Now $\widehat{V^\epsilon}(\omega,t) \to \widehat{V}(\omega,t)$ uniformly as $\epsilon \to 0$, so that $V(R,t)$ will be the limit in the topology of tempered distribution of $V^\epsilon(R,t)$. Now $\widehat{V^\epsilon}(\omega,t) \in \mathcal{L}^1(\mathbb{R})$, thus

$$\begin{aligned}
V^\epsilon(R,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R}\widehat{V^\epsilon}(\omega,t)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R}e^{\left(\frac{\sigma^2}{2}\omega^2+\left(r-\frac{\sigma}{2}\right)i\omega+r\right)t-\frac{\sigma^2}{2}\omega^2\epsilon}\widehat{f}(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R}e^{-(\epsilon-t)\frac{\sigma^2}{2}\omega^2+i\left(r-\frac{\sigma}{2}\right)\omega t}\widehat{f}(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp\left[-(\epsilon-t)\frac{\sigma^2}{2}\omega^2 + \left(\left(r-\frac{\sigma}{2}\right)t + (R-y)\right)i\omega\right]f(y)dyd\omega \\
&= \frac{e^{rt}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp\left[-(\epsilon-t)\frac{\sigma^2}{2}\left(\omega - \left(\frac{\left(r-\frac{\sigma}{2}\right)t + (R-y)}{(\epsilon-t)\sigma^2}\right)i\right)^2\right] \\
&\qquad exp\frac{\left[-\left((R-y)+\left(r-\frac{\sigma}{2}\right)t\right)^2\right]}{2(\epsilon-t)\sigma^2}f(y)dyd\omega
\end{aligned}$$

for $0 \leq t < \epsilon < 1$. Let $u = \sqrt{\frac{\epsilon-t}{2}}\cdot\sigma\left[\omega - \frac{\left(\left(r-\frac{\sigma}{2}\right)t+(R-y)\right)i}{(\epsilon-t)\sigma^2}\right]$, then $du = \sigma\sqrt{\frac{\epsilon-t}{2}}d\omega$, $d\omega = \frac{1}{\sigma}\sqrt{\frac{2}{\epsilon-t}}$. Thus

$$\begin{aligned}
V^\epsilon(R,t) &= \frac{e^{rt}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2}\cdot\frac{1}{\sigma}\sqrt{\frac{2}{\epsilon-t}}du\, exp\frac{\left[-\left((R-y)+\left(r-\frac{\sigma}{2}\right)t\right)^2\right]}{2(\epsilon-t)\sigma^2}f(y)dy \\
&= \frac{e^{rt}}{2\pi}\frac{1}{\sigma}\sqrt{\frac{2}{\epsilon-t}}\sqrt{\pi}\int_{-\infty}^{\infty} exp\frac{\left[-\left((R-y)+\left(r-\frac{\sigma}{2}\right)t\right)^2\right]}{2(\epsilon-t)\sigma^2}f(y)dy
\end{aligned}$$

$$= \frac{e^{rt}}{\sqrt{2\pi(\epsilon - t)\sigma^2}} \int_{-\infty}^{\infty} exp \frac{\left[-\left((R-y) + \left(r - \frac{\sigma}{2}\right)t\right)^2\right]}{2(\epsilon - t)\sigma^2} f(y)dy$$

for $0 \leq t < \epsilon < 1$.

Now $R = \ln s$, thus $V^\epsilon(R,t) = V^\epsilon(\ln s, t) = u^\epsilon(s,t)$

$$u^\epsilon(s,t) = \frac{e^{rt}}{\sqrt{2\pi(\epsilon - t)\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-\left[\left((\ln s - y) + \left(\frac{\sigma}{2} - r\right)t\right)^2\right]}{2(\epsilon - t)\sigma^2}} f(y)dy.$$

Thus, we obtain (3.3)as required. Since we have

$$\widehat{V}(\omega,t) = \widehat{f}(\omega) exp\left(\left[\frac{\sigma^2}{2}\omega^2 + \left(r - \frac{\sigma}{2}\right)i\omega + r\right]t\right)$$

$$\widehat{V^\epsilon}(\omega,t) = e^{-\frac{\sigma^2}{2}\omega^2\epsilon}\widehat{V}(\omega,t) \quad \text{for} \quad 0 \leq t < \epsilon < 1.$$

Thus

$$\begin{aligned} V^\epsilon(R,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R}\widehat{V^\epsilon}(\omega,t)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R} e^{\left(\frac{\sigma^2}{2}\omega^2 + \left(r - \frac{\sigma}{2}\right)i\omega + r\right)t - \frac{\sigma^2}{2}\omega^2\epsilon}\widehat{f}(\omega)d\omega \end{aligned}$$

$$\begin{aligned} \mid V^\epsilon(R,t) \mid &\leq \frac{1}{2\pi} \cdot e^{rt} \int_{-\infty}^{\infty} \mid \widehat{f}(\omega) \mid e^{-(\epsilon - t)\frac{\sigma^2}{2}\omega^2} d\omega \\ &\leq \frac{M}{2\pi} \cdot e^{rt} \int_{-\infty}^{\infty} e^{-(\epsilon - t)\frac{\sigma^2}{2}\omega^2} d\omega \quad \text{for} \quad 0 \leq t < \epsilon < 1 \end{aligned}$$

where $M = \max_{\omega \epsilon \mathbb{R}} \mid \widehat{f}(\omega) \mid$. Put $W = \sqrt{\frac{\epsilon - t}{2}}\sigma\omega$, then $d\omega = \frac{1}{\sigma}\sqrt{\frac{2}{\epsilon - t}}dW$. Thus

$$\begin{aligned} \mid V^\epsilon(R,t) \mid &\leq \frac{M}{2\pi} \cdot e^{rt} \int_{-\infty}^{\infty} e^{-W^2} dW \cdot \frac{1}{\sigma}\sqrt{\frac{2}{\epsilon - t}} \\ &= \frac{Me^{rt}}{2\pi} \cdot \frac{1}{\sigma}\sqrt{\frac{2}{\epsilon - t}} \cdot \sqrt{\pi} \\ &= \frac{M}{\sigma} \cdot \frac{e^{rt}}{\sqrt{2\pi(\epsilon - t)}} \qquad \text{for} \quad 0 \leq t < \epsilon < 1. \end{aligned}$$

Thus for any fixed t, $\left(\sqrt{\epsilon - t}\right) \mid V^\epsilon(R,t) \mid \leq \frac{Me^{rt}}{\sigma\sqrt{2\pi}}$, then for $\epsilon \to 0$, $V(R,t) = O\left((\epsilon - t)^{-\frac{1}{2}}\right)$. Since $V(R,t) = V(\ln s, t) = u(s,t)$ where $R = \ln s$, thus $u(s,t) = O\left((\epsilon - t)^{-\frac{1}{2}}\right)$ as $\epsilon \to 0$.

Now, (3.3) can be written in the convolution form $u^\epsilon(s,t) = K^\epsilon(s,t) * f(s)$, where

$$K^\epsilon(s,t) = \frac{e^{rt}}{\sqrt{2\pi(\epsilon-t)\sigma^2}} \cdot e^{\frac{-\left[\left((\ln s)+\left(\frac{\sigma}{2}-r\right)t\right)^2\right]}{2(\epsilon-t)\sigma^2}}$$

is the kernel of (1.1). It can be shown that $\lim_{\epsilon\to 0} K^\epsilon(s,0) = \delta(s)$, [see 2, pp 36-37].

Since $0 \le t < \epsilon < 1$, as $\epsilon \to 0, t = 0$. Now, $u^\epsilon(s,t) \to u(s,t)$ uniformly as $\epsilon \to 0$, since $\widehat{u}^\epsilon(\omega,t) = e^{-\frac{\sigma^2}{2}\omega^2\epsilon}\widehat{u}(\omega,t) \to \widehat{u}(\omega,t)$ uniformly. $u(s,0) = \lim_{\epsilon\to 0} u^\epsilon(s,t) = \delta(s) * f(s) = f(s)$. Thus (3.2) hold for $0 \le t \le 1$. It follows that $u^\epsilon(s,t)$ in (3.3) is the solution of (3.1) which satisfies (3.2). ☐

Corollary 3.2. *In (3.3) of theorem (3.1), the conditions on f are given as follows.*

(i) If f is a bounded function on $\mathbb{R}$, then $| u^\epsilon(s,t) | \le Me^{rt}$ for $0 \le t < \epsilon < 1$ where $M = \max | f(s) |$,

(ii) If $f \in H_3(\mathbb{R})-$the sobolev space of order 3 which is given in definition (2.2) with k=3. Then $u^\epsilon(s,t) \in H_3(\mathbb{R} \times [0,\infty))$ with the sobolev norm

$$\| u^\epsilon(s,t) \|_3 = \left(\int_{\mathbb{R}} | \widehat{u}(\omega,t) |^2 \left(1 + | \omega^2 |\right)^3 d\omega \right)^{1/2} \quad for \ 0 \le t < \epsilon < 1.$$

Proof. (i) From (3.3)

$$| u^\epsilon(s,t) | \le M \frac{e^{rt}}{\sqrt{2\pi(\epsilon-t)\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{\left[(y-\ln s)-\left(\frac{\sigma}{2}-r\right)t\right]^2}{2(\epsilon-t)\sigma^2}} dy$$

where $M = \max | f(s) |$.

Put $W = \frac{(y-\ln s)-\left(\frac{\sigma}{2}-r\right)t}{\sqrt{2(\epsilon-t)}\sigma}$, then $dW = \frac{1}{\sqrt{2(\epsilon-t)}\sigma}$, $dy = \sigma\sqrt{2(\epsilon-t)}dW$. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\frac{\left[(y-\ln s)-\left(\frac{\sigma}{2}-r\right)t\right]^2}{2(\epsilon-t)\sigma^2}} dy &= \sigma\sqrt{2(\epsilon-t)} \int_{-\infty}^{\infty} e^{-W^2} dW \\ &= \sigma\sqrt{2(\epsilon-t)}\sqrt{\pi} \\ &= \sigma\sqrt{2\pi(\epsilon-t)}.\end{aligned}$$

Then

$$\begin{aligned}| u^\epsilon(s,t) | &\le M \frac{e^{rt}}{\sqrt{2\pi(\epsilon-t)\sigma^2}} \cdot \sqrt{2\pi(\epsilon-t)\sigma^2} \\ &= Me^{rt} \qquad for \ 0 \le t < \epsilon < 1.\end{aligned}$$

That means that the option price $u^\epsilon(s,t)$ is bounded by the values of the money M with the interest rate of r for the time t $(0 \le t \le 1)$.

(ii) By the plancherel theorem, $f \in H_3(\mathbb{R})$ if and only if $\left(1+ \mid \omega \mid^2\right)^3 \widehat{f}(\omega) \in \mathcal{L}^2(\mathbb{R})$. Now, from (3.9) and (3.10)

$$\widehat{V^\epsilon}(\omega, t) = \widehat{f}(\omega) exp\left[-(\epsilon - t)\frac{\sigma^2}{2}\omega^2 + i\left(r - \frac{\sigma}{2}\right)\omega t\right].$$

We know that $\widehat{\frac{\partial^2}{\partial R^2} \cdot \frac{\partial}{\partial t} V^\epsilon}(\omega, t) = (-i\omega^2)\frac{\partial}{\partial t}\widehat{V^\epsilon}(\omega, t)$. Thus

$$\begin{aligned}\widehat{\frac{\partial^2}{\partial R^2} \cdot \frac{\partial}{\partial t} V^\epsilon}(\omega, t) &= -\omega^2 \widehat{f}(\omega)\left[i\left(r - \frac{\sigma}{2}\right)\right] exp\left[-(\epsilon - t)\frac{\sigma^2}{2}\omega^2 + i\left(r - \frac{\sigma}{2}\right)\omega t\right] \\ &= -\omega^3 i\left(r - \frac{\sigma}{2}\right) exp\left[-(\epsilon - t)\frac{\sigma^2}{2}\omega^2 + i\left(r - \frac{\sigma}{2}\right)\omega t\right]\widehat{f}(\omega).\end{aligned}$$

Now, $exp\left[-(\epsilon - t)\frac{\sigma^2}{2}\omega^2\right] \leq \omega^2$. Thus

$$\begin{aligned}-\omega^3 i\left(r - \frac{\sigma}{2}\right) exp\left[-(\epsilon - t)\frac{\sigma^2}{2}\omega^2\right] exp\left[i\left(r - \frac{\sigma}{2}\right)\omega t\right]\widehat{f}(\omega)& \\ \leq \omega^5 \mid\mid r - \frac{\sigma}{2} \mid\mid \widehat{f}(\omega) \mid& \\ \leq r - \frac{\sigma}{2} \mid \left(1+ \mid \omega \mid^2\right)^3 \mid \widehat{f}(\omega) \mid .&\end{aligned}$$

Since $f(R) \in H_3(\mathbb{R})$, thus $\left(r - \frac{\sigma}{2}\right)\left(1+ \mid \omega \mid^2\right)^3 \widehat{f}(\omega) \in L^2(\mathbb{R})$. It follows that $\widehat{\frac{\partial^2}{\partial R^2} \cdot \frac{\partial}{\partial t} V^\epsilon}(\omega, t) \in L^2(\mathbb{R})$. By the plancherel theorem $\frac{\partial^2}{\partial R^2} \cdot \frac{\partial}{\partial t} V^\epsilon(R, t) \in L^2(\mathbb{R})$. It follow that $V^\epsilon(R, t) \in H_3(\mathbb{R} \times [0, \infty))$. Since $V^\epsilon(R, t) = V^\epsilon(\ln s, t) = u^\epsilon(s, t)$, thus $u^\epsilon(s, t) \in H_3(\mathbb{R} \times [0, \infty))$ with the sobolev norm

$$\| u^\epsilon(s, t) \|_3 = \left(\int_{\mathbb{R}} \mid \widehat{u}(\omega, t) \mid^2 \left(1+ \mid \omega \mid^2\right)^3 d\omega\right)^{1/2}$$

for $0 \leq t < \epsilon < 1$. □

4 Conclusion

In the main results of this paper that we obtained the solution of *(3.3)* which is the solution focusing on the mathematical sense, particularly in the financial mathematics area. But, the solution of *(1.3)* which is the Black-Scholes formula that can be applied in the real world application which is different purpose compare with *(3.3)*.

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References

[1] M. Baxter, A. Rennie, *Financial Calculus*, Cambridge University Press, 1998.

[2] I.M. Gel'fand, G.E. Shilov, *Generalized Function*, Vol 1, Academic Press, 1972.

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