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Ideal Convergence Sequence Spaces Defined by a Musielak-Orlicz Function

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Abstract : In the present paper we introduce some sequence spaces using ideal convergence and Musielak-Orlicz function $\mathcal{M} = (M_k)$ and examine some properties of the resulting sequence spaces.

Keywords : paranorm space; I-convergence; difference sequence spaces; Orlicz function; Musielak-Orlicz function.
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1 Introduction and Preliminaries

The notion of ideal convergence was first introduced by Kostyrko [1] as a generalization of statistical convergence which was further studied in topological spaces by Das et al., see [2]. More applications of ideals can be seen in ([2, 3]). We continue in this direction and introduce I-convergence of generalized sequences with respect to Musielak-Orlicz function.

A family $\mathcal{I} \subset 2^X$ of subsets of a non empty set X is said to be an ideal in X if

1. $\phi \in \mathcal{I}$
2. $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$
3. $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$,

while an admissible ideal $\mathcal{I}$ of X further satisfies $\{x\} \in \mathcal{I}$ for each $x \in X$ see [1]. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : ||x_n - x|| \geq \epsilon\}$ belongs to $\mathcal{I}$, see [1]. For more details about ideal convergence sequence spaces (see [4–9]) and references therein.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [10, Theorem 10.4.2, p. 183]).

An orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Lindenstrauss and Tzafriri [11] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

It is shown in [11] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq k(L)M(x)$ for all values of $x \geq 0$, and for $L > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see ([12, 13]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \ldots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\},$$

$$h_{\mathcal{M}} = \{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\left\{k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1\right\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\left\{\frac{1}{k}\left(1 + I_{\mathcal{M}}(kx)\right) : k > 0\right\}.$$

The notion of difference sequence spaces was introduced by Kizmaz [14], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [15] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$.

Let n be non-negative integers, then for $Z = c, c_0$ and l_∞, we have sequence spaces

$$Z(\Delta^n) = \{x = (x_k) \in w : (\Delta^n x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_k)$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+v}.$$

Taking $n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ studied by Kizmaz [14]. For more details about sequence spaces (see [16–19]) and references therein.

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. We define the following sequence spaces in the present paper:

$$c^I(\mathcal{M}, u, p, \Delta^n) = \left\{x = (x_k) \in w : I - \lim_k M_k\left(\frac{|u_k \Delta^n x_k - L|}{\rho}\right)^{p_k} = 0, \right.$$
$$\left. \text{for some } L \text{ and} \rho > 0\right\},$$

$$c_0^I(\mathcal{M}, u, p, \Delta^n) = \left\{x = (x_k) \in w : I\text{-}\lim_k M_k\left(\frac{|u_k \Delta^n x_k|}{\rho}\right)^{p_k} = 0, \text{ for some } \rho > 0\right\}$$

and

$$l_\infty(\mathcal{M}, u, p, \Delta^n) = \left\{x = (x_k) \in w : \sup_k M_k\left(\frac{|u_k \Delta^n x_k|}{\rho}\right)^{p_k} < \infty, \text{ for some } \rho > 0\right\}.$$

We can write

$$m^I(\mathcal{M}, u, p, \Delta^n) = c^I(\mathcal{M}, u, p, \Delta^n) \cap l_\infty(\mathcal{M}, u, p, \Delta^n)$$

and

$$m_0^I(\mathcal{M},u,p,\Delta^n)=c_0^I(\mathcal{M},u,p,\Delta^n)\cap l_\infty(\mathcal{M},u,p,\Delta^n).$$

If we take $p=(p_k)=1$ for all $k\in\mathbb{N}$, we have

$$c^I(\mathcal{M},u,\Delta^n)=\left\{x=(x_k)\in w: I-\lim_k M_k\left(\frac{|u_k\Delta^n x_k-L|}{\rho}\right)=0,\right.$$
$$\left.\text{for some } L \text{ and } \rho>0\right\},$$

$$c_0^I(\mathcal{M},u,\Delta^n)=\left\{x=(x_k)\in w: I-\lim_k M_k\left(\frac{|u_k\Delta^n x_k|}{\rho}\right)=0,\ \text{for some } \rho>0\right\}$$

and

$$l_\infty(\mathcal{M},u,\Delta^n)=\left\{x=(x_k)\in w: \sup_k M_k\left(\frac{|u_k\Delta^n x_k|}{\rho}\right)<\infty,\ \text{for some } \rho>0\right\}.$$

If we take $u=(u_k)=1$ for all $k\in\mathbb{N}$, we have

$$c^I(\mathcal{M},p,\Delta^n)=\left\{x=(x_k)\in w: I-\lim_k M_k\left(\frac{|\Delta^n x_k-L|}{\rho}\right)^{p_k}=0,\right.$$
$$\left.\text{for some } L \text{ and } \rho>0\right\},$$

$$c_0^I(\mathcal{M},p,\Delta^n)=\left\{x=(x_k)\in w: I-\lim_k M_k\left(\frac{|\Delta^n x_k|}{\rho}\right)^{p_k}=0,\ \text{for some } \rho>0\right\}$$

and

$$l_\infty(\mathcal{M},p,\Delta^n)=\left\{x=(x_k)\in w: \sup_k M_k\left(\frac{|\Delta^n x_k|}{\rho}\right)^{p_k}<\infty,\ \text{for some } \rho>0\right\}.$$

If we take $n=0$, we have

$$c^I(\mathcal{M},u,p)=\left\{x=(x_k)\in w: I-\lim_k M_k\left(\frac{|u_k x_k-L|}{\rho}\right)^{p_k}=0,\right.$$
$$\left.\text{for some } L \text{ and } \rho>0\right\},$$

$$c_0^I(\mathcal{M},u,p)=\left\{x=(x_k)\in w: I-\lim_k M_k\left(\frac{|u_k x_k|}{\rho}\right)^{p_k}=0,\ \text{for some } \rho>0\right\}$$

and

$$l_\infty(\mathcal{M},u,p)=\left\{x=(x_k)\in w:\sup_k M_k\left(\frac{|u_kx_k|}{\rho}\right)^{p_k}<\infty,\ \text{for some}\ \rho>0\right\}.$$

If we take $\mathcal{M}(x)=M(x)$, $p=(p_k)=1$, $u=(u_k)=1$ and $n=0$, we get the spaces which were studied by Tripathy and Hazarika [20]. The following inequality will be used throughout the paper. If $0\leq p_k\leq\sup p_k=H$, $D=\max(1,2^{H-1})$ then

$$|a_k+b_k|^{p_k}\leq D\{|a_k|^{p_k}+|b_k|^{p_k}\}\tag{1.1}$$

for all k and $a_k,b_k\in\mathbb{C}$. Also $|a|^{p_k}\leq\max(1,|a|^H)$ for all $a\in\mathbb{C}$.

The main aim of this paper is to study some ideal convergence sequence spaces defined by a Musielak-Orlicz function. We also make an effort to study some topological properties and prove some inclusion relation between these spaces.

2 Main Results

Theorem 2.1. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be any sequence of strictly positive real numbers. Then $c^I(\mathcal{M},u,p,\Delta^n), c_0^I(\mathcal{M},u,p,\Delta^n), m^I(\mathcal{M},u,p,\Delta^n)$ and $m_0^I(\mathcal{M},u,p,\Delta^n)$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. Let $x=(x_k),y=(y_k)\in c^I(\mathcal{M},u,p,\Delta^n)$ and let α,β be scalars. Then there exist positive numbers ρ_1 and ρ_2 such that

$$I-\lim_k M_k\left(\frac{|u_k\Delta^nx_k-L_1|}{\rho_1}\right)^{p_k}=0,\ \text{for some}\ L_1\in\mathbb{C}$$

and

$$I-\lim_k M_k\left(\frac{|u_k\Delta^ny_k-L_2|}{\rho_2}\right)^{p_k}=0,\ \text{for some}\ L_2\in\mathbb{C}.$$

For a given $\epsilon>0$, we have

$$D_1=\left\{k\in\mathbb{N}:M_k\left(\frac{|u_k\Delta^nx_k-L_1|}{\rho_1}\right)^{p_k}>\frac{\epsilon}{2}\right\}\tag{2.1}$$

$$D_2=\left\{k\in\mathbb{N}:M_k\left(\frac{|u_k\Delta^ny_k-L_2|}{\rho_2}\right)^{p_k}>\frac{\epsilon}{2}\right\}.\tag{2.2}$$

Let $\rho_3=\max\left\{2|\alpha|\rho_1,2|\beta|\rho_2\right\}$. Since $\mathcal{M}=(M_k)$ is non-decreasing and convex function, we have

$$\lim_k M_k\left(\frac{|u_k\Delta^n((\alpha x_k+\beta y_k)-(\alpha L_1+\beta L_2))|}{\rho_3}\right)^{p_k}$$
$$\leq \lim_k M_k\left(\frac{|\alpha||u_k\Delta^n x_k-L_1|}{\rho_3}+\frac{|\beta||u_k\Delta^n y_k-L_2|}{\rho_3}\right)^{p_k}$$
$$\leq \lim_k M_k\left(\frac{|u_k\Delta^n x_k-L_1|}{\rho_1}\right)^{p_k}+\lim_k M_k\left(\frac{|u_k\Delta^n y_k-L_2|}{\rho_2}\right)^{p_k}.$$

Now by (2.1) and (2.2), we have

$$\left\{k\in\mathbb{N}:\lim_k M_k\left(\frac{|u_k\Delta^n((\alpha x_k+\beta y_k)-(\alpha L_1+\beta L_2))|}{\rho_3}\right)^{p_k}>\epsilon\right\}\subset D_1\cup D_2.$$

Therefore $\alpha x_k+\beta y_k\in c^I(\mathcal{M},u,p,\Delta^n)$. Hence $c^I(\mathcal{M},u,p,\Delta^n)$ is a linear space. Similarly we can prove that $c_0^I(\mathcal{M},u,p,\Delta^n), m^I(\mathcal{M},u,p,\Delta^n)$ and $m_0^I(\mathcal{M},u,p,\Delta^n)$ are linear spaces. □

Theorem 2.2. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be any sequence of strictly positive real numbers. Then $l_\infty(\mathcal{M},u,p,\Delta^n)$ is a paranormed space with paranorm defined by*

$$g(x)=\inf\left\{\rho>0:\sup_k M_k\left(\frac{|u_k\Delta^n x_k|}{\rho}\right)^{p_k}\leq 1\right\}.$$

Proof. It is clear that $g(x)=g(-x)$. Since $M_k(0)=0$, we get $g(0)=0$. Let us take $x=(x_k)$ and $y=(y_k)$ in $l_\infty(\mathcal{M},u,p,\Delta^n)$. Let

$$B(x)=\left\{\rho>0:\sup_k M_k\left(\frac{|u_k\Delta^n x_k|}{\rho}\right)^{p_k}\leq 1,\right\},$$

$$B(y)=\left\{\rho>0:\sup_k M_k\left(\frac{|u_k\Delta^n y_k|}{\rho}\right)^{p_k}\leq 1,\right\}.$$

Let $\rho_1\in B(x)$ and $\rho_2\in B(y)$. Then if $\rho=\rho_1+\rho_2$, then we have

$$\sup_k M_k\left(\frac{|u_k\Delta^n(x_k+y_k)|}{\rho}\right)\leq\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_k M_k\left(\frac{|u_k\Delta^n x_k|}{\rho_1}\right)$$
$$+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_k M_k\left(\frac{|u_k\Delta^n y_k|}{\rho_2}\right).$$

Thus $\sup_k M_k\Big(\dfrac{|u_k\Delta^n(x_k+y_k)|}{\rho_1+\rho_2}\Big)^{p_k} \leq 1$ and

$$\begin{aligned} g(x+y) &\leq \inf\{(\rho_1+\rho_2)>0 : \rho_1\in B(x),\ \rho_2\in B(y)\} \\ &\leq \inf\{\rho_1>0 : \rho_1\in B(x)\} + \inf\{\rho_2>0 : \rho_2\in B(y)\} \\ &= g(x)+g(y). \end{aligned}$$

Let $\sigma^s \to \sigma$ where $\sigma, \sigma^s \in \mathbb{C}$ and let $g(x^s - x) \to 0$ as $s \to \infty$. We have to show that $g(\sigma^s x^s - \sigma x) \to 0$ as $s \to \infty$. Let

$$B(x^s) = \left\{\rho_s > 0 : \sup_k M_k\left(\frac{|u_k\Delta^n(x_k^s)|}{\rho_s}\right)^{p_k} \leq 1, \right\},$$

$$B(x^s - x) = \left\{\rho_s' > 0 : \sup_k M_k\left(\frac{|u_k\Delta^n(x_k^s - x_k)|}{\rho_s'}\right)^{p_k} \leq 1, \right\}.$$

If $\rho_s \in B(x^s)$ and $\rho_s' \in B(x^s - x)$ then we observe that

$$\begin{aligned} M_k\left(\frac{|u_k\Delta^n(\sigma^s x_k^s - \sigma x_k)}{\rho_s|\sigma^s-\sigma| + \rho_s'|\sigma|}\right) &\leq M_k\left(\frac{|u_k\Delta^n(\sigma^s x_k^s - \sigma x_k^s)|}{\rho_s|\sigma^s-\sigma| + \rho_s'|\sigma|} + \frac{|(\sigma x_k^s - \sigma x_k)|}{\rho_s|\sigma^s-\sigma| + \rho_s'|\sigma|}\right) \\ &\leq \frac{|\sigma^s-\sigma|\rho_s}{\rho_s|\sigma^s-\sigma| + \rho_s'|\sigma|} M_k\left(\frac{(|u_k\Delta^n x_k^s|)}{\rho_s}\right) \\ &\quad + \frac{|\sigma|\rho_s'}{\rho_s|\sigma^s-\sigma| + \rho_s'|\sigma|} M_k\left(\frac{|u_k\Delta^n(x_k^s - x_k)|}{\rho_s'}\right). \end{aligned}$$

From the above inequality, it follows that

$$M_k\left(\frac{|u_k\Delta^n(\sigma^s x_k^s - \sigma x_k)|}{\rho_s|\sigma^s-\sigma| + \rho_s'|\sigma|}\right)^{p_k} \leq 1$$

and consequently,

$$\begin{aligned} g(\sigma^s x^s - \sigma x) &\leq \inf\{(\rho_s|\sigma^s-\sigma| + \rho_s'|\sigma|) > 0 : \rho_s \in B(x^s), \rho_s' \in B(x^s - x)\} \\ &\leq (|\sigma^s-\sigma|) > 0 \inf\{\rho > 0 : \rho_s \in B(x^s)\} \\ &\quad\ + (|\sigma|) > 0 \inf\{(\rho_s')^{\frac{p_n}{H}} : \rho_s' \in B(x^s - x)\} \\ &\longrightarrow 0 \ \text{ as } \ s \longrightarrow \infty. \end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ are Musielak-orlicz functions that satisfies the Δ_2-condition. Then*

(i) $Z(\mathcal{M}'', u, p, \Delta^n) \subseteq Z(\mathcal{M}' \circ \mathcal{M}'', u, p, \Delta^n)$,

(ii) $Z(\mathcal{M}', u, p, \Delta^n) \cap Z(\mathcal{M}'', u, p, \Delta^n) \subseteq Z(\mathcal{M}' + \mathcal{M}'', u, p, \Delta^n)$ *for* $Z = c^I$, c_0^I, m^I, m_0^I.

Proof. Let $x = (x_k) \in c_0^I(\mathcal{M}'', u, p, \Delta^n)$. Then there exist $\rho > 0$ such that

$$I - \lim_k M_k''\left(\frac{|u_k \Delta^n x_k|}{\rho}\right)^{p_k} = 0. \tag{2.3}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k'(t) < \epsilon$ for $0 \le t \le \delta$. Write $y_k = M_k''(\frac{|u_k \Delta^n x_k|}{\rho})^{p_k}$ and consider

$$\lim_{0 \le y_k \le \delta, k \in \mathbb{N}} M_k'(y_k) = \lim_{y_k \le \delta, k \in \mathbb{N}} M_k'(y_k) + \lim_{y_k > \delta, k \in \mathbb{N}} M_k'(y_k).$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2-condition, we have

$$\lim_{y_k \le \delta, k \in \mathbb{N}} M_k'(y_k) \le M_k'(2) \lim_{y_k \le \delta, k \in \mathbb{N}} (y_k). \tag{2.4}$$

For $y_k > \delta$, we have

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$

Since $\mathcal{M}' = (M_k')$ is non-decreasing and convex, it follows that

$$M_k'(y_k) < M_k'(1 + \frac{y_k}{\delta}) < \frac{1}{2} M_k'(2) + \frac{1}{2} \frac{M_k'(2y_k)}{\delta}.$$

Since $\mathcal{M}' = (M_k')$ is satisfies Δ_2-condition, we have

$$M_k'(y_k) < \frac{1}{2} K \frac{y_k}{\delta} M_k'(2) + \frac{1}{2} K \frac{y_k}{\delta} M_k'(2) = K \frac{y_k}{\delta} M_k'(2).$$

Hence

$$\lim_{y_k > \delta, k \in \mathbb{N}} M_k'(y_k) \le \max(1, K\delta^{-1} M_k'(2)) \lim_{y_k \le \delta, k \in \mathbb{N}} (y_k) \tag{2.5}$$

from equation (2.3), (2.4) and (2.5), we have $x = (x_k) \in c_0^I(\mathcal{M}' \circ \mathcal{M}'', u, p, \Delta^n)$. Thus $c_0^I(\mathcal{M}'', u, p, \Delta^n) \subseteq c_0^I(\mathcal{M}' \circ \mathcal{M}'', u, p, \Delta^n)$. Similarly we can prove the other cases.

(ii) Let $x = (x_k) \in c_0^I(\mathcal{M}', u, p, \Delta^n) \cap c_0^I(\mathcal{M}'', u, p, \Delta^n)$. Then there exist $\rho > 0$ such that

$$I - \lim_k M_k'\left(\frac{|u_k \Delta^n x_k|}{\rho}\right)^{p_k} = 0$$

and

$$I - \lim_k M_k''\left(\frac{|u_k \Delta^n x_k|}{\rho}\right)^{p_k} = 0.$$

The rest of the proof follows from the following equality

$$\lim_{k \in \mathbb{N}} (M_k' + M_k'')\left(\frac{|u_k \Delta^n x_k|}{\rho}\right)^{p_k} = \lim_{k \in \mathbb{N}} M_k'\left(\frac{|u_k \Delta^n x_k|}{\rho}\right)^{p_k} + \lim_{k \in \mathbb{N}} M_k''\left(\frac{|u_k \Delta^n x_k|}{\rho}\right)^{p_k}.$$

□

Corollary 2.4. *Let $\mathcal{M} = (M_k)$ be a Musielak orlicz function which satisfies Δ_2-condition. Then $Z(u, p, \Delta^n) \subseteq Z(\mathcal{M}, u, p, \Delta^n)$ for $Z = c^I, c_0^I, m^I$ and m_0^I.*

Proof. The proof follows from Theorem 2.3 by putting $M_k''(x) = x$ and $M_k'(x) = M_k(x) \forall x \in [0, \infty)$. □

Theorem 2.5. *The spaces $c_0^I(\mathcal{M}, u, p, \Delta^n)$ and $m_0^I(\mathcal{M}, u, p, \Delta^n)$ are solid.*

Proof. We shall prove for the space $c_0^I(\mathcal{M}, u, p, \Delta^n)$. Let $x = (x_k) \in c_0^I(\mathcal{M}, u, p, \Delta^n)$. Then there exist $\rho > 0$ such that

$$I - \lim_k M_k \left(\frac{|u_k \Delta^n x_k|}{\rho} \right)^{p_k} = 0. \tag{2.6}$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1 \ \forall \ k \in \mathbb{N}$. Then, the result follows from the following inequality

$$\lim_k M_k \left(\frac{|u_k \Delta^n \alpha_k x_k|}{\rho} \right)^{p_k} \leq \lim_k M_k \left(\frac{|u_k \Delta^n x_k|}{\rho} \right)^{p_k}$$

and this completes the proof.

Similarly we can prove for the space $m_0^I(\mathcal{M}, u, p, \Delta^n)$. □

Corollary 2.6. *The spaces $c_0^I(\mathcal{M}, u, p, \Delta^n)$ and $m_0^I(\mathcal{M}, u, p, \Delta^n)$ are monotone.*

Proof. It is easy to prove so we omit the details. □

Theorem 2.7. *The spaces $c^I(\mathcal{M}, u, p, \Delta^n)$ and $c_0^I(\mathcal{M}, u, p, \Delta^n)$ are sequence algebra.*

Proof. Let $x = (x_k), y = (y_k) \in c_0^I(\mathcal{M}, u, p, \Delta^n)$. Then

$$I - \lim_k M_k \left(\frac{|u_k \Delta^n x_k|}{\rho_1} \right)^{p_k} = 0, \text{ for some } \rho_1 > 0$$

and

$$I - \lim_k M_k \left(\frac{|u_k \Delta^n y_k|}{\rho_2} \right)^{p_k} = 0, \text{ for some } \rho_2 > 0.$$

Let $\rho = \rho_1 + \rho_2$. Then we can show that

$$I - \lim_k M_k \left(\frac{|u_k \Delta^n (x_k \cdot y_k)|}{\rho} \right)^{p_k} = 0.$$

Thus $(x_k \cdot y_k) \in c_0^I(\mathcal{M}, u, p, \Delta^n)$. Hence $c_0^I(\mathcal{M}, u, p, \Delta^n)$ is a sequence algebra. Similarly, we can prove that $c^I(\mathcal{M}, u, p, \Delta^n)$ is a sequence algebra. □

Theorem 2.8. *Let $\mathcal{M} = (M_k)$ be Musielak orlicz function. Then*

$$c_0^I(\mathcal{M}, u, p, \Delta^n) \subset c^I(\mathcal{M}, u, p, \Delta^n) \subset \ell_\infty(\mathcal{M}, u, p, \Delta^n).$$

Proof. Let $x = (x_k) \in c^I(\mathcal{M}, u, p, \Delta^n)$. Then there exist $L \in \mathbb{C}$ and $\rho > 0$ such that

$$I - \lim_k M_k\left(\frac{|u_k \Delta^n x_k - L|}{\rho}\right)^{p_k} = 0.$$

We have

$$M_k\left(\frac{|u_k \Delta^n x_k|}{2\rho}\right)^{p_k} \leq \frac{1}{2} M_k\left(\frac{|u_k \Delta^n x_k - L|}{\rho}\right)^{p_k} + M_k \frac{1}{2}\left(\frac{|L|}{\rho}\right)^{p_k}$$

taking supremum over k on both sides, we get $x = (x_k) \in \ell_\infty(\mathcal{M}, u, p, \Delta^n)$. The inclusion $c_0^I(\mathcal{M}, u, p, \Delta^n) \subset c^I(\mathcal{M}, u, p, \Delta^n)$ is obvious. Thus

$$c_0^I(\mathcal{M}, u, p, \Delta^n) \subset c^I(\mathcal{M}, u, p, \Delta^n) \subset \ell_\infty(\mathcal{M}, u, p, \Delta^n).$$

This completes the proof of the theorem. □

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