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Isomorphisms via Congruences on Γ-Semigroups and Γ-Ideals

H. Hedayati

Department of Mathematics
Babol University of Technology, Babol, Iran
e-mail : hedayati143@yahoo.com;
hedayati143@gmail.com

Abstract : The notion of Γ-semigroup introduced in [1] as a generalization of a semigroup. In this paper, we investigate the formation of quotient and generated Γ-semigroups and Γ-ideals. Also, by a congruence relation ρ on a Γ-semigroup S, we construct the quotient Γ-semigroup $S : \rho$ and discuss on the behavior of some diagrams of quotients and their commutativity. In particular, we prove that the product of quotient Γ-semigroups is isomorphic to the quotient of product of Γ-semigroups.

Keywords : Γ-semigroup; Γ-ideal; generated Γ-ideal; congruence relation.
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1 Introduction and Preliminaries

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation. The formal study of semigroups began in the early 20th century. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. In 1986, Sen and Saha [1] defined the notion of a Γ-semigroup as a generalization of a semigroup. Many classical notions of semigroups have been extended to Γ-semigroups and a lot of results on Γ-semigroups are published by a lot of mathematicians, for instance, Chattopadhyay [2, 3], Chinram and Tinpun [4], Hila [5–10], Saha [11], Sen et al. [1, 12–15].

Let S and Γ be two non-empty sets. S is called a Γ-*semigroup* ([11, 1]) if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ written as $(x, \gamma, y) \mapsto x\gamma y$ satisfying $(x\gamma y)\beta z = x\gamma(y\beta z)$ for all $x, y, z \in S$ and $\gamma, \beta \in \Gamma$. In this case by (S, Γ) we mean S is a Γ-semigroup.

Let S be a Γ-semigroup. A non-empty subset A of S is called a Γ-*sub-semigroup* of S, if $A\Gamma A \subseteq A$. A non-empty Γ-sub-semigroup I of S is called a Γ-*ideal* of S, if $I\Gamma S \subseteq I$ and $S\Gamma I \subseteq I$. Also, S is called a *commutative* Γ-semigroup, if $x\gamma y = y\gamma x$ for all $x, y \in S$ and $\gamma \in \Gamma$.

Example 1.1.

(1) Let $S = [0, 1]$ and $\Gamma = \{\frac{1}{n} |\ n$ is a positive integer$\}$. Then S is a commutative Γ-semigroup under the usual multiplication.

(2) Let S be the set of all 3×2 matrices and Γ be the set of all 2×3 matrices over a field. Then for $A, B \in S$, the product AB can not be defined i.e., S is not a semigroup under the usual matrix multiplication. But for all $A, B, C \in S$ and $P, Q \in \Gamma$ we have $APB \in S$ and since the matrix multiplication is associative, we have $(APB)QC = AP(BQC)$. Hence S is a Γ-semigroup.

In what follows, S is a Γ-semigroup unless otherwise specified.

Lemma 1.2. *Let Λ be a non-empty index set and $\{I_\lambda\}_{\lambda\in\Lambda}$ be a family of Γ-ideals of (S, Γ). Then $\cap_{\lambda\in\Lambda} I_\lambda$ is a Γ-ideal of (S, Γ).*

Proof. It is easy to verify that $\cap_{\lambda\in\Lambda} I_\lambda$ is a Γ-sub-semigroup of S. Also, we have $(\cap_{\lambda\in\Lambda} I_\lambda)\Gamma S = \cap_{\lambda\in\Lambda}(I_\lambda \Gamma S) \subseteq \cap_{\lambda\in\Lambda} I_\lambda$ and $S\Gamma(\cap_{\lambda\in\Lambda} I_\lambda) = \cap_{\lambda\in\Lambda}(S\Gamma I_\lambda) \subseteq \cap_{\lambda\in\Lambda} I_\lambda$. Therefore, $\cap_{\lambda\in\Lambda} I_\lambda$ is a Γ-ideal of (S, Γ). □

In the next theorem, we see that the lattice of Γ-ideals of a Γ-semigroup S, is a complete lattice.

Theorem 1.3. *Let $\mathcal{L}$ be the set of all Γ-ideals of (S, Γ). Then $(\mathcal{L}, \subseteq, \wedge, \vee)$ is a complete lattice, where $I \wedge J = I \cap J$ and $I \vee J = \langle I \cup J\rangle$ the unique smallest Γ-ideal containing $I \cup J$.*

Proof. It is a consequence of Lemma 1.2. □

Let S be a Γ-semigroup. An equivalence relation ρ on S is called *congruence* if $x\rho y$ implies that $(x\gamma z)\rho(y\gamma z)$ and $(z\gamma x)\rho(z\gamma y)$ for all $x, y, z \in S$ and $\gamma \in \Gamma$, where by $x\rho y$ we mean $(x, y) \in \rho$.

Let S_1 be a Γ_1-semigroup and S_2 a Γ_2-semigroup. Then $(f, g) : (S_1, \Gamma_1) \longrightarrow (S_2, \Gamma_2)$ is called a *homomorphism* if $f : S_1 \longrightarrow S_2$ and $g : \Gamma_1 \longrightarrow \Gamma_2$ are functions and $f(x\gamma y) = f(x)g(\gamma)f(y)$ for all $x, y \in S_1$ and $\gamma \in \Gamma_1$.

2 Properties of Γ-Ideals and Γ-Semigroups

Some comprehensive studies on Γ-semigroups and some related extended structures can be found in [16–27]. In this section, we investigate some interesting properties of Γ-semigroups and Γ-ideals. For instance, the following theorems are theorem of quotient Γ-semigroup and correspondence theorem.

Theorem 2.1. *Let S be a commutative Γ-semigroup and I a non-empty subset of S. Then $S/I = \{x\Gamma I \mid x \in S\}$ is a Γ-semigroup.*

Proof. We define $* : S/I \times \Gamma \times S/I \longrightarrow S/I$ by $(x\Gamma I) * \gamma * (y\Gamma I) = x\gamma y\Gamma I$ for all $x\Gamma I$, $y\Gamma I \in S/I$ and $\gamma \in \Gamma$. We prove that $*$ is well-defined. Let $x\Gamma I = x'\Gamma I$, $\gamma = \gamma'$ and $y\Gamma I = y'\Gamma I$. We have

$$\begin{aligned}(x\Gamma I) * \gamma * (y\Gamma I) &= x\gamma y\Gamma I = x\gamma' y'\Gamma I \\ &= y'\gamma' x\Gamma I = y'\gamma' x'\Gamma I \\ &= x'\gamma' y'\Gamma I = (x'\Gamma I) * \gamma' * (y'\Gamma I).\end{aligned}$$

Thus $*$ is well-defined. Also, for all $x, y, z \in S$ and $\gamma, \beta \in \Gamma$, we have

$$\begin{aligned}((x\gamma I) * \gamma * (y\Gamma I)) * \beta * (z\Gamma I) &= ((x\gamma y)\Gamma I) * \beta * (z\Gamma I) \\ &= (x\gamma y)\beta z\Gamma I = x\gamma(y\beta z)\Gamma I \\ &= x\gamma((y\Gamma I) * \beta * (z\Gamma I)) \\ &= (x\Gamma I) * \gamma * ((y\Gamma I) * \beta * (z\Gamma I)).\end{aligned}$$

Therefore, S/I is a Γ-semigroup. □

Theorem 2.2 (Correspondence Theorem)**.** *Let S be a commutative Γ-semigroup and J a Γ-ideal of S such that $\varnothing \neq I \subseteq J$. Then J/I is a Γ-ideal of $(S/I, \Gamma)$. Conversely, let K be a Γ-ideal of $(S/I, \Gamma)$. Then there exists a Γ-ideal J of (S, Γ) such that $I \subseteq J$ and $K = J/I$.*

Proof. We prove $J/I * \Gamma * S/I \subseteq J/I$. Let $j\Gamma I \in J/I$, $\gamma \in \Gamma$ and $x\Gamma I \in S/I$. We have $(j\Gamma I) * \gamma * (x\Gamma I) = j\gamma x\Gamma I \in J/I$. Similarly, we can prove that $S/I * \Gamma * J/I \subseteq J/I$. Therefore, J/I is a Γ-ideal of S/I.

Conversely, let K be a Γ-ideal of $(S/I, \Gamma)$. Put $J = \{x \in S \mid x\Gamma I \in K\}$. We prove that $J\Gamma S \subseteq J$. Suppose that $x \in J$, $r \in S$ and $\gamma \in \Gamma$. Then $x\Gamma I \in K$ and $r\gamma I \in S/I$. Hence $x\gamma r\Gamma I = (x\Gamma I) * \gamma * (r\Gamma I) \in K$. Thus $x\gamma r \in J$. Similarly, we can prove that $S\Gamma J \subseteq J$. Therefore, J is a Γ-ideal of (S, Γ). □

Theorem 2.3. *Let A be a Γ-ideal and B a Γ-sub-semigroup of a commutative Γ-semigroup S. Then $A \cap B$ and $A \cup B$ are Γ-sub-semigroups of (S, Γ). Moreover, there is a homomorphism from $B/(A \cap B)$ to $(A \cup B)/A$.*

Proof. We prove $A \cap B$ and $A \cup B$ are Γ-sub-semigroups of (S, Γ). We have

$$\begin{aligned}(A \cap B)\Gamma(A \cap B) &= A\Gamma A \cap A\Gamma B \cap B\Gamma A \cap B\Gamma B \\ &\subseteq A \cap A\Gamma B \cap B\Gamma A \cap B \subseteq A \cap B.\end{aligned}$$

Similarly, we can prove that $(A\cup B)\Gamma(A\cup B)\subseteq A\cup B$. Now, we define $\psi : B/(A\cap B)\longrightarrow (A\cup B)/A$ by $\psi(b\Gamma(A\cap B))=b\Gamma A$ for all $b\in B$. Let $1_\Gamma:\Gamma\longrightarrow\Gamma$ be the identity map. If $b\Gamma(A\cap B)$, $b'\Gamma(A\cap B)\in B/(A\cap B)$ and $\gamma\in\Gamma$, then we have

$$\begin{aligned}\psi(b\Gamma(A\cap B)*\gamma*b'\Gamma(A\cap B)) &= \psi(b\gamma b'\Gamma(A\cap B))\\ &= b\gamma b'\Gamma A=(b\Gamma A)*\gamma*(b'\Gamma A)\\ &= \psi(b\Gamma(A\cap B))*1_\Gamma(\gamma)*\psi(b'\Gamma(A\cap B)).\end{aligned}$$

Therefore, $(\psi,1_\Gamma):(B/(A\cap B),\Gamma)\longrightarrow((A\cup B)/A,\Gamma)$ is a homomorphism. □

Let A be a non-empty subset of a Γ-semigroup S. Put

$$X=\{B\mid B \text{ is a } \Gamma\text{-ideal of } (S,\Gamma) \text{ containing } A\}.$$

Then $X\neq\varnothing$, because $S\in X$. Let $\langle A\rangle=\cap_{B\in X}B$. Clearly, $A\subseteq\langle A\rangle$. By Lemma 1.2, $\langle A\rangle$ is a Γ-ideal of (S,Γ). Also, $\langle A\rangle$ is the smallest Γ-ideal of (S,Γ) containing A. $\langle A\rangle$ is called the Γ-ideal of S *generated* by A.

Theorem 2.4. *If A is a non-empty subset of (S,Γ), then $\langle A\rangle=A\cup S\Gamma A\cup A\Gamma S\cup S\Gamma A\Gamma S$.*

Proof. We prove that $B=A\cup S\Gamma A\cup A\Gamma S\cup S\Gamma A\Gamma S$ is a Γ-ideal of (S,Γ). We have

$$\begin{aligned}S\Gamma B &= S\Gamma(A\cup S\Gamma A\cup A\Gamma S\cup S\Gamma A\Gamma S)\\ &= S\Gamma A\cup S\Gamma S\Gamma A\cup S\Gamma A\Gamma S\cup S\Gamma S\Gamma A\Gamma S\\ &\subseteq S\Gamma A\cup S\Gamma A\cup S\Gamma A\Gamma S\cup S\Gamma A\Gamma S\\ &= S\Gamma A\cup S\Gamma A\Gamma S\subseteq B.\end{aligned}$$

Also, we have

$$\begin{aligned}B\Gamma S &= (A\cup S\Gamma A\cup A\Gamma S\cup S\Gamma A\Gamma S)\Gamma S\\ &= A\Gamma S\cup S\Gamma A\Gamma S\cup A\Gamma S\Gamma S\cup S\Gamma A\Gamma S\Gamma S\\ &\subseteq A\Gamma S\cup S\Gamma A\Gamma S\cup A\Gamma S\cup S\Gamma A\Gamma S\\ &= A\Gamma S\cup S\Gamma A\Gamma S\cup A\Gamma S\subseteq B.\end{aligned}$$

Therefore, B is a Γ-ideal of (S,Γ). Since $A\subseteq B$, then $\langle A\rangle\subseteq B$. Let I be a Γ-ideal of (S,Γ) such that $A\subseteq I$. Then $S\Gamma A\subseteq S\Gamma I\subseteq I$, $A\Gamma S\subseteq I\Gamma S\subseteq I$ and $S\Gamma A\Gamma S\subseteq S\Gamma I\Gamma S\subseteq I$, which imply that $B=A\cup S\Gamma A\cup A\Gamma S\cup S\Gamma A\Gamma S\subseteq\langle A\rangle$. Therefore, $B=\langle A\rangle$ and the proof is completed. □

Theorem 2.5. *Let S be a commutative Γ-semigroup, $A\subseteq S$ and I a Γ-ideal of (S,Γ). Then $(I:A)=\{x\in S\mid x\gamma a\in I$, for all $a\in A$ and $\gamma\in\Gamma\}$ is a Γ-ideal of (S,Γ).*

Proof. Suppose that $x \in (I : A)$, $r \in S$ and $\beta \in \Gamma$. Then $x\gamma a \in I$ for all $a \in A$ and $\gamma \in \Gamma$. It implies that $(r\beta x)\gamma a \in I$. Thus $r\beta x \in (I : A)$. Similarly, we can prove that $x\beta r \in (I : A)$. Therefore, $(I : A)$ is a Γ-ideal of (S, Γ). □

Theorem 2.6. *Let S be a commutative Γ-semigroup. If I is a Γ-ideal of (S, Γ), $\varnothing \neq A \subseteq S$ and $\gamma \in \Gamma$, then the following statements hold:*

(1) $I \subseteq (I : A) \subseteq (I : A\Gamma A) \subseteq (I : A\gamma A)$.

(2) If $A \subseteq I$, then $(I : A) = S$.

Proof. (1) If $x \in I$, then $x\Gamma A \subseteq I\Gamma S \subseteq I$. Thus $x \in (I : A)$. If $x \in (I : A)$, then $x\Gamma(A\Gamma A) = (x\Gamma A)\Gamma A \subseteq I\Gamma A \subseteq I$. Thus $x \in (I : A\Gamma A)$. Finally, if $x \in (I : A\Gamma A)$, then $x\Gamma(A\gamma A) \subseteq x\Gamma(A\Gamma A) \subseteq I$. Hence $x \in (I : A\gamma A)$.

(2) Let $A \subseteq I$ and $x \in S$. Then $x\Gamma A \subseteq S\Gamma I \subseteq I$, so $x \in (I : A)$. Thus $(I : A) = S$. □

Theorem 2.7. *Let S be a commutative Γ-semigroup. If I is a Γ-ideal of (S, Γ) and $\varnothing \neq A \subseteq S$, then $(I : A) = \cap_{a\in A}(I : a) = (I : A\backslash I)$.*

Proof. We have $(I : A) \subseteq \cap_{a\in A}(I : a)$. Let $x \in \cap_{a\in A}(I : a)$. Then $x\Gamma a \subseteq I$ for all $a \in A$. So $\cap_{a\in A}(I : a) \subseteq (I : A)$. Hence $(I : A) = \cap_{a\in A}(I : a)$. Again by Theorem 2.6, we have $(I : A) = \cap_{a\in A}(I : a) = (I : A\backslash I)$. □

3 Congruence Relations on the Product of Γ-Semigroups

Let ρ be a congruence relation on (S, Γ). By $S : \rho$ we mean the set of all equivalence classes of the elements of S with respect to ρ, that is $S : \rho = \{\rho(x) \mid x \in S\}$.

Lemma 3.1. *Let ρ be a congruence relation on (S, Γ). Then $\rho(x\gamma y) = \rho(\rho(x)\gamma\rho(y))$ for all $x, y \in S$ and $\gamma \in \Gamma$.*

Proof. Clearly $\rho(x\gamma y) \subseteq \rho(\rho(x)\gamma\rho(y))$. Let $z \in \rho(\rho(x)\gamma\rho(y))$, then $z \in \rho(x'\gamma y')$ for some $x' \in \rho(x)$ and $y' \in \rho(y)$. In other hand, ρ is a congruence relation on (S, Γ), so $\rho(x\gamma y) = \rho(x'\gamma y')$. This implies that $z \in \rho(x\gamma y)$. Therefore, $\rho(\rho(x)\gamma\rho(y)) \subseteq \rho(x\gamma y)$. □

In the next theorem, we demonstrate how to construct a new Γ-semigroups by using congruence relations.

Theorem 3.2. *Let ρ be a congruence relation on (S, Γ). Then $S : \rho$ is a Γ-semigroup.*

Proof. Define the map $\odot : (S : \rho) \times \Gamma \times (S : \rho) \longrightarrow (S : \rho)$ by $\rho(x) \odot \gamma \odot \rho(y) = \rho(x\gamma y)$ for all $x, y \in S$ and $\gamma \in \Gamma$. Let $\rho(x) = \rho(x')$ and $\rho(y) = \rho(y')$. By Lemma 3.1, we have

$$\begin{aligned}\rho(x) \odot \gamma \odot \rho(y) &= \rho(x\gamma y) = \rho(\rho(x)\gamma\rho(y)) \\ &= \rho(\rho(x')\gamma\rho(y')) = \rho(x') \odot \gamma \odot \rho(y').\end{aligned}$$

Thus $\odot$ is well-defined. Also for all $\rho(x), \rho(y), \rho(z) \in S : \rho$ and $\gamma, \beta \in \Gamma$ we have

$$\begin{aligned}(\rho(x) \odot \gamma \odot \rho(y)) \odot \beta \odot \rho(z) &= \rho(x\gamma y) \odot \beta \odot \rho(z) \\ &= \rho((x\gamma y)\beta z) = \rho(x\gamma(y\beta z)) \\ &= \rho(x) \odot \gamma \odot \rho(y\beta z) \\ &= \rho(x) \odot \gamma \odot (\rho(y) \odot \beta \odot \rho(z)).\end{aligned}$$

Therefore, $S : \rho$ is a Γ-semigroup. □

Lemma 3.3. *If $\Pi_S : S \longrightarrow S : \rho$ is defined by $\Pi_S(x) = \rho(x)$ and 1_Γ is the identity map on Γ, then $(\Pi_S, 1_\Gamma) : (S, \Gamma) \longrightarrow (S : \rho, \Gamma)$ is an epimorphism.*

Proof. Let $x, y \in S$ and $\gamma \in \Gamma$. By Theorem 3.2, we have

$$\Pi_S(x\gamma y) = \rho(x\gamma y) = \rho(x) \odot \gamma \odot \rho(y) = \Pi_S(x) \odot 1_\Gamma(\gamma) \odot \Pi_S(y).$$

Clearly, Π_S is onto. Therefore, $(\Pi_S, 1_\Gamma)$ is an epimorphism. □

In the following, we show how to use a Γ-ideal and a congruence relation on a Γ-semigroup S to construct a new Γ-ideal of S and to investigate the relationship between them.

Theorem 3.4. *Let ρ be a congruence relation on (S, Γ). If I is a Γ-ideal of (S, Γ), then $C_I = \{x \in S \mid x\rho a, \exists a \in I\}$ is a Γ-ideal of (S, Γ) and $I \subseteq C_I$.*

Proof. Clearly $I \subseteq C_I$. Let $x \in C_I$, $r \in S$ and $\gamma \in \Gamma$. Then $x\rho a$ for some $a \in I$. In other hand ρ is a congruence relation which implies that $(x\gamma r)\rho(a\gamma r)$. Thus $x\gamma r \in C_I$. Similarly, we can prove that $r\gamma x \in C_I$. Therefore C_I is a Γ-ideal of (S, Γ). □

Theorem 3.5. *If I is a Γ-ideal of (S, Γ), then $I : \rho$ is a Γ-ideal of $(S : \rho, \Gamma)$.*

Proof. Suppose that $\rho(x) \in I : \rho$, $\rho(r) \in S : \rho$ and $\gamma \in \Gamma$. Then, by Theorem 3.2, $\rho(x) \odot \gamma \odot \rho(r) = \rho(x\gamma y) \in I : \rho$. Similarly, we can prove that $\rho(r) \odot \gamma \odot \rho(x) \in I : \rho$. Therefore, $I : \rho$ is a Γ-ideal of $(S : \rho, \Gamma)$. □

Theorem 3.6. *If J is a Γ-ideal of $(S : \rho, \Gamma)$, then there exists a Γ-ideal I of (S, Γ) such that $J = I : \rho$.*

Proof. Put $I = \{x \in S \mid \rho(x) \in J\}$. We have

$$\rho(x) \in J \Longrightarrow x \in I \Longrightarrow \rho(x) \in I : \rho,$$

and

$$\rho(x) \in I : \rho \Longrightarrow \exists a \in I, \rho(x) = \rho(a) \Longrightarrow \rho(x) = \rho(a) \in J.$$

Thus $J = I : \rho$. Now, suppose that $x \in I$, $r \in S$ and $\gamma \in \Gamma$. Then $\rho(x) \in J$ and by Theorem 3.2, $\rho(x\gamma y) = \rho(x) \odot \gamma \odot \rho(r) \in J$. Hence $x\gamma r \in I$. Similarly, we can prove that $r\gamma x \in I$. Therefore, I is a Γ-ideal of (S, Γ). □

Lemma 3.7. *Let S_λ be a Γ_λ-semigroup $(\lambda \in \Lambda)$. Then $\prod_{\lambda\in\Lambda} S_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} \mid x_\lambda \in S_\lambda\}$ is a $\prod_{\lambda\in\Lambda} \Gamma_\lambda$-semigroup.*

Proof. Define $\circ : (\prod_{\lambda\in\Lambda} S_\lambda) \times (\prod_{\lambda\in\Lambda} \Gamma_\lambda) \times (\prod_{\lambda\in\Lambda} S_\lambda) \longrightarrow \prod_{\lambda\in\Lambda} S_\lambda$ by $(x_\lambda)_{\lambda\in\Lambda} \circ (\gamma_\lambda)_{\lambda\in\Lambda} \circ (y_\lambda)_{\lambda\in\Lambda} = (x_\lambda\gamma_\lambda y_\lambda)_{\lambda\in\Lambda}$ for all $(x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} S_\lambda$ and $(\gamma_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} \Gamma_\lambda$. It is easy to verify that $\circ$ is well-defined. We have

$$\begin{aligned} ((x_\lambda)_{\lambda\in\Lambda} \circ (\gamma_\lambda)_{\lambda\in\Lambda} \circ (y_\lambda)_{\lambda\in\Lambda}) \circ (\beta_\lambda)_{\lambda\in\Lambda} \circ (z_\lambda)_{\lambda\in\Lambda} &= \\ (x_\lambda\gamma_\lambda y_\lambda)_{\lambda\in\Lambda} \circ (\beta_\lambda)_{\lambda\in\Lambda} \circ (z_\lambda)_{\lambda\in\Lambda} &= \\ ((x_\lambda\gamma_\lambda y_\lambda)\beta_\lambda z_\lambda)_{\lambda\in\Lambda} = (x_\lambda\gamma_\lambda(y_\lambda\beta_\lambda z_\lambda))_{\lambda\in\Lambda} &= \\ (x_\lambda)_{\lambda\in\Lambda} \circ (\gamma_\lambda)_{\lambda\in\Lambda} \circ (y_\lambda\beta_\lambda z_\lambda)_{\lambda\in\Lambda} &= \\ (x_\lambda)_{\lambda\in\Lambda} \circ (\gamma_\lambda)_{\lambda\in\Lambda} \circ ((y_\lambda)_{\lambda\in\Lambda} \circ (\beta_\lambda)_{\lambda\in\Lambda} \circ (z_\lambda)_{\lambda\in\Lambda}) & \end{aligned}$$

for all $(x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}, (z_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} S_\lambda$ and $(\gamma_\lambda)_{\lambda\in\Lambda}, (\beta_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} \Gamma_\lambda$. Therefore, $\prod_{\lambda\in\Lambda} S_\lambda$ is a $\prod_{\lambda\in\Lambda} \Gamma_\lambda$-semigroup. □

In the next lemma, we investigate the behavior of congruence relations on the product of Γ-semigroups.

Lemma 3.8. *Let ρ_λ be a congruence relation on $(S_\lambda, \Gamma_\lambda)$ for all $\lambda \in \Lambda$. Then ρ is a congruence relation on $(\prod_{\lambda\in\Lambda} S_\lambda, \prod_{\lambda\in\Lambda} \Gamma_\lambda)$ where $(a_\lambda)_{\lambda\in\Lambda}\rho(b_\lambda)_{\lambda\in\Lambda}$ if and only if $a_\lambda\rho_\lambda b_\lambda$ for all $a_\lambda, b_\lambda \in S_\lambda$ and $\lambda \in \Lambda$.*

Proof. If $(x_\lambda)_{\lambda\in\Lambda}\rho(y_\lambda)_{\lambda\in\Lambda}$, then $x_\lambda\rho_\lambda y_\lambda$ for all $\lambda \in \Lambda$. Hence $(x_\lambda\gamma_\lambda z_\lambda)\rho_\lambda(y_\lambda\gamma_\lambda z_\lambda)$ for all $z_\lambda \in S_\lambda$, $\gamma_\lambda \in \Gamma_\lambda$ and $\lambda \in \Lambda$. Hence

$$((x_\lambda)_{\lambda\in\Lambda} \circ (\gamma_\lambda)_{\lambda\in\Lambda} \circ (z_\lambda)_{\lambda\in\Lambda})\rho((y_\lambda)_{\lambda\in\Lambda} \circ (\gamma_\lambda)_{\lambda\in\Lambda} \circ (z_\lambda)_{\lambda\in\Lambda}).$$

Similarly, we can prove that

$$((z_\lambda)_{\lambda\in\Lambda} \circ (\gamma_\lambda)_{\lambda\in\Lambda} \circ (x_\lambda)_{\lambda\in\Lambda})\rho((z_\lambda)_{\lambda\in\Lambda} \circ (\gamma_\lambda)_{\lambda\in\Lambda} \circ (y_\lambda)_{\lambda\in\Lambda}).$$

Therefore, ρ is a congruence relation on $(\prod_{\lambda\in\Lambda} S_\lambda, \prod_{\lambda\in\Lambda} \Gamma_\lambda)$. □

In the following theorem, we prove an isomorphism theorem on the product of Γ-semigroups.

Theorem 3.9. *Let ρ_λ be a congruence relation on $(S_\lambda, \Gamma_\lambda)$ for all $\lambda \in \Lambda$ and ρ the congruence relation on $(\prod_{\lambda\in\Lambda} S_\lambda, \prod_{\lambda\in\Lambda} \Gamma_\lambda)$ defined in Lemma 3.8. Then,*

$$\left(\prod_{\lambda\in\Lambda}(S_\lambda : \rho_\lambda), \prod_{\lambda\in\Lambda}\Gamma_\lambda\right) \cong \left((\prod_{\lambda\in\Lambda} S_\lambda) : \rho, \prod_{\lambda\in\Lambda}\Gamma_\lambda\right).$$

Proof. By Theorem 3.2 and Lemmas 3.7 and 3.8, $\prod_{\lambda\in\Lambda}(S_\lambda : \rho_\lambda)$ and $(\prod_{\lambda\in\Lambda} S_\lambda) : \rho$ are $\prod_{\lambda\in\Lambda}\Gamma_\lambda$-semigroups. Define $\psi : \prod_{\lambda\in\Lambda}(S_\lambda : \rho_\lambda) \longrightarrow (\prod_{\lambda\in\Lambda} S_\lambda) : \rho$ by $\psi((\rho_\lambda(x_\lambda))_{\lambda\in\Lambda}) = \rho((x_\lambda)_{\lambda\in\Lambda})$ for all $x_\lambda \in S_\lambda$ $(\lambda \in \Lambda)$. We show that $(\psi, 1_{\prod_{\lambda\in\Lambda}\Gamma_\lambda})$ is an isomorphism between $(\prod_{\lambda\in\Lambda}(S_\lambda : \rho_\lambda), \prod_{\lambda\in\Lambda}\Gamma_\lambda)$ and $((\prod_{\lambda\in\Lambda} S_\lambda) : \rho, \prod_{\lambda\in\Lambda}\Gamma_\lambda)$. We have

$$\begin{aligned}
(\rho_\lambda(x_\lambda))_{\lambda\in\Lambda} = (\rho_\lambda(y_\lambda)_{\lambda\in\Lambda} &\iff \\
\rho_\lambda(x_\lambda) = \rho_\lambda(y_\lambda), \quad \forall\lambda\in\Lambda \iff x_\lambda\rho_\lambda y_\lambda, \quad \forall\lambda\in\Lambda &\iff \\
(x_\lambda)_{\lambda\in\Lambda}\rho(y_\lambda)_{\lambda\in\Lambda} \iff \rho((x_\lambda)_{\lambda\in\Lambda}) = \rho((y_\lambda)_{\lambda\in\Lambda})) &\iff \\
\psi((\rho_\lambda(x_\lambda))_{\lambda\in\Lambda}) = \psi((\rho_\lambda(y_\lambda))_{\lambda\in\Lambda}).&
\end{aligned}$$

Hence $(\psi, 1_{\prod_{\lambda\in\Lambda}\Gamma_\lambda})$ is well-defined and one to one. Clearly, $(\psi, 1_{\prod_{\lambda\in\Lambda}\Gamma_\lambda})$ is onto. Now, we prove that $(\psi, 1_{\prod_{\lambda\in\Lambda}\Gamma_\lambda})$ is a homomorphism. We have

$$\begin{aligned}
\psi((\rho_\lambda(x_\lambda))_{\lambda\in\Lambda} \circ (\gamma_\lambda)_{\lambda\in\Lambda} \circ (\rho_\lambda(y_\lambda))_{\lambda\in\Lambda}) &= \\
\psi((\rho_\lambda(x_\lambda) \odot \gamma_\lambda \odot \rho_\lambda(y_\lambda))_{\lambda\in\Lambda}) &= \\
\psi((\rho_\lambda(x_\lambda\gamma_\lambda y_\lambda))_{\lambda\in\Lambda}) = \rho((x_\lambda\gamma_\lambda y_\lambda)_{\lambda\in\Lambda}) &= \\
\rho((x_\lambda)_{\lambda\in\Lambda}) \circ (\gamma_\lambda)_{\lambda\in\Lambda} \circ \rho((y_\lambda)_{\lambda\in\Lambda}) &= \\
\psi((\rho_\lambda(x_\lambda))_{\lambda\in\Lambda}) \odot 1_{\prod_{\lambda\in\Lambda}\Gamma_\lambda}((\gamma_\lambda)_{\lambda\in\Lambda}) \odot \psi((\rho_\lambda(y_\lambda))_{\lambda\in\Lambda}).&
\end{aligned}$$

Therefore, $(\psi, 1_{\prod_{\lambda\in\Lambda}\Gamma_\lambda})$ is an isomorphism. □

In the next theorems, we consider the congruence relation induced by homomorphisms and investigate the corresponding results and properties associated with this congruence relation.

Theorem 3.10. *Let $(\varphi, g) : (S_1, \Gamma_1) \longrightarrow (S_2, \Gamma_2)$ be a homomorphism. Define the relation $\rho_{(\varphi,g)}$ on (S_1, Γ_1) as follows: $x\rho_{(\varphi,g)}y$ if and only if $\varphi(x) = \varphi(y)$. Then $\rho_{(\varphi,g)}$ is a congruence relation on (S_1, Γ_1).*

Proof. Clearly, $\rho_{(\varphi,g)}$ is an equivalence relation. Suppose that $x\rho_{(\varphi,g)}y$. We have

$$\varphi(x) = \varphi(y) \Longrightarrow \varphi(x)g(\gamma)\varphi(z) = \varphi(y)g(\gamma)\varphi(z) \Longrightarrow \varphi(x\gamma z) = \varphi(y\gamma z)$$

for all $z \in S_1$ and $\gamma \in \Gamma_1$. Thus $(x\gamma z)\rho_{(\varphi,g)}(y\gamma z)$. Similarly, we can prove that $(z\gamma x)\rho_{(\varphi,g)}(z\gamma y)$. Therefore, $\rho_{(\varphi,g)}$ is a congruence relation on (S_1, Γ_1). □

Theorem 3.11. *Let $(\varphi, g) : (S_1, \Gamma_1) \longrightarrow (S_2, \Gamma_2)$ be a homomorphism. Set $A = \{I \subseteq S_1 \mid \rho_{(\varphi,g)} \subseteq I \times I\}$ and $B = \{J \mid J \subseteq S_2\}$. Then there exists an one to one map from A to B.*

Proof. Define $\psi : A \longrightarrow B$ by $\psi(I) = \varphi(I)$. Clearly, ψ is well-defined. Suppose that $\psi(I_1) = \psi(I_2)$, then $\varphi(I_1) = \varphi(I_2)$. Also we have

$$\begin{aligned} x \in I_1 &\Longrightarrow \varphi(x) \in \varphi(I_1) = \varphi(I_2) \\ &\Longrightarrow \exists y \in I_2, \varphi(x) = \varphi(y) \\ &\Longrightarrow (x, y) \in \rho_{(\varphi,g)} \subseteq I_2 \times I_2 \\ &\Longrightarrow x \in I_2 \Longrightarrow I_1 \subseteq I_2. \end{aligned}$$

Similarly, we can prove that $I_2 \subseteq I_1$. Thus $I_1 = I_2$ and therefore ψ is one to one. □

Theorem 3.12. *Let* $(S_1, \Gamma_1) \xrightarrow{(\varphi_1,g_1)} (S_2, \Gamma_2) \xrightarrow{(\varphi_2,g_2)} (S_3, \Gamma_3)$ *be a sequence of homomorphisms. Then*

$$(\psi, g) : (S_1 \times S_1, \Gamma_1 \times \Gamma_1) \longrightarrow (S_2 \times S_2, \Gamma_2 \times \Gamma_2)$$

defined by $\psi(x, y) = (\varphi_1(x), \varphi_1(y))$ *and* $g(\gamma, \beta) = (g_1(\gamma), g_1(\beta))$ *for all* $x, y \in S_1$ *and* $\gamma, \beta \in \Gamma_1$ *is a homomorphism such that* $\psi(\rho_{(\varphi_1,g_1)}) \subseteq \rho_{(\varphi_2,g_2)}$. *Moreover, if* (φ_1, g_1) *is onto and* (φ_2, g_2) *is one to one, then* $\psi(\rho_{(\varphi_1,g_1)}) = \rho_{(\varphi_2,g_2)}$.

Proof. It is easy to verify that (ψ, g) is a homomorphism. We have

$$\begin{aligned} \psi(a, b) \in \psi(\rho_{(\varphi_1,g_1)}), (a, b) \in \rho_{(\varphi_1,g_1)} &\Longrightarrow \\ \varphi_1(a) = \varphi_1(b) \Longrightarrow \varphi_2(\varphi_1(a)) = \varphi_2(\varphi_1(b)) &\Longrightarrow \\ (\varphi_1(a), \varphi_1(b)) \in \rho_{(\varphi_2,g_2)} \Longrightarrow \psi(a, b) \in \rho_{(\varphi_2,g_2)}. \end{aligned}$$

Thus $\psi(\rho_{(\varphi_1,g_1)}(x)) \subseteq \rho_{(\varphi_2,g_2)}$. Now, if (φ_1, g_1) is onto and (φ_2, g_2) is one to one, then we prove that $\psi(\rho_{(\varphi_1,g_1)}(x)) = \rho_{(\varphi_2,g_2)}$. It is enough we prove that $\rho_{(\varphi_2,g_2)} \subseteq \psi(\rho_{(\varphi_1,g_1)}(x))$. We have

$$\begin{aligned} (t, t') \in \rho_{(\varphi_2,g_2)} &\Longrightarrow \varphi_2(t) = \varphi_2(t') \\ &\Longrightarrow \exists a, b \in S,\ \varphi_1(a) = t, \varphi_1(b) = t' \\ &\Longrightarrow (t, t') = \psi(a, b) = (\varphi_1(a), \varphi_1(b)) \\ &\Longrightarrow (t, t') \in \psi(\rho_{(\varphi_1,g_1)}). \end{aligned}$$

Therefore $\rho_{(\varphi_2,g_2)} \subseteq \psi(\rho_{(\varphi_1,g_1)}(x))$, which completes the proof. □

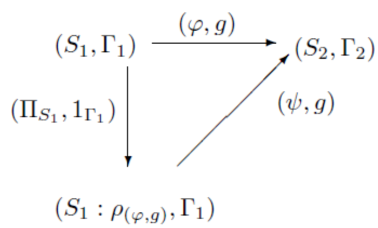
Theorem 3.13. *Let* $(S_1, \Gamma_1) \xrightarrow{(\varphi_1,g_1)} (S_2, \Gamma_2) \xrightarrow{(\varphi_2,g_2)} (S_3, \Gamma_3)$ *be a sequence of homomorphisms. Then* $Im\varphi_1 \times Im\varphi_1 \subseteq \rho_{(\varphi_2,g_2)}$ *if and only if* $\varphi_2 \circ \varphi_1$ *is constant.*

Proof. ($\Longleftarrow$): Let $(x, y) \in Im\varphi_1 \times Im\varphi_1$, then $x = \varphi_1(a)$ and $y = \varphi_1(b)$ for some $a, b \in S_1$. By hypothesis, we have $\varphi_2(\varphi_1(a)) = \varphi_2(\varphi_1(b))$, which implies that $(\varphi_1(a), \varphi_1(b)) = (x, y) \in \rho_{(\varphi_2,g_2)}$. Therefore $Im\varphi_1 \times Im\varphi_1 \subseteq \rho_{(\varphi_2,g_2)}$.

($\Longrightarrow$): Let $x, y \in S_1$, then $(\varphi_1(x), \varphi_1(y)) \in Im\varphi_1 \times Im\varphi_1 \subseteq \rho_{(\varphi_2,g_2)}$. Hence $\varphi_2(\varphi_1(x)) = \varphi_2(\varphi_1(y))$. Therefore, $\varphi_2 \circ \varphi_1$ is constant. □

Finally, by the congruence relation induced by homomorphisms, we are able to establish some isomorphism theorems and investigate the commutativity of some diagrams.

Theorem 3.14 (Isomorphism Theorem)**.** *If* $(\varphi, g) : (S_1, \Gamma_1) \longrightarrow (S_2, \Gamma_2)$ *is an homomorphism, then there exists an unique isomorphism* $(\psi, g) : (S_1 : \rho_{(\varphi,g)}, \Gamma_1) \longrightarrow (S_2, \Gamma_2)$ *such that the following diagram commutes:*



where $\Pi_{S_1} : S_1 \longrightarrow (S_1 : \rho_{(\varphi,g)})$ *is defined by* $\Pi_{S_1}(x) = \rho_{(\varphi,g)}(x)$ *for all* $x \in S_1$ *and* 1_{Γ_1} *is the identity map on* Γ_1*.*

Proof. Define $\psi : (S_1 : \rho_{(\varphi,g)}) \longrightarrow S_2$ by $\psi(\rho_{(\varphi,g)}(x)) = \varphi(x)$ for all $x \in S_1$. Then, we have

$$\rho_{(\varphi,g)}(x) = \rho_{(\varphi,g)}(y) \Longleftrightarrow x\rho_{(\varphi,g)}y \Longleftrightarrow \varphi(x) = \varphi(y),$$

which implies that ψ is well-defined and one to one. Clearly, ψ is onto. Now, we prove (ψ, g) is a homomorphism. We have

$$\begin{aligned} \psi(\rho_{(\varphi,g)}(x) \odot \gamma \odot \rho_{(\varphi,g)}(y)) &= \psi(\rho_{(\varphi,g)}(x\gamma y)) \\ &= \varphi(x\gamma y) = \varphi(x)g(\gamma)\varphi(y) \\ &= \psi(\rho_{(\varphi,g)}(x))g(\gamma)\psi(\rho_{(\varphi,g)}(y)), \end{aligned}$$

for all $\rho_{(\varphi,g)}(x), \rho_{(\varphi,g)}(y) \in S_1 : \rho_{(\varphi,g)}$ and $\gamma \in \Gamma_1$. Therefore, (ψ, g) is a homomorphism. Also $\varphi(x) = \psi(\rho_{(\varphi,g)}(x)) = \psi(\Pi_{S_1}(x))$ and $g \circ 1_{\Gamma_1} = g$, which imply that the diagram is commutative. Let $(\overline{\psi}, g) : (S_1 : \rho_{(\varphi,g)}, \Gamma_1) \longrightarrow (S_2, \Gamma_2)$ be a homomorphism such that $\overline{\psi} \circ \Pi_{S_1} = \varphi$. We have

$$\overline{\psi}(\rho_{(\varphi,g)}(x)) = \overline{\psi}(\Pi_{S_1}(x)) = \varphi(x) = \psi(\Pi_{S_1}(x)) = \psi(\rho_{(\varphi,g)}(x)).$$

Therefore, (ψ, g) is unique and the proof is completed. □

Theorem 3.15. *Let* $(S_1, \Gamma_1) \xrightarrow{(\varphi_1,g_1)} (S_2, \Gamma_2) \xrightarrow{(\varphi_2,g_2)} (S_3, \Gamma_3)$ *be a sequence of homomorphisms. Then there exists an unique homomorphism*

$$(\psi, g_1) : (S_1 : \rho_{(\varphi_1,g_1)}, \Gamma_1) \longrightarrow (S_2 : \rho_{(\varphi_2,g_2)}, \Gamma_2)$$

such that the following diagram is commutative:

$$\begin{array}{ccccc} & (S_1,\Gamma_1) & \overset{(\varphi_1,g_1)}{- - - \longrightarrow} & (S_2,\Gamma_2) & \\ (\Pi_{S_1},1_{\Gamma_1}) & \Big\downarrow & & \Big\downarrow & (\Pi_{S_2},1_{\Gamma_2}) \\ & (S_1:\rho_{(\varphi_1,g_1)},\Gamma_1) & \overset{(\psi,g_1)}{- - - \longrightarrow} & (S_2:\rho_{(\varphi_2,g_2)},\Gamma_2) & \end{array}$$

Moreover, if (φ_1, g_1) is onto and (φ_2, g_2) is one to one, then (ψ, g_1) is an isomorphism.

Proof. Define $\psi : S_1 : \rho_{(\varphi_1,g_1)} \longrightarrow S_2 : \rho_{(\varphi_2,g_2)}$ by $\psi(\rho_{(\varphi_1,g_1)}(x)) = \rho_{(\varphi_2,g_2)}(\varphi_1(x))$ for all $\rho_{(\varphi_1,g_1)}(x) \in S_1 : \rho_{(\varphi_1,g_1)}$. We prove that ψ is well-defined:

$$\begin{aligned} \rho_{(\varphi_1,g_1)}(x) = \rho_{(\varphi_1,g_1)}(y) &\Longrightarrow \varphi_1(x) = \varphi_1(y) \\ &\Longrightarrow \rho_{(\varphi_2,g_2)}(\varphi_1(x)) = \rho_{(\varphi_2,g_2)}(\varphi_1(y)) \\ &\Longrightarrow \psi(\rho_{(\varphi_1,g_1)}(x)) = \psi(\rho_{(\varphi_2,g_2)}(y)). \end{aligned}$$

Now, we prove that (ψ, g_1) is a homomorphism. We have

$$\begin{aligned} \psi(\rho_{(\varphi_1,g_1)}(x) \odot \gamma \odot \rho_{(\varphi_1,g_1)}(y)) &= \psi(\rho_{(\varphi_1,g_1)}(x\gamma y)) \\ &= \rho_{(\varphi_2,g_2)}(\varphi_1(x\gamma y)) \\ &= \rho_{(\varphi_2,g_2)}(\varphi_1(x)g_1(\gamma)\varphi_1(y)) \\ &= \rho_{(\varphi_2,g_2)}(\varphi_1(x)) \odot g_1(\gamma) \odot \rho_{(\varphi_2,g_2)}(\varphi_1(y)) \\ &= \psi(\rho_{(\varphi_1,g_1)}(x)) \odot g_1(\gamma) \odot \psi(\rho_{(\varphi_1,g_1)}(y)). \end{aligned}$$

Therefore, (ψ, g_1) is a homomorphism. Also, we have

$$\psi(\Pi_{S_1}(x)) = \psi(\rho_{(\varphi_1,g_1)}(x)) = \rho_{(\varphi_2,g_2)}(\varphi_1(x)) = \Pi_{S_2}(\varphi_1(x)),$$

and $g_1 \circ 1_{\Gamma_1} = 1_{\Gamma_2} \circ g_1$. Therefore, the diagram is commutative. Let $(\overline{\psi}, g_1) : (S_1 : \rho_{(\varphi_1,g_1)}, \Gamma_1) \longrightarrow (S_2 : \rho_{(\varphi_2,g_2)}, \Gamma_2)$ be a homomorphism which makes the diagram commutative. We have

$$\overline{\psi}(\rho_{(\varphi_1,g_1)}(x)) = \overline{\psi}(\Pi_{S_1}(x)) = \Pi_{S_2}(\varphi_1(x)) = \rho_{(\varphi_2,g_2)}(\varphi_1(x)) = \psi(\rho_{(\varphi_1,g_1)}(x)).$$

Therefore, (ψ, g_1) is unique and the proof is completed. □

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