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Some Structural Properties of Vector Valued Orlicz Sequence Space

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Abstract : In this work, we introduce the vector valued sequence space $F(X_k, M, p, s)$ and study the closed subspace of it. We examine various algebraic and topological properties of this space and also investigate some inclusion relations on it.

Keywords : Orlicz function, Orlicz sequence space, vector valued sequence space, paranormed space.
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1 Introduction

Orlicz sequence spaces are one of the most natural generalizations of classical spaces ℓ_p, $p \geqslant 1$. They were first considered by W. Orlicz in 1936. Afterwards, J. Lindenstrauss and L. Tzafriri [4] used the idea of Orlicz function M to construct the sequence space ℓ_M of all sequences of scalars (x_n) such that

$$\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0.$$

The space ℓ_M becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$, $(1 \leqslant p < \infty)$. In the present note, we introduce and examine some properties of a sequence space defined by using Orlicz function M, which generalizes the well known Orlicz sequence space ℓ_M. Before introducing this sequence space, let us give some basic concepts :

An algebra X is a linear space together with an internal operation of multiplication of elements of X, such that $xy \in X$, $x(yz) = (xy)z$, $x(y+z) = xy + xz$, $(x+y)z = xz + yz$ and $\lambda(xy) = (\lambda x)y = x(\lambda y)$, for any scalar λ, and a normed algebra is an algebra which is normed, as a linear space, and in which $\|xy\| \leqslant \|x\| \, \|y\|$ for all x, y; [6].

Let F be a sequence space and x, y be the arbitrary elements of F. Then F is called a sequence algebra if it is closed under the multiplication defined by $xy = (x_k y_k)$. The space F is called normal or solid if $y = (y_k) \in F$ whenever

$|y_k| \leqslant |x_k|$, $k \in \mathbb{N}$, for some $x = (x_k) \in F$. If F is both normal and sequence algebra then it is called a normal sequence algebra. For example, w, ℓ_∞, c_0 and ℓ_p $(0 < p < \infty)$ are normal sequence algebras. c is a sequence algebra but not normal.

A norm $\|\cdot\|$ on a normal sequence space F is said to be absolutely monotone if $x = (x_k), y = (y_k) \in F$ and $|x_k| \leqslant |y_k|$ for all $k \in \mathbb{N}$ implies $\|x\| \leqslant \|y\|$, [5]. The norm

$$\|x\|_\infty = \sup |x_k|$$

over ℓ_∞, c, c_0 and the norm

$$\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$$

over ℓ_p for $p \geqslant 1$ are absolutely monotone.

We recall [3, 4] that an Orlicz function is a function $M\colon [0, \infty) \longrightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for all $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function M can always be represented in the following integral form:

$$M(x) = \int_0^x p(t)\, dt,$$

where p, known as the kernel of M.

We remark that $M_1 + M_2$ and $M_1 \circ M_2$ are Orlicz functions when M_1 and M_2 are Orlicz functions.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leqslant KM(u)$, $u \geqslant 0$. It is easy to see always that $K > 2$. The Δ_2-condition is equivalent to the inequality $M(\ell u) \leqslant K(\ell) M(u)$ which holds for all values of u and $\ell > 1$; [3].

We now introduce the vector valued sequence space $F(X_k, M, p, s)$ using Orlicz function M.

Let X_k be seminormed space over the complex field $\mathbb{C}$ with seminorm q_k for each $k \in \mathbb{N}$, and F be a normal sequence algebra with absolutely monotone norm $\|\cdot\|_F$ and having a Schauder basis (e_k), where $e_k = (0, \ldots, 0, 1, 0, \ldots)$, with 1 in k-th place. Let $p = (p_k)$ be any sequence of strictly positive real numbers and s be any non-negative real number. By $s(X_k)$, we denote the linear space of all sequences $x = (x_k)$ with $x_k \in X_k$ for each $k \in \mathbb{N}$ under the usual coordinatewise operations:

$$\alpha x = (\alpha x_k) \quad \text{and} \quad x + y = (x_k + y_k)$$

for each $\alpha \in \mathbb{C}$. Let $x \in s(X_k)$ and $\lambda = (\lambda_k)$ is a scalar sequence such that

$\lambda x = (\lambda_k x_k)$. We define for an Orlicz function M,

$$F(X_k, M, p, s) = \Big\{ x = (x_k) \in s(X_k) : x_k \in X_k \text{ for each } k \text{ and } \left(k^{-s} \left[M\left(\frac{q_k(x_k)}{\rho} \right) \right]^{p_k} \right) \in F \text{ for some } \rho > 0 \Big\}.$$

Y. Yılmaz, M. K. Özdemir and İ. Solak [8] introduced a generalization of Minkowski Inequality to normal sequence algebras with absolutely monotone seminorm. We will use Lemma 1 which states this extension to put forward a topology of the space $F(X_k, M, p, s)$. For $x = (x_k) \in F(X_k, M, p, s)$, we define

$$g(x) = \inf \left\{ \rho^{p_n/H} > 0 \colon \left\| \left(k^{-s} \left[M\left(\frac{q_k(x_k)}{\rho} \right) \right]^{p_k} \right) \right\|_F^{1/H} \leqslant 1, \, n \in \mathbb{N} \right\}, \quad (1.1)$$

where $H = \max(1, \sup p_k)$. It is shown that $F(X_k, M, p, s)$ turns out to be a complete paranormed space with the paranorm defined by (1.1) whenever the seminormed space X_k is complete under the seminorm q_k for each $k \in \mathbb{N}$.

It can be seen that for suitable choice of the sequence space F, the seminormed space X_k, the sequence of strictly positive real numbers (p_k), $s \geqslant 0$ and Orlicz function M, the space $F(X_k, M, p, s)$ reduces to the many number of known ordinary sequence spaces and as well as vector valued sequence spaces, as a particular case. For example, choosing F to be ℓ_1, $X_k = X$ (a vector space over $\mathbb{C}$) and $q_k = q$ to be a seminorm on X in $F(X_k, M, p, s)$ one gets the scalar valued sequence space $\ell_M(p, q, s)$ defined by Ç. A. Bektaş & Y. Altın [1].

If X_k is normed space, $p_k = 1$ for each $k \in \mathbb{N}$ and $s = 0$, then the class $F(X_k, M, p, s)$ gives the class $F(X_k, M)$ defined by D. Ghosh & P. D. Srivastava [2]. Furthermore, if $F = \ell_1$, $X_k = \mathbb{C}$ and $s = 0$ in $F(X_k, M, p, s)$, then one obtains the space $\ell_M(p)$ defined by S. D. Parashar & B. Choudhary [7]. Thus, the generalized sequence space $F(X_k, M, p, s)$ yields several spaces studied by several authors.

2 Linear Topological Structure of $F(X_k, M, p, s)$

Now, we examine some algebraic and topological properties of $F(X_k, M, p, s)$ and investigate some inclusion relations on it. In order to discuss the properties of $F(X_k, M, p, s)$, we assume that (p_k) is bounded. We will henceforth denote by h and C, the real numbers $\sup p_k$ and $\max(1, 2^{h-1})$, respectively.

Theorem 2.1 *$F(X_k, M, p, s)$ is a linear space over the complex field $\mathbb{C}$.*

Proof. Let $x = (x_k), y = (y_k) \in F(X_k, M, p, s)$ and $\alpha, \beta \in \mathbb{C}$. So, there exist $\rho_1, \rho_2 > 0$ such that

$$\left(k^{-s} \left[M\left(\frac{q_k(x_k)}{\rho_1} \right) \right]^{p_k} \right), \left(k^{-s} \left[M\left(\frac{q_k(y_k)}{\rho_2} \right) \right]^{p_k} \right) \in F.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing and convex,

$$\begin{aligned} k^{-s}\left[M\left(\frac{q_k(\alpha x_k+\beta y_k)}{\rho_3}\right)\right]^{p_k} &\leqslant k^{-s}\left[M\left(|\alpha|\frac{q_k(x_k)}{\rho_3}+|\beta|\frac{q_k(y_k)}{\rho_3}\right)\right]^{p_k} \\ &\leqslant k^{-s}\left[M\left(\frac{q_k(x_k)}{\rho_1}\right)+M\left(\frac{q_k(y_k)}{\rho_2}\right)\right]^{p_k} \\ &\leqslant C\left\{k^{-s}\left[M\left(\frac{q_k(x_k)}{\rho_1}\right)\right]^{p_k}\right. \\ &\qquad \left.+k^{-s}\left[M\left(\frac{q_k(y_k)}{\rho_2}\right)\right]^{p_k}\right\}. \end{aligned}$$

Since F is a normal space, we have

$$\left(k^{-s}\left[M\left(\frac{q_k(\alpha x_k+\beta y_k)}{\rho_3}\right)\right]^{p_k}\right) \in F$$

which shows that $\alpha x+\beta y \in F(X_k, M, p, s)$. □

Theorem 2.2 *$F(X_k, M, p, s)$ is a topological linear space, paranormed by*

$$g(x) = \inf\left\{\rho^{p_n/H} > 0\colon \left\|\left(k^{-s}\left[M\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}\right)\right\|_F^{1/H} \leqslant 1,\, n \in \mathbb{N}\right\},$$

where $H = \max(1, h)$.

To prove this theorem we need the following lemma.

Lemma 1 *Let F be a normal sequence algebra, $\|\cdot\|_F$ be an absolutely monotone seminorm on F and let $p > 1$. Then*

$$\|(u+v)^p\|_F^{1/p} \leqslant \|u^p\|_F^{1/p} + \|v^p\|_F^{1/p},$$

for every $u = (u_n)$, $v = (v_n) \in F$; [8].

Proof. [Proof of Theorem 2.2] Let $x = (x_k)$, $y = (y_k) \in F(X_k, M, p, s)$. It is easy to see that $g(x) = g(-x)$ and $g(\theta) = 0$ for $\theta = (\theta_1, \theta_2, \ldots)$ the null element of $F(X_k, M, p, s)$ (where θ_i is the zero element of X_i for each i).

We shall now show the subadditivity of g. By taking $\alpha = \beta = 1$ in Theorem 2.1, we have

$$\begin{aligned}k^{-s}\left[M\left(\frac{q_k\left(x_k+y_k\right)}{\rho_3}\right)\right]^{p_k} &\leqslant k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{\rho_1}\right)+M\left(\frac{q_k\left(y_k\right)}{\rho_2}\right)\right]^{p_k}\\ &=\left(k^{-s/H}\left[M\left(\frac{q_k\left(x_k\right)}{\rho_1}\right)+M\left(\frac{q_k\left(y_k\right)}{\rho_2}\right)\right]^{p_k/H}\right)^H\\ &\leqslant\left(k^{-s/H}\left[M\left(\frac{q_k\left(x_k\right)}{\rho_1}\right)\right]^{p_k/H}\right.\\ &\qquad\left.+k^{-s/H}\left[M\left(\frac{q_k\left(y_k\right)}{\rho_2}\right)\right]^{p_k/H}\right)^H.\end{aligned}$$

Considering Lemma 1, we get

$$\begin{aligned}\left\|\left(k^{-s}\left[M\left(\frac{q_k\left(x_k+y_k\right)}{\rho_3}\right)\right]^{p_k}\right)\right\|_F^{1/H} &\leqslant\left\|\left(k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{\rho_1}\right)\right]^{p_k}\right)\right\|_F^{1/H}\\ &\quad+\left\|\left(k^{-s}\left[M\left(\frac{q_k\left(y_k\right)}{\rho_2}\right)\right]^{p_k}\right)\right\|_F^{1/H}\end{aligned}$$

which means that $g(x+y)\leqslant g(x)+g(y)$.

Finally, we show that the scalar multiplication is continuous. Let λ be any complex number. By (1.1), we have

$$g(\lambda x)=\inf\left\{\rho^{p_n/H}>0\colon\ \left\|\left(k^{-s}\left[M\left(\frac{q_k\left(\lambda x_k\right)}{\rho}\right)\right]^{p_k}\right)\right\|_F^{1/H}\leqslant 1,\ n\in\mathbb{N}\right\}.$$

Then

$$g(\lambda x)=\inf\left\{\left(|\lambda|\,r\right)^{p_n/H}>0\colon\ \left\|\left(k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{r}\right)\right]^{p_k}\right)\right\|_F^{1/H}\leqslant 1,\ n\in\mathbb{N}\right\},$$

where $r=\rho/|\lambda|$. Since $|\lambda|^{p_n}\leqslant\max\left(1,|\lambda|^{\sup p_n}\right)$, we have

$$\begin{aligned}g(\lambda x)&=\max\left(1,|\lambda|^{\sup p_n}\right)^{1/H}\\ &\quad.\inf\left\{r^{p_n/H}>0\colon\ \left\|\left(k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{r}\right)\right]^{p_k}\right)\right\|_F^{1/H}\leqslant 1,\ n\in\mathbb{N}\right\},\end{aligned}$$

which converges to zero whenever x converges to zero in $F(X_k,M,p,s)$.

Suppose that $\lambda_n\to 0$ and x is fixed in $F(X_k,M,p,s)$. Then,

$$t=(t_k)=\left(k^{-s}\left[M\left(\frac{q_k\left(x_k\right)}{\rho}\right)\right]^{p_k}\right)\in F$$

for some $\rho > 0$. For arbitrary $\varepsilon > 0$, let N be a positive integer such that

$$\left\| t - \sum_{k=1}^{N} t_k e_k \right\|_F = \left\| \sum_{k=N+1}^{\infty} t_k e_k \right\|_F < \left(\frac{\varepsilon}{2}\right)^H,$$

since (e_k) is a Schauder basis for F. Let $0 < |\lambda| < 1$, using convexity of M and absolutely monotonicity of $\|\cdot\|_F$ we get

$$\begin{aligned} \left\| \sum_{k=N+1}^{\infty} k^{-s} \left[M\left(\frac{q_k(\lambda x_k)}{\rho}\right)\right]^{p_k} e_k \right\|_F &\leqslant \left\| \sum_{k=N+1}^{\infty} k^{-s} \left[|\lambda| M\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k} e_k \right\|_F \\ &< \left(\frac{\varepsilon}{2}\right)^H. \end{aligned}$$

Since M is continuous everywhere in $[0,\infty)$, then

$$f(u) =: \left\| \sum_{k=1}^{N} k^{-s} \left[M\left(\frac{q_k(u x_k)}{\rho}\right)\right]^{p_k} e_k \right\|_F$$

is continuous at 0. So there is $0 < \delta < 1$ such that $f(u) < (\varepsilon/2)^H$ for $0 < u < \delta$. Let K be a positive integer such that $|\lambda_n| < \delta$ for $n > K$, then for $n > K$

$$\left\| \sum_{k=1}^{N} k^{-s} \left[M\left(\frac{q_k(\lambda_n x_k)}{\rho}\right)\right]^{p_k} e_k \right\|_F^{1/H} < \frac{\varepsilon}{2}.$$

Thus

$$\left\| \left(k^{-s} \left[M\left(\frac{q_k(\lambda_n x_k)}{\rho}\right)\right]^{p_k} \right) \right\|_F^{1/H} < \frac{\varepsilon}{2}$$

for $n > K$, so that $g(\lambda x) \to 0$ as $\lambda \to 0$.

This completes the proof of Theorem 2.2. □

Remark 1 It can be easily verified that when $F = \ell_1$, $(X_k, q_k) = (\mathbb{C}, |\cdot|)$, $p_k = 1$ for each $k \in \mathbb{N}$ and $s = 0$ the paranorms defined on $F(X_k, M, p, s)$ and $\ell_M(p)$ are the same, and also taking $q_k = \|\cdot\|_{X_k}$, $p_k = 1$ for each $k \in \mathbb{N}$ and $s = 0$ in (1.1), one obtains the norm of $F(X_k, M)$.

Theorem 2.3 *$F(X_k, M, p, s)$ is complete with the paranorm (1.1) if X_k is complete under the seminorm q_k for each $k \in \mathbb{N}$.*

Proof. Let (x^i) be any Cauchy sequence in $F(X_k, M, p, s)$. We get by (1.1) that

$$\left\| \left(k^{-s} \left[M\left(\frac{q_k\left(x_k^i - x_k^j\right)}{g(x^i - x^j)}\right)\right]^{p_k} \right) \right\|_F^{1/H} \leqslant 1.$$

Since F is a normal space and (e_k) is a Schauder basis of F, it follows that

$$k^{-s}\left[M\left(\frac{q_k(x_k^i-x_k^j)}{g(x^i-x^j)}\right)\right]^{p_k}\|e_k\|_F \leqslant \left\|\left(k^{-s}\left[M\left(\frac{q_k(x_k^i-x_k^j)}{g(x^i-x^j)}\right)\right]^{p_k}\right)\right\|_F \leqslant 1.$$

We choose γ with $\gamma^H\|e_k\|_F > 1$ and $x_0 > 0$, such that

$$\gamma^H\|e_k\|_F\frac{x_0^H}{2}\left[p\left(\frac{x_0}{2}\right)\right]^{p_k} \geqslant 1,$$

where p is the kernel associated with M. Hence,

$$k^{-s}\left[M\left(\frac{q_k(x_k^i-x_k^j)}{g(x^i-x^j)}\right)\right]^{p_k}\|e_k\|_F \leqslant \gamma^H\|e_k\|_F\frac{x_0^H}{2}\left[p\left(\frac{x_0}{2}\right)\right]^{p_k}$$

for each $k \in \mathbb{N}$. Using the integral representation of Orlicz function M, we get

$$k^{-s}\left[q_k(x_k^i-x_k^j)\right]^{p_k} \leqslant \gamma^H x_0^H\left[g(x^i-x^j)\right]^H. \tag{2.1}$$

For given $\varepsilon > 0$, we choose an integer i_0 such that

$$g(x^i-x^j) < \frac{\varepsilon^{1/H}}{\gamma x_0} \text{ for all } i,j > i_0. \tag{2.2}$$

¿From (2.1) and (2.2) we get

$$k^{-s}\left[q_k(x_k^i-x_k^j)\right]^{p_k} < \varepsilon \text{ for all } i,j > i_0$$

and so,

$$q_k(x_k^i-x_k^j) < \varepsilon \text{ for all } i,j > i_0.$$

Hence, there exists a sequence $x = (x_k)$ such that $x_k \in X_k$ for each $k \in \mathbb{N}$ and

$$q_k(x_k^i-x_k) < \varepsilon \text{ as } i \to \infty,$$

for each fixed $k \in \mathbb{N}$. For given $\varepsilon > 0$, choose an integer $n > 1$ such that $g(x^i-x^j) < \varepsilon/2$, for all $i,j > n$ and a $\rho > 0$, such that $g(x^i-x^j) < \rho < \varepsilon/2$. Since F is a normal space and (e_k) is a Schauder basis of F,

$$\left\|\sum_{k=1}^{n}k^{-s}\left[M\left(\frac{q_k(x_k^i-x_k^j)}{\rho}\right)\right]^{p_k}e_k\right\|_F \leqslant \left\|\left(k^{-s}\left[M\left(\frac{q_k(x_k^i-x_k^j)}{\rho}\right)\right]^{p_k}\right)\right\|_F \leqslant 1.$$

Since M is continuous, so by taking $j \to \infty$ and $i,j > n$ in the above inequality we get

$$\left\|\sum_{k=1}^{n}k^{-s}\left[M\left(\frac{q_k(x_k^i-x_k)}{2\rho}\right)\right]^{p_k}e_k\right\|_F < 1.$$

Letting $n \to \infty$, we get $g(x^i - x) < 2\rho < \varepsilon$ for all $i > n$. That is to say that $\left(x^i\right)$ converges to x in the paranorm of $F(X_k, M, p, s)$. Now, we should show that $x \in F(X_k, M, p, s)$. Since $x^i = \left(x_k^i\right) \in F(X_k, M, p, s)$, there exists a $\rho > 0$ such that

$$\left(k^{-s}\left[M\left(\frac{q_k(x_k^i)}{\rho}\right)\right]^{p_k}\right) \in F.$$

Since $q_k(x_k^i - x_k) \to 0$ as $i \to \infty$, for each fixed k we can choose a positive number δ_k^i satisfying $0 < \delta_k^i < 1$ such that

$$k^{-s}\left[M\left(\frac{q_k(x_k^i - x_k)}{\rho}\right)\right]^{p_k} < \delta_k^i k^{-s}\left[M\left(\frac{q_k(x_k^i)}{\rho}\right)\right]^{p_k}.$$

Consider

$$\begin{aligned} M\left(\frac{q_k(x_k)}{2\rho}\right) &= M\left(\frac{q_k(x_k^i + x_k - x_k^i)}{2\rho}\right) \\ &\leqslant M\left(\frac{q_k(x_k^i)}{\rho}\right) + M\left(\frac{q_k(x_k^i - x_k)}{\rho}\right) \end{aligned}$$

Hence,

$$\begin{aligned} k^{-s}\left[M\left(\frac{q_k(x_k)}{2\rho}\right)\right]^{p_k} &\leqslant k^{-s}\left[M\left(\frac{q_k(x_k^i)}{\rho}\right) + M\left(\frac{q_k(x_k^i - x_k)}{\rho}\right)\right]^{p_k} \\ &\leqslant Ck^{-s}\left\{\left[M\left(\frac{q_k(x_k^i)}{\rho}\right)\right]^{p_k} + \left[M\left(\frac{q_k(x_k^i - x_k)}{\rho}\right)\right]^{p_k}\right\} \\ &< C\left(1 + \delta_k^i\right) k^{-s}\left[M\left(\frac{q_k(x_k^i)}{\rho}\right)\right]^{p_k}. \end{aligned}$$

Since F is normal,

$$\left(k^{-s}\left[M\left(\frac{q_k(x_k)}{2\rho}\right)\right]^{p_k}\right) \in F,$$

that is, $x = (x_k) \in F(X_k, M, p, s)$. This step completes the proof. □

Theorem 2.4 *Let M and M_1 be two Orlicz functions. If M satisfies the Δ_2-condition, then*

$$F\left(X_k, M_1, p, s\right) \subseteq F\left(X_k, M \circ M_1, p, s\right).$$

Proof. Let $x \in F(X_k, M_1, p, s)$. Then

$$\left(k^{-s}\left[M_1\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}\right) \in F$$

for some $\rho > 0$. Since M satisfies the Δ_2-condition, we have

$$\begin{aligned} k^{-s}\left[M\left(M_1\left(\frac{q_k(x_k)}{\rho}\right)\right)\right]^{p_k} &\leqslant k^{-s}\left[KM_1\left(\frac{q_k(x_k)}{\rho}\right)M(1)\right]^{p_k} \\ &\leqslant \max\left(1, [KM(1)]^h\right) k^{-s}\left[M_1\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}. \end{aligned}$$

Thus, we obtain by the normality of F that $x \in F(X_k, M \circ M_1, p, s)$. □

Theorem 2.5 *Let M_1 and M_2 be two Orlicz functions. Then the following inclusions are hold for non-negative real numbers s_1, s_2, s:*

(i) $F(X_k, M_1, p, s) \cap F(X_k, M_2, p, s) \subseteq F(X_k, M_1 + M_2, p, s)$,

(ii) *If* $\limsup_{t\to\infty} M_1(t)/M_2(t) < \infty$, *then* $F(X_k, M_2, p, s) \subseteq F(X_k, M_1, p, s)$,

(iii) *If* $s_1 \leqslant s_2$, *then* $F(X_k, M_1, p, s_1) \subseteq F(X_k, M_1, p, s_2)$,

(iv) *If* $F_1 \subseteq F_2$, *then* $F_1(X_k, M_1, p, s) \subseteq F_2(X_k, M_1, p, s)$.

Proof. (i) Let $x \in F(X_k, M_1, p, s) \cap F(X_k, M_2, p, s)$. Then there exist some $\rho_1, \rho_2 > 0$ such that

$$\left(k^{-s}\left[M_1\left(\frac{q_k(x_k)}{\rho_1}\right)\right]^{p_k}\right), \left(k^{-s}\left[M_2\left(\frac{q_k(x_k)}{\rho_2}\right)\right]^{p_k}\right) \in F.$$

Letting $\rho = \max(\rho_1, \rho_2)$, we get

$$\begin{aligned} k^{-s}\left[(M_1+M_2)\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k} &\leqslant k^{-s}\left[M_1\left(\frac{q_k(x_k)}{\rho_1}\right) + M_2\left(\frac{q_k(x_k)}{\rho_2}\right)\right]^{p_k} \\ &\leqslant C\left\{k^{-s}\left[M_1\left(\frac{q_k(x_k)}{\rho_1}\right)\right]^{p_k}\right. \\ &\qquad \left. + k^{-s}\left[M_2\left(\frac{q_k(x_k)}{\rho_2}\right)\right]^{p_k}\right\}. \end{aligned}$$

Since F is a normal space, $x \in F(X_k, M_1 + M_2, p, s)$.

(ii) We can find $K > 0$ such that $M_1(t)/M_2(t) \leqslant K$ for all $t \geqslant 0$, since $\limsup_{t\to\infty} M_1(t)/M_2(t) < \infty$. Let $x \in F(X_k, M_2, p, s)$. There exists a $\rho > 0$ such that

$$\frac{M_1\left(\frac{q_k(x_k)}{\rho}\right)}{M_2\left(\frac{q_k(x_k)}{\rho}\right)} \leqslant K.$$

Hence

$$k^{-s}\left[M_1\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k} \leqslant \max\left(1, K^h\right) k^{-s}\left[M_2\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}.$$

Since F is normal, $x \in F(X_k, M_1, p, s)$.

The proofs of the cases (iii) and (iv) are trivial. □

Corollary 1 *We have*

(i) $F(X_k, p, s) \subseteq F(X_k, M, p, s)$ *for any Orlicz function M satisfying the Δ_2-condition,*

(ii) $F(X_k, M, p) \subseteq F(X_k, M, p, s)$ *for any Orlicz function M.*

3 A Closed Subspace of $F(X_k, M, p, s)$

We define $[F(X_k, M, p, s)]$ by

$$[F(X_k, M, p, s)] = \Big\{x = (x_k)\colon x_k \in X_k \text{ for each } k \in \mathbb{N} \text{ and } \left(k^{-s}\left[M\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}\right) \in F \text{ for every } \rho > 0\Big\}.$$

The space $[F(X_k, M, p, s)]$ is clearly a subspace of $F(X_k, M, p, s)$, and its topology is introduced by the paranorm of $F(X_k, M, p, s)$ given by (1.1).

Theorem 3.1 *$[F(X_k, M, p, s)]$ is a complete paranormed space with the paranorm given by (1.1) if (X_k, q_k) is complete seminormed space for each $k \in \mathbb{N}$.*

Proof. Since $F(X_k, M, p, s)$ is just shown that a complete paranormed space under the paranorm (1.1) and $[F(X_k, M, p, s)]$ is a subspace of $F(X_k, M, p, s)$, it is sufficient to show that it is closed. For this let us consider $\left(x^i\right) = \left(\left(x_k^i\right)\right) \in [F(X_k, M, p, s)]$ such that $g\left(x^i - x\right) \to 0$ as $i \to \infty$, where $x = (x_k) \in F(X_k, M, p, s)$. So for given $\xi > 0$, we can choose an integer i_0 such that

$$g\left(x^i - x\right) < \xi/2, \ \forall i > i_0.$$

Consider

$$\begin{aligned} k^{-s}\left[M\left(\frac{q_k(x_k)}{\xi}\right)\right]^{p_k} &\leqslant k^{-s}\left[\frac{1}{2}M\left(\frac{q_k(x_k^i - x_k)}{\xi/2}\right) + \frac{1}{2}M\left(\frac{q_k(x_k^i)}{\xi/2}\right)\right]^{p_k} \\ &\leqslant Ck^{-s}\left\{\left[M\left(\frac{q_k(x_k^i - x_k)}{g\left(x^i - x\right)}\right)\right]^{p_k} + \left[M\left(\frac{q_k(x_k^i)}{\xi/2}\right)\right]^{p_k}\right\}. \end{aligned}$$

Since

$$\left(k^{-s}\left[M\left(\frac{q_k(x_k^i - x_k)}{g\left(x^i - x\right)}\right)\right]^{p_k}\right), \left(k^{-s}\left[M\left(\frac{q_k(x_k^i)}{\xi/2}\right)\right]^{p_k}\right) \in F$$

and F is normal space,

$$\left(k^{-s}\left[M\left(\frac{q_k(x_k)}{\xi}\right)\right]^{p_k}\right) \in F.$$

This implies $x = (x_k) \in [F(X_k, M, p, s)]$ which shows that $[F(X_k, M, p, s)]$ is complete. □

Proposition 1 *$[F(X_k, M, p, s)]$ is an AK-space.*

Proof. Let $x = (x_k) \in [F(X_k, M, p, s)]$. Therefore,

$$\left(k^{-s}\left[M\left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}\right) \in F$$

for every $\rho > 0$. Since (e_k) is a Schauder basis of F, for a given $\varepsilon \in (0,1)$, we can find an arbitrary positive integer m_0 such that

$$\left\| \sum_{k=m_0}^{\infty} k^{-s} \left[M \left(\frac{q_k(x_k)}{\varepsilon} \right) \right]^{p_k} e_k \right\|_F < 1. \tag{3.1}$$

Using the definition of the paranorm, we have

$$g\left(x - x^{[m]}\right) = \inf \left\{ \xi^{p_n/H} > 0 \colon \left\| \sum_{k=m+1}^{\infty} k^{-s} \left[M \left(\frac{q_k(x_k)}{\xi} \right) \right]^{p_k} e_k \right\|_F^{1/H} \leqslant 1,\, n \in \mathbb{N} \right\},$$

where $x^{[m]}$ denotes the m-th section of x. ¿From this equality and (3.1), it is obvious that

$$g\left(x - x^{[m]}\right) < \varepsilon \text{ for all } m > m_0.$$

Therefore $[F(X_k, M, p, s)]$ is an AK-space. □

Theorem 3.2 *Let* $\left(x^i\right) = \left(\left(x_k^i\right)\right)$ *be a sequence of the elements of* $[F(X_k, M, p, s)]$ *and* $x = (x_k) \in [F(X_k, M, p, s)]$. *Then* $x^i \to x$ *in* $[F(X_k, M, p, s)]$ *iff*

(i) $x_k^i \to x_k$ *in* X_k *for each* $k \geqslant 1$,

(ii) $g(x^i) \to g(x)$ *as* $i \to \infty$.

Proof. The necessity part is obvious.

Sufficiency. Suppose that (i) and (ii) hold, and let m be an arbitrary positive integer. Then

$$g(x^i - x) \leqslant g(x^i - x^{i[m]}) + g(x^{i[m]} - x^{[m]}) + g(x^{[m]} - x),$$

where $x^{i[m]}$, $x^{[m]}$ denote the m-th sections of x^i and x, respectively. Letting $i \to \infty$, we get

$$\begin{aligned} \limsup_{i\to\infty} g(x^i - x) &\leqslant \limsup_{i\to\infty} g(x^i - x^{i[m]}) + \limsup_{i\to\infty} g(x^{i[m]} - x^{[m]}) + g(x^{[m]} - x) \\ &\leqslant 2g(x^{[m]} - x). \end{aligned}$$

Since m is arbitrary, letting $m \to \infty$, we get $\limsup_{i\to\infty} g\left(x^i - x\right) = 0$, i.e. $g\left(x^i - x\right) \to 0$ as $i \to \infty$. □

Theorem 3.3 $[F(X_k, M, p, s)]$ *is separable if for each* $k \in \mathbb{N}$, X_k *is.*

Proof. Suppose X_k is separable for each $k \in \mathbb{N}$. Then, there exists a countable dense subset U_k of X_k. Let Z denotes the set of finite sequences $z = (z_k)$ where $z_k \in U_k$ for each $k \in \mathbb{N}$ and

$$(z_k) = (z_1, z_2, \ldots, z_m, \theta_{m+1}, \theta_{m+2}, \ldots)$$

for arbitrary $m \in \mathbb{N}$. Obviously, Z is a countable subset of $[F(X_k, M, p, s)]$. We shall prove that Z is dense in $[F(X_k, M, p, s)]$. Let $x \in [F(X_k, M, p, s)]$. Since $[F(X_k, M, p, s)]$ is an AK-space, $g\left(x - x^{[m]}\right) \to 0$ as $m \to \infty$. So for a given $\varepsilon > 0$, there exists an integer $m_1 > 1$ such that

$$g\left(x - x^{[m]}\right) < \varepsilon/2 \text{ for all } m \geqslant m_1.$$

If we take $m = m_1$, then

$$g\left(x - x^{[m_1]}\right) < \varepsilon/2.$$

Let us choose $y = (y_k) = (y_1, y_2, \ldots, y_{m_1}, \theta_{m_1+1}, \theta_{m_1+2}, \ldots) \in Z$ such that

$$q_k\left(x_k^{[m_1]} - y_k\right) < \frac{\varepsilon}{2M\left(1\right)m_1\left\|e_k\right\|_F} \text{ for each } k \in \mathbb{N}.$$

Now

$$\begin{aligned} g\left(x - y\right) &= g\left(x - x^{[m_1]} + x^{[m_1]} - y\right) \\ &\leqslant g\left(x - x^{[m_1]}\right) + g\left(x^{[m_1]} - y\right) < \varepsilon. \end{aligned}$$

This implies that Z is dense in $[F(X_k, M, p, s)]$. Hence $[F(X_k, M, p, s)]$ is separable. □

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