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k-Rotundity of Quotient Spaces

C. Yunan and D. LiFen [1]

Abstract: In this paper, we prove that if M is a closed and proximinal subspace of Banach space X , $[x] \in S(X/M)$ and every point on $[x] \cap S(X)$ is a k–extreme point of $B(X)$, then $[x]$ is a k–extreme point of $B(X/M)$. Moreover, we get that if X is k–rotund Banach space and M is a closed and proximinal subspace of X, then the quotient space X/M is also k–rotund. It is shown that if Φ does not satisfy the $\triangle_2$–condition, then E^0_Φ is not proximinal in L^0_Φ and the quotient space L^0_Φ/E^0_Φ is not k–rotund (even if L^0_Φ is rotund.)

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1 Introduction

In 1960, Singer[1] introduced the k–rotund Banach spaces. The definition play a important role in approximation, control theory and so on. Let X be a Banach space and X^* be its dual space. By $B(X)$ and $S(X)$ we denote the closed unit ball and the unit sphere of X, respectively. A point $x \in S(X)$ is called a *k–extreme point* of $B(X)$ provided that $\{x_i\}_{i=1}^{k+1} \subset S(X)$, $x = \frac{x_1+\cdots+x_k}{k+1}$ imply that $\{x_i\}_{i=1}^{k+1}$ are linearly dependent. Obviously, if every point on $S(X)$ is a k–extreme point, then X is a k–rotund space.

Let M be a closed subspace of a Banach space X. We denote by X/M the quotient space of X modulo M. It is well known that X/M equipped with the norm $\|[x]\| = \inf\{\|y\| : y \in [x]\}$, where $[x] = \{y \in X : y - x \in M\}$ is also a Banach space. The subspace M of X is called proximal in X if for any $x \in X$ there is $y \in X$ such that $\|[x]\| = \|x - y\|$.

A mapping $\Phi : R \to [0, \infty)$ is said to be an Orlicz function if Φ vanishes only at zero, Φ is even, convex and left continuous on whole nonnegative line R^+. We define its complementary function $\Psi : R \to [0, \infty)$ by the

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formula

$$\Psi(x) = \sup\{|x|\, y - \Phi(y) : y \geq 0\}.$$

Denote by $p(x)$, $q(x)$ the right derivative of Φ and Ψ, respectively.

Let $(G, \sum, \mu)$ be a measure space with a $\sigma-$finite, nonatomic and complete measure μ and $L^0(\mu)$ be the set of all μ-equivalence classes of real and $\sum$–measurable function defined on G. For a given Orlicz function Φ we define on $L^0(\mu)$ the convex modular I_Φ by

$$I_\Phi\,(x) = \int_G \Phi(x(t))dt.$$

The linear space L_Φ defined by

$$L_\Phi = \{x \in L^0(\mu) : I_\Phi\,(cx) < \infty, \text{ for some } c > 0 \text{ depending on } x\}$$

is called the Orlicz space generated by Φ. We consider L_Φ equipped with the Amemiya-Orlicz norm

$$\|x\|^0 = \inf\{\frac{1}{h}(1 + I_\Phi(hx)) : h > 0\}.$$

To simplify the notation we write L^0_Φ in place of $(L^0_\Phi\,, \|\cdot\|^0)$. The Luxemburg norm in L_Φ is defined by

$$\|x\| = \inf\{\lambda > 0 : I_\Phi(\frac{x}{\lambda}) \leq 1\}.$$

For any x in $L^0_\Phi \setminus\{0\}$, the set of all numbers $h > 0$ such that $\|x\| = \frac{1}{h}(1 + I_\Phi(hx))$ is denoted by $H(x)$. It is well known that $H(x) = [h^*_x, h^{**}_x]$, where $h^*_x = \inf\{h > 0 : I_\Psi(p(h \mid x \mid)) \geq 1\}$ and $h^{**}_x = \sup\{h > 0 : I_\Psi(p(h \mid x \mid)) \leq 1\}$ if $h^*_x < \infty$ and $H(x) = \phi$ if $h^*_x = \infty$. It is also known that if Φ satisfied the condition : $\frac{\Phi(u)}{u} \to +\infty$ as $u \to +\infty$, then $H(x) \neq \phi$ for any $x \in L_\Phi \setminus \{0\}$.(see [2]). And if $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = a < \infty$ then $\|x\|^0 = a \int_G |x(t)|\, dt$ (see [3]).

We say an Orlicz function Φ satisfies the Δ_2–condition ($\Phi \in \Delta_2$ for short) if there are $l \geq 2$ and $u_0 > 0$ such that $\Phi(2u) \leq l\Phi(u)$ whenever $\mid u \mid \geq u_0$. In the sequel E_Φ denotes the space of these $x \in L_\Phi$ that $I_\Phi\,(cx) < \infty$ for all $c > 0$. We write shortly E^0_Φ in place of $(E^0_\Phi, \|\cdot\|^0)$. It is well known that $L_\Phi = E_\Phi$ if and only if $\Phi \in \Delta_2$.

We say a point w is a point of strict convexity of Φ (we write $w \in SC(\Phi)$) if for every $u, v \in R$ such that $u \neq v$ and $w = \frac{1}{2}(u + v)$ there holds

$$\Phi(\frac{u+v}{2}) < \frac{1}{2}(\Phi(u) + \Phi(v)).$$

2 Main Results

Theorem 2.1 *If M is a closed and proximal subspace of Banach space X , $[x] \in S(X/M)$ and every point on $[x] \cap S(X)$ is a k–extreme point of $B(X)$, then $[x]$ is a k–extreme point of $B(X/M)$.*

Proof. Suppose $[x_1], \cdots, [x_{k+1}] \in S(X/M)$, $[x] = \frac{[x_1]+\cdots+[x_{k+1}]}{k+1}$. We have to show that $\{[x_i]\}_{i=1}^{k+1}$ are linearly dependent. By the proximation of M in X, there is $x_i^{'} \in [x_i]$ such that $\left\|x_i^{'}\right\| = \|[x_i]\| = 1$. Then

$$[x] = \frac{[x_1^{'}] + \cdots + [x_{k+1}^{'}]}{k+1} = [\frac{x_1^{'} + \cdots + x_{k+1}^{'}}{k+1}]$$

and

$$1 = \|[x]\| = \left\|[\frac{x_1^{'} + \cdots + x_{k+1}^{'}}{k+1}]\right\| \leq \left\|\frac{x_1^{'} + \cdots + x_{k+1}^{'}}{k+1}\right\|$$
$$\leq \frac{\left\|x_1^{'}\right\| + \cdots + \left\|x_{k+1}^{'}\right\|}{k+1} = 1$$

which implies

$$\left\|\frac{x_1^{'} + \cdots + x_{k+1}^{'}}{k+1}\right\| = 1.$$

This shows

$$\frac{x_1^{'} + \cdots + x_{k+1}^{'}}{k+1} \in [x] \cap S(X).$$

By the assumption, $\frac{x_1^{'}+\cdots+x_{k+1}^{'}}{k+1}$ is a k–extreme point of $B(X)$. It follows that $\{x_i^{'}\}_{i=1}^{k+1}$ are linearly dependent. Therefore, $\{[x_i]\}_{i=1}^{k+1}$ are linearly dependent. The proof is finished. □

By Theorem 2.1, we may get the following Theorem 2.2 directed.

Theorem 2.2 *If X is a k–rotund Banach space and M is its closed proximal subspace, then the Banach space X/M is also k–rotund.*

Theorem 2.3 *Let Φ is an Orlicz function with $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = \infty$. Then $x \in S(L_\Phi^0)$ is a k-extreme point of $B(L_\Phi^0)$ if and only if x is an extreme point, i.e., $\mu\{t \in G : hx(t) \notin SC(\Phi)\} = 0$ for any $h \in H(x)$.*

Proof. We only need to prove necessity. Suppose $\mu\{t \in G : hx(t) \notin SC(\Phi)\} > 0$. Since complementary set of $SC(\Phi)$ is the union of at most countably many open intervals, there exists an interval (a, b) such that

$$\mu\{t \in G : hx(t) \in (a+\varepsilon, b-\varepsilon)\} > 0 \ (\varepsilon > 0)$$

and that Φ is affine on $[a, b] : \Phi(u) = Au + B$ for $u \in [a, b]$. Divide the set $G_0 = \{t \in G : hx(t) \in (a+\varepsilon, b-\varepsilon)\}$ into k+1 sets $G_0^1, \cdots, G_0^{k+1}$ with $\mu(G_0^1) = \cdots = \mu(G_0^{k+1})$ and let

$$x_j(t) = x\chi_{G\backslash G_0}(t) + (x+\varepsilon)\chi_{G_0^j}(t) + (x - \frac{\varepsilon}{k})\chi_{G_0\backslash G_0^j}(t)$$

for $j = 1, \cdots, k+1$. Then $x = \frac{x_1+\cdots+x_{k+1}}{k+1}$ and $\{x_i\}_{i=1}^{k+1}$ are linearly independent. Otherwise, we may assume $x_{k+1}(t) = \sum_{j=1}^{k} \beta_j x_j(t)$. Then we have $x(t) = \sum_{j=1}^{k} \beta_j x(t)$ when $t \in G\backslash G_0$ and so $\sum_{j=1}^{k} \beta_j = 1$. Thus we get a contradiction:

$$x(t) + \varepsilon = \sum_{j=1}^{k} \beta_j (x(t) - \frac{\varepsilon}{k}) = x(t) - \frac{\varepsilon}{k},$$

when $t \in G_0^{k+1}$. Notice that

$$\int_{G_0} \Phi(hx_j(t))dt = \int_{G_0^j} [A(x(t)+\varepsilon) + B]dt$$

$$+ \int_{G_0\backslash G_0^j} [A(x(t) - \frac{\varepsilon}{k}) + B]dt = \int_{G_0} \Phi(hx(t))dt$$

for $j = 1, \cdots, k+1$,we find

$$\|x_j\|^0 \le \frac{1}{h}[1 + \int_G \Phi(hx_j(t))dt] = \frac{1}{h}[1 + \int_G \Phi(hx(t))dt] = \|x\|^0 = 1.$$

This shows a contradiction to the fact that x is a k–extreme point. □

Theorem 2.4 *Let Φ be arbitrary Orlicz function. Then L_Φ^0 is k–rotund if and only if Φ is strictly convex in the whole line and $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = \infty$.*

Proof. We only need to prove that $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = \infty$. If $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = a < \infty$, then there exists $\delta > 0$ such that

$$\|x\|^0 = a \int_G |x(t)|\, dt$$

whenever $\mu(\text{supp}(x)) \leq \delta$. By the continuous of integral, there exist $G_1, \cdots, G_{k+1} \subset\text{supp}(x)$ with $G_i \cap G_j = \phi$ for $i, j = 1, \cdots, k+1$ and $i \neq j$ such that

$$\int_{G_i} |x(t)|\, dt = \frac{1}{k+1} \int_G |x(t)|\, dt$$

for $i = 1, \cdots k+1$. Put

$$x_i = a(k+1)x(t)\chi_{G_i}$$

for $i = 1, \cdots k+1$. Then

$$\|x_i\|^0 = a(k+1) \int_{G_i} |x(t)|\, dt = a \int_G |x(t)|\, dt = 1$$

and

$$\|x_1 + x_2 + \cdots + x_k\|^0 = a(k+1) \int_{\sum\limits_{i=1}^{k+1} G_i} |x(t)|\, dt = \sum_{i=1}^{k+1} a \int_G |x(t)|\, dt = k+1.$$

It is clear that $\{x_i\}_{i=1}^{k+1}$ are linearly independent, a contradiction. □

Remark: If $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} < \infty$, then $\Phi \in \Delta_2$.

The next theorem shows that the assumption in Theorem 2.2 that M is proximal is essential in general.

Theorem 2.5 *If Φ is an arbitrary Orlicz function such that $\Phi \notin \Delta_2$, then E_Φ^0 is not proximal in L_Φ^0 and L_Φ^0/E_Φ^0 is not k–rotund.*

Proof. It is well known that the spaces L_Φ/E_Φ and L_Φ^0/E_Φ^0 are isometric under the identity operator. Recall that we know that

$$\|x\| < \|x\|^0 \text{ for any } x \in L_\Phi \backslash \{0\}$$

Assuming that E_Φ^0 is proximal in L_Φ^0 and taking any $x \in L_\Phi^0/E_\Phi^0$, we can find $y \in E_\Phi^0$ such that$\|[x]\| = \| x - y \|^0$. Hence, we get

$$\|x - y\|^0 = \|[x]\| \leq \|x - y\| \ ,$$

which yields the equality $\|x - y\|^0 = \|x - y\|$, contradicting inequality $\|x - y\|^0 < \|x - y\|$ whence the proof of the fact that E_Φ^0 is not proximal in L_Φ^0 will be finished.

Now, we claim the fact that L^0_Φ/E^0_Φ is not k–rotund if $\Phi \notin \Delta_2$ (even if Φ is strictly convex). It is well known that if $\Phi \notin \Delta_2$ then there exist $\alpha_j \uparrow \infty$ such that $\Phi(\alpha_1) \geq 1/\mu F$ and

$$\Phi((1+\frac{1}{j})\alpha_j) > 2^j \Phi(\alpha_j) \ (j \in N)$$

where $F \in \Sigma$ with $\mu F > 0$ are given previously.. Select a sequence $\{F_j\}$ of disjoint subsets of F such that

$$\Phi(\alpha_j)\mu F_j = 2^{-j} \ (j \in N)$$

and define

$$x(t) = \sum_{j=1}^{\infty} \alpha_j \chi_{F_j}(t) \ .$$

Then

$$I_\Phi(x) = \int_G \Phi(x(t))dt = \sum_{j=1}^{\infty} \Phi(\alpha_j)\mu F_j = 1 < +\infty.$$

But for any $\lambda < 1$, let $n_0 \in N$ satisfy $\frac{1}{\lambda} \geq 1 + 1/n_0$. We have

$$I_\Phi(\frac{x}{\lambda}) = \int_G \Phi(\frac{x(t)}{\lambda})dt = \sum_{j=1}^{\infty} \Phi(\frac{1}{\lambda}\alpha_j)\mu F_j$$

$$> \sum_{j=n_0}^{\infty} \Phi((1+\frac{1}{j})\alpha_j)\mu F_j > \sum_{j=n_0}^{\infty} 2^j \Phi(\alpha_j)\mu F_j = +\infty.$$

This shows $x \in L^0_\Phi/E^0_\Phi$.

Let

$$y_i(t) = \sum_{j=1}^{2^{k+1}} \varepsilon_{i,j}\alpha_j \chi_{F_j}(t) + \sum_{j=2^{k+1}+1}^{2^{k+2}} \varepsilon_{i,j-2^{k+1}}\alpha_j \chi_{F_j}(t) + \cdots .$$

for $i = 1, \cdots, k+1$, where $\varepsilon_{i,j} = \pm 1, i = 1, \cdots, k+1, j = 1, \cdots, 2^{k+1}$. For convenience, we may assume that $\varepsilon_{i,1} = 1$ for $i = 1, \cdots, k+1$. Then $I_\Phi(y_i) = I_\Phi(x)$ and so $y_i \in L^0_\Phi/E^0_\Phi$,

$$\|[y_i]\| = \inf\{\lambda > 0 : I_\Phi(\frac{y_i}{\lambda}) < \infty\} = 1$$

for $i = 1, \cdots, k+1$. Moreover

$$I_\Phi\left(\frac{\sum\limits_{i=1}^{k+1} y_i}{k+1}\right) \leq \frac{1}{k+1}\sum_{i=1}^{k+1} I_\Phi(y_i) = 1$$

and for any $\lambda < 1$

$$I_\Phi\left(\frac{\sum\limits_{i=1}^{k+1} y_i}{\lambda(k+1)}\right) \geq \sum_{s=0}^{\infty}\Phi\left(\frac{1}{\lambda}\alpha_{s2^{k+1}+1}\right)\mu F_{s2^{k+1}+1} = +\infty,$$

which shows that

$$\left\|\sum_{i=1}^{k+1}[y_i]\right\| = \left\|\left[\sum_{i=1}^{k+1} y_i\right]\right\| = k+1.$$

It is clear that $\{y_i\}_{i=1}^{k+1}$ are linearly independent. Next, we will prove that $\{[y_i]\}_{i=1}^{k+1}$ are also linearly independent. Suppose that $\{[y_i]\}_{i=1}^{k+1}$ are linearly dependent. Then there exists $\{l_i\}_{i=1}^{k+1} \subset R$ in which there is at least one element which is nozero such that

$$l_1[y_1] + l_2[y_2] + \cdots + l_{k+1}[y_{k+1}] = E_\Phi^0.$$

This shows that $l_1y_1 + l_2y_2 + \cdots + l_{k+1}y_{k+1} \in E_\Phi^0$.

For convenience, we may assume that $l_1 \neq 0$. Hence

$$I_\Phi\left(\sum_{i=1}^{k+1} l_iy_i\right) \geq \sum_{s=0}^{\infty}\Phi((l_1 + l_2 + \cdots + l_{k+1})\alpha_{s2^{k+1}+1})\mu F_{s2^{k+1}+1}.$$

In order to satisfying $l_1y_1 + l_2y_2 + \cdots + l_{k+1}y_{k+1} \in E_\Phi^0$, we have $\sum_{i=1}^{k+1} l_i = 0$. Let $\lambda_0 = \sum_{i=2}^{k+1} l_i - l_1$. Then $\lambda_0 \neq 0$. Take some $1 \leq j_0 \leq 2^{k+1}$such that $\varepsilon_{1,j_0} = -1$, $\varepsilon_{i,j_0} = 1$ for $i = 2, \cdots k+1$. Then

$$I_\Phi\left(\frac{\sum_{i=1}^{k+1} l_iy_i}{2^{-1}\lambda_0}\right) \geq \sum_{s=0}^{\infty}\Phi\left(\frac{\alpha_{s2^{k+1}+j_0}}{2^{-1}}\right)\mu F_{s2^{k+1}+j_0} = +\infty.$$

So $l_1y_1 + l_2y_2 + \cdots + l_{k+1}y_{k+1} \notin E_\Phi^0$. This contradict with $l_1y_1 + l_2y_2 + \cdots + l_{k+1}y_{k+1} \in E_\Phi^0$. Whence $\{[y_i]\}_{i=1}^{k+1}$ are also linearly independent.

Hence, L_Φ^0/E_Φ^0 is not k–rotund. □

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Cui Yunan
Department of Mathematics,
Harbin University of Science and Technology,
HARBIN,150080, P.R. China
E-mail:Cuiya@mail.hrbust.edu.cn

Duan LiFen
Department of Mathematics, Tonghua Normal College
TONGHUA,134002, P.R. China