Thai Journal of Mathematics
Volume 11 (2013) Number 3 : 543–562
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Ruin Probability-Based Initial Capital of the Discrete-Time Surplus Process in Insurance under Reinsurance as a Control Parameter

A. Intarasit[†,1], **K. Chuarkham**[‡], **P. Sattayatham**[§]

[†]Department of Mathematics and Computer Science
Prince of Songkla University, Pattani 94000, Thailand
e-mail : iarthit@bunga.pn.psu.ac.th

[‡]Faculty of Commerce and Management, Prince of Songkla University
Trang 92000, Thailand

[§]School of Mathematics, Suranaree University of Technology
Nakhon Ratchasima 30000, Thailand

Abstract : This paper studied an insurance model under the condition that the claims can be control by reinsurance and an insurance company requires a sufficient initial capital to ensure a ruin probability will not exceed a given quantity α.The objective is to find the minimum initial capital for a given ruin probability under the condition that the claims can be controlled by reinsurance. The existence of the minimum initial capital was proved and an example in approximating the minimum initial capital for exponential claims was finally given.

Keywords : insurance; reinsurance; capital reserve; ruin probability.
2013 Mathematics Subject Classification : 91B30; 93E20.

1 Introduction

In this paper, we assume that all processes are defined in a probability space $(\Omega, \Im, P)$. Claims happen at the times T_i, satisfying $0 = T_0 \leq T_1 \leq T_2 \leq \cdots$. We call them *arrivals*. The n^{th} claim arriving at time T_n causes the *claim size* Y_n.

[1]Corresponding author.

The *interarrival*, $Z_n := T_n - T_{n-1}$ is the length of time between the $(n-1)^{th}$ claim and the n^{th} claim. By a *period* n, we shall mean the random interval $[T_{n-1}, T_n), n \geq 1$.

Now let a constant c_0 represent the premium rate for one unit time; the random variable $c_0 \sum_{i=1}^{n} Z_i = c_0 T_n$ describes the inflow of capital into the business in $[0, T_n]$, and $\sum_{i=1}^{n} Y_i$ describes the outflow of capital due to payments for claims occurring in $[0, T_n]$. Therefore, the quantity

$$X_0 = x,\ X_n = x + c_0 \sum_{i=1}^{n} Z_i - \sum_{i=1}^{n} Y_i,\ n = 1, 2, 3, \ldots \tag{1.1}$$

is the discrete-time surplus process at time T_n with the constant $x \geq 0$ as initial capital.

The general approach to studying ruin probability in the discrete-time surplus process; Chan and Zhang [3], Pavlovao and Willmot [11], Dickson [4] Li [8][9] and Rongming and Haifeng [12]. The researcher studied the ruin probability in term of initial capital x.

In this paper, we study the minimum initial capital x for the discrete-time surplus process under the condition that the claims can be controlled by reinsurance and the given boundary of the ruin probability.

2 Model Descriptions

Let $\{X_n, n \geq 0\}$ be the surplus process which can be controlled by choosing a retention level $b \in [\underline{b}, \overline{b}], 0 \leq \underline{b} \leq b \leq \overline{b} \leq \infty$, of a reinsurance for one period. Next, for each level b, an insurer pays a premium rate to a reinsurer which is deducted from c_0. As a result, the insurer's income rate will be represented by a function $c(b)$. The level $\overline{b}$ stands for the control action without reinsurance, so that $c_0 = c(\overline{b})$ and the level $\underline{b}$ is the smallest retention level which can be chosen. As a consequence, we obtain the *net income rate* $c(b)$ where $0 \leq c(b) \leq c_0$ for all $b \in [\underline{b}, \overline{b}]$ and $c(b)$ is non-decreasing. The premium rate for one unit time c_0 and the net income rate $c(b)$ are assumed to be satisfied the following:

$$c_0 > \frac{E[Y]}{E[Z]} \quad \text{and} \quad c(b) > \frac{E[h(b,Y)]}{E[Z]} \tag{2.1}$$

where Y is a claim size and Z is an interarrival.

Moreover, by the *expected value principle* c_0 and $c(b)$ can be calculated as follows:

$$\begin{aligned} c_0 &= (1+\theta_0)\frac{E[Y]}{E[Z]} \quad \text{and} \\ c(b) &= c_0 - (1+\theta_1)\frac{E[Y - h(b,Y)]}{E[Z]} \end{aligned} \tag{2.2}$$

where $0 < \theta_0 < 1$ and $0 < \theta_1 < 1$ are the *safety loadings* of the insurer and the reinsurer respectively. The measurable function $h(b, y)$ is the part of the claim size y paid by the insurer, and the remaining part $y - h(b, y)$ which is called *reinsurance recovery* paid by the reinsurer. In the case of an *excess of loss reinsurance*, we have

$$h(b, y) = \min\{b, y\} \text{ with retention level } 0 \le \underline{b} \le b \le \overline{b} = \infty.$$

In the case of a *proportional reinsurance*, we have

$$h(b, y) = by \text{ with retention level } 0 \le \underline{b} \le b \le \overline{b} = 1.$$

For each $n \in \{1, 2, 3, \ldots\}$, let b_{n-1} be a retention level (control action) at the time T_{n-1} and let $Z_n = 1$. Therefore, we can modify the surplus process (1.1) to be the following:

$$X_n = x + \sum_{i=1}^{n} c(b_{i-1}) - \sum_{i=1}^{n} h(b_{i-1}, Y_i) \tag{2.3}$$

where $X_0 = x$.

We see that the process $\{X_n, n \ge 0\}$ is driven by the sequence of retention level (control actions) $\{b_{n-1}, n \ge 1\}$ and the sequence of claims $\{Y_n, n \ge 1\}$. So, we make the following assumption:

Assumption 1. Independence Assumption (IA)
The sequence of claims $\{Y_n, n \ge 1\}$ is independent and identically distributed (iid) random variables.

From Assumption IA, it follows that $\{h(b_{n-1}, Y_n), n \ge 1\}$ is an independent sequence.

Definition 2.1. *Let $N \in \{1, 2, 3, \ldots\}$ be a time horizon (number of periods). A plan for the time N is a (finite) sequence $\pi = \{b_{n-1}\}_{n=1}^{N}$ of $b_{n-1} \in [\underline{b}, \overline{b}]$ for $n = 1, 2, 3, \ldots, N$. A set of all plans for the time horizon N over a control space $[\underline{b}, \overline{b}]$ is denoted by $\mathcal{P}(N, [\underline{b}, \overline{b}])$. A plan $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ is said to be stationary, if $b_0 = b_1 = \cdots = b_{N-1}$.*

3 Main Results

In this section, we consider a finite-time ruin probability of the discrete-time surplus process in equation (2.3) where the sequence of claims $\{Y_n, n \ge 1\}$ satisfy Assumption IA. Let F_{Y_1} be the distribution function of Y_1, i.e.,

$$F_{Y_1}(y) = P(Y_1 \le y).$$

Let $N \in \{1, 2, 3, \ldots\}$ be a time horizon and $x \ge 0$ be an initial capital. The *survival probability* at a time $n \in \{1, 2, 3, \ldots, N\}$ is defined by

$$\varphi_n(x, \pi) := P(X_1 \ge 0, X_2 \ge 0, X_3 \ge 0, \ldots, X_n \ge 0 | X_0 = x) \tag{3.1}$$

where $\pi \in \mathcal{P}(N, [\underline{b}, \bar{b}])$. Moreover, the *ruin probability* at a time $n \in \{1, 2, 3, \ldots, N\}$ is defined by

$$\Phi_n(x, \pi) = 1 - \varphi_n(x, \pi). \tag{3.2}$$

Definition 3.1. *Let $\{X_n, n \geq 0\}$ be the surplus process in equation (2.3), driven by the sequence of control actions $\{b_{n-1}, n \geq 1\}$ and the sequence of claims $\{Y_n, n \geq 1\}$. Let $\{c(b_{n-1})\}_{n\geq 1}$ be a sequence of net income rates and $x \geq 0$ be an initial capital. For each time horizon $N \in \{1, 2, 3, \ldots\}$, let $\pi \in \mathcal{P}(N, [\underline{b}, \bar{b}])$ and $\alpha \in (0, 1)$. If $\Phi_N(x, \pi) \leq \alpha$, then x is called an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$. Particularly, if*

$$x^* = \min_{x\geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\}$$

exists, x^* is called the minimum initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$.

3.1 Ruin and Survival Probability

We defined a *total claim process* by

$$S_n := h(b_0, Y_1) + h(b_1, Y_2) + \cdots + h(b_{n-1}, Y_n)$$

for all $n \in \{1, 2, 3, \ldots\}$. The survival probability at the time horizon N as mentioned in equation (3.1) can be expressed as follows:

$$\begin{aligned}\varphi_N(x, \pi) &= P\left(S_1 \leq x + c(b_0), S_2 \leq x + \sum_{n=1}^{2} c(b_{n-1}), \ldots, S_N \leq x + \sum_{n=1}^{N} c(b_{n-1})\right) \\ &= P\left(\bigcap_{n=1}^{N} \left\{S_n \leq x + \sum_{k=1}^{n} c(b_{k-1})\right\}\right). \end{aligned} \tag{3.3}$$

From equation (3.3), we have

$$\varphi_N(x, \pi) = E\left[\prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n - \sum_{k=1}^{n} c(b_{k-1}) - x\right)\right],$$

where

$$1_A(x) = \begin{cases} 1 & , \ x \in A \\ 0 & , \ \text{else} , \end{cases}$$

for all $A \subseteq R$. For each $a \in R$ and $x \geq 0$, we obtain

$$1_{(-\infty,0]}(a - x) = \begin{cases} 1 & , \ x \geq a \\ 0 & , \ x < a. \end{cases}$$

Then $1_{(-\infty,0]}(a-x)$ is non-decreasing in x and right continuous on $[0,\infty)$. This implies that $\prod_{n=1}^{N} 1_{(-\infty,0]}(a_n-x)$ is also non-decreasing in x and right continuous on $[0,\infty)$ where $a_n \in R, n=1,2,3,\ldots,N$. For each plan $\pi=\{b_0,b_1,b_2,\ldots,b_{N-1}\}$, by the Dominated Convergence Theorem, we get

$$\begin{aligned}
\lim_{u\to x^+}\varphi_N(u,\pi) &= \lim_{u\to x^+} E\left[\prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n-\sum_{k=1}^{n}c(b_{k-1})-u\right)\right]\\
&= E\left[\lim_{u\to x^+}\prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n-\sum_{k=1}^{n}c(b_{k-1})-u\right)\right]\\
&= E\left[\prod_{n=1}^{N} 1_{(-\infty,0]}\left(S_n-\sum_{k=1}^{n}c(b_{k-1})-x\right)\right]\\
&= \varphi_N(x,\pi).
\end{aligned}$$

Therefore, $\varphi_N(x,\pi)$ is non-decreasing in x and right continuous on $[0,\infty)$. This implies that $\Phi_N(x,\pi)=1-\varphi_N(x,\pi)$ is non-increasing in x and also right continuous on $[0,\infty)$.

Theorem 3.2. *Let $N\in\{1,2,3,\ldots\}$, $\pi\in\mathcal{P}(N,[\underline{b},\overline{b}])$, and let $x\ge 0$ be given. Then*

$$\lim_{x\to\infty}\varphi_N(x,\pi)=1 \text{ and } \lim_{x\to\infty}\Phi_N(x,\pi)=0.$$

Proof : Firstly, we will show the following relation

$$\bigcap_{n=1}^{N}\{\omega : h(b_{n-1},Y_n)(\omega)\le x+c(b_{n-1})\}\subseteq\bigcap_{n=1}^{N}\left\{\omega : S_n(\omega)\le Nx+\sum_{k=1}^{n}c(b_{k-1})\right\}. \tag{3.4}$$

Let $\omega_0\in\bigcap_{n=1}^{N}\{\omega : h(b_{n-1},Y_n)(\omega)\le x+c(b_{n-1})\}$ be given. For each $n\in\{1,2,3,\ldots,N\}$, we have $h(b_{n-1},Y_n)(\omega_0)\le x+c(b_{n-1})$. Thus,

$$\begin{aligned}
S_n(\omega_0) &= \sum_{k=1}^{n}h(b_{k-1},Y_k)(\omega_0)\\
&\le nx+\sum_{k=1}^{n}c(b_{k-1})\\
&\le Nx+\sum_{k=1}^{n}c(b_{k-1}).
\end{aligned}$$

That is $\omega_0\in\left\{\omega : S_n(\omega)\le Nx+\sum_{k=1}^{n}c(b_{k-1})\right\}$. Therefore (3.4) follows. By Assumption IA, the process $\{h(b_{n-1},Y_n), n\ge 1\}$ is an independent sequence, then we have

$$P\left(\bigcap_{n=1}^{N}\{h(b_{n-1},Y_n)\leq x+c(b_{n-1})\}\right)=\prod_{n=1}^{N}P\left(h(b_{n-1},Y_n)\leq x+c(b_{n-1})\right). \tag{3.5}$$

Note that $Y_n \geq h(b_{n-1}, Y_n)$ for all $n \in \{1, 2, 3, \ldots, N\}$, then

$$\{\omega : Y_n(\omega) \leq x + c(b_{n-1})\} \subseteq \{\omega : h(b_{n-1}, Y_n)(\omega) \leq x + c(b_{n-1})\}.$$

From equation (3.5), we get

$$\begin{aligned} P\left(\bigcap_{n=1}^{N}\{h(b_{n-1},Y_n)\leq x+c(b_{n-1})\}\right) &\geq \prod_{n=1}^{N}P\left(Y_n\leq x+c(b_{n-1})\right) \\ &= \prod_{n=1}^{N}F_{Y_n}\left(x+c(b_{n-1})\right). \end{aligned} \tag{3.6}$$

Moreover, it follows from equation (3.3) that

$$\varphi_N(Nx,\pi)=P\left(\bigcap_{n=1}^{N}\left\{S_n\leq Nx+\sum_{k=1}^{n}c(b_{k-1})\right\}\right). \tag{3.7}$$

Thus

$$\begin{aligned} \prod_{n=1}^{N}F_{Y_n}\left(x+c(b_{n-1})\right) &\leq P\left(\bigcap_{n=1}^{N}\{h(b_{n-1},Y_n)\leq x+c(b_{n-1})\}\right) \\ &\leq P\left(\bigcap_{n=1}^{N}\left\{S_n\leq Nx+\sum_{k=1}^{n}c(b_{k-1})\right\}\right) \quad \text{(By (3.4))} \\ &= \varphi_N(Nx,\pi)\leq 1. \quad \text{(By equation (3.7))} \end{aligned}$$

Since $F_{Y_n}(x + c(b_{n-1})) \to 1$ as $x \to \infty$ for $n = 1, 2, 3, \ldots, N$, then

$$\prod_{n=1}^{N}F_{Y_n}\left(x+c(b_{n-1})\right)\to 1 \text{ as } x\to\infty.$$

Hence $\varphi_N(x, \pi) \to 1$ and $\Phi_N(x, \pi) = 1 - \varphi_N(x, \pi) \to 0$ for $x \to \infty$. The proof is now complete.

Corollary 3.3. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N,[\underline{b},\overline{b}])$, $\alpha \in (0,1)$, and let $x \geq 0$ be given. Then there exists $\tilde{x} \geq 0$ such that, for all $x \geq \tilde{x}$, x is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$.*

Proof : Let

$$\tilde{x} = sup\{x \geq 0 \mid \Phi_N(x,\pi) > \alpha\}.$$

We consider by cases:

Case 1. $\Phi_N(\tilde{x},\pi) > \alpha$. Since $\Phi_N(x,\pi)$ is non-increasing in x, then $\Phi_N(x,\pi) \leq \alpha$ for all $x > \tilde{x}$, i.e., $\Phi_N(x,\pi) \leq \alpha$ on $(\tilde{x},\infty)$. Thus

$$\lim_{x\to\tilde{x}^+} \Phi_N(x,\pi) \leq \alpha.$$

Since $\Phi_N(x,\pi)$ right continuous on $(\tilde{x},\infty)$ and non-increasing in x, then

$$\alpha < \Phi_N(\tilde{x},\pi) = \lim_{x\to\tilde{x}^+} \Phi_N(x,\pi) \leq \alpha.$$

Hence, $\Phi_N(\tilde{x},\pi) = \alpha$. As a result $\tilde{x}$ is a smallest real constant such that, for all $x \geq \tilde{x}$, x is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$.

Case 2. $\Phi_N(\tilde{x},\pi) \leq \alpha$. Since $\Phi_N(x,\pi)$ is non-increasing in x, then $\Phi_N(x,\pi) > \alpha$ for all $x < \tilde{x}$ and $\Phi_N(x,\pi) \leq \alpha$ for all $x \geq \tilde{x}$, i.e., $\tilde{x}$ is a smallest real constant such that, for all $x \geq \tilde{x}$, x is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq 1}, \{h(b_{n-1}, Y_n)\}_{n\geq 1})$.

3.2 Bounds of the Ruin Probability

In this part, we shall describe the upper bound of the ruin probability with negative exponential. In order to prove the following lemma, we shall use an equivalent definition of the ruin probability which will be given as follows:

$$\Phi_n(x,\pi) = P\left(\max_{1\leq k\leq n}\left(\sum_{i=1}^{k}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x\right),\ n = 1,2,3,\ldots.$$

Lemma 3.4. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N,[\underline{b},\overline{b}])$ be stationary, $\alpha \in (0,1)$, and let $x \geq 0$ be given. Then the ruin probability at the time N satisfies the following equation*

$$\Phi_N(x,\pi) = \Phi_1(x,\pi) + \int\limits_{\{y:0\leq h(b_0,y)\leq x+c(b_0)\}} \Phi_{N-1}(x + c(b_0) - h(b_0,y),\pi)dF_{Y_1}(y) \tag{3.8}$$

where $\Phi_0(x,\pi) = 0$.

Proof : We prove equation (3.8) by induction. We start with $N = 1$. Since $\Phi_0(x,\pi) = 0$ for all $x \geq 0$, then

$$\int\limits_{\{y:0\leq h(b_0,y)\leq x+c(b_0)\}} \Phi_0(x + c(b_0) - h(b_0, y), \pi) dF_{Y_1}(y) = 0.$$

This proves equation (3.8) for $N = 1$. Now assume that equation (3.8) holds for $1 < n \leq N - 1$. Then

$$\Phi_N(x,\pi) = P\left(\max_{1\leq n\leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x\right)$$
$$= P\left(\left\{\max_{1\leq n\leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x\right\}\bigcap\Omega\right)$$
$$= P\left(\left\{\max_{1\leq n\leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x\right\}\bigcap\left\{\left\{h(b_0,Y_1) - c(b_0) > x\right\}\right.\right.$$
$$\left.\left.\bigcup\left\{h(b_0,Y_1) - c(b_0) \leq x\right\}\right\}\right)$$
$$= P\left(\max_{1\leq n\leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x, h(b_0,Y_1) - c(b_0) > x\right)$$
$$+ P\left(\max_{1\leq n\leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x, h(b_0,Y_1) - c(b_0) \leq x\right).$$

Since π is stationary and $\{Y_n\}_{n\geq1}$ is an iid sequence, then

$$\left\{\omega\in\Omega : \max_{1\leq n\leq N}\left(\sum_{i=1}^{n}(h(b_{i-1},Y_i)(\omega) - c(b_{i-1}))\right) > x, h(b_0,Y_1)(\omega) - c(b_0) > x\right\}$$
$$= \left\{\omega\in\Omega : h(b_0,Y_1)(\omega) - c(b_0) > x\right\}.$$

This result implies

$$\Phi_N(x,\pi) = P(h(b_0,Y_1) - c(b_0) > x)$$
$$+P\left(\max_{2\leq n\leq N}\left(h(b_0,Y_1) - c(b_0) + \sum_{i=2}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x, h(b_0,Y_1) - c(b_0) \leq x\right)$$
$$= \Phi_1(x,\pi)$$
$$+P\left(h(b_0,Y_1) - c(b_0) + \max_{2\leq n\leq N}\left(\sum_{i=2}^{n}(h(b_{i-1},Y_i) - c(b_{i-1}))\right) > x, h(b_0,Y_1) - c(b_0) \leq x\right)$$
$$= \Phi_1(x,\pi)$$
$$+ E\left[1_{h(b_0,Y_1)-c(b_0)\leq x,\ h(b_0,Y_1)-c(b_0)+\max_{2\leq n\leq N}\left(\sum_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)>x}\right]$$

$$= \Phi_1(x,\pi)$$
$$+ E\left[1_{h(b_0,Y_1)-c(b_0)\le x} \cdot 1_{h(b_0,Y_1)-c(b_0)+\max\limits_{2\le n\le N}\left(\sum\limits_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)>x}\right]$$
$$= \Phi_1(x,\pi)$$
$$+ E\left[E\left[1_{h(b_0,Y_1)-c(b_0)\le x} \cdot 1_{h(b_0,Y_1)-c(b_0)+\max\limits_{2\le n\le N}\left(\sum\limits_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)>x}|\sigma(Y_1)\right]\right]$$
$$= \Phi_1(x,\pi)$$
$$+E\left[1_{h(b_0,Y_1)\le x+c(b_0)} \cdot E\left[1_{\max\limits_{2\le n\le N}\left(\sum\limits_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)+(h(b_0,Y_1)-x-c(b_0))>0}|\sigma(Y_1)\right]\right]$$
$$= \Phi_1(x,\pi) + E\left[1_{h(b_0,Y_1)\le x+c(b_0)} \cdot E\left[1_{(0,\infty)}(Z+W)|\sigma(Y_1)\right]\right] \tag{3.9}$$

where $Z = \max\limits_{2\le n\le N}\left(\sum\limits_{i=2}^{n}(h(b_{i-1},Y_i)-c(b_{i-1}))\right)$ and $W = h(b_0,Y_1)-x-c(b_0)$. Since $\{h(b_{n-1},Y_n)\}_{n\ge 1}$ is an independent sequence, then Z and W are independent. It follows from [5, exercise 9, page 341] that

$$\begin{aligned} E\left[1_{(0,\infty)}(Z+W)|\sigma(Y_1)\right] &= \int\limits_{\omega\in\ \Omega} 1_{(0,\infty)}(Z(\omega)+W|\sigma(Y_1))dP_Z(\omega) \\ &= \int\limits_{R} 1_{(0,\infty)}(z+W)dF_Z(z). \end{aligned}$$

This implies that

$$\Phi_N(x,\pi) = \Phi_1(x,\pi) + E\left[1_{h(b_0,Y_1)\le x+c(b_0)} \cdot \left(\int\limits_R 1_{(0,\infty)}(z+W)dF_Z(z)\right)\right]$$
$$= \Phi_1(x,\pi) + E\left[1_{h(b_0,Y_1)\le x+c(b_0)} \cdot \left(\int\limits_R 1_{(0,\infty)}(z+h(b_0,Y_1)-x-c(b_0))dF_Z(z)\right)\right]$$
$$= \Phi_1(x,\pi) + \int\limits_{\{\omega\in\Omega: h(b_0,Y_1)(\omega)\in[0,x+c(b_0)]\}} \left(\int\limits_R 1_{(0,\infty)}(z+h(b_0,Y_1)(\omega)-x-c(b_0))dF_Z(z)\right) dP(\omega)$$
$$= \Phi_1(x,\pi) + \int\limits_{\{\omega\in\Omega: h(b_0,Y_1)(\omega)\in[0,x+c(b_0)]\}} E\left[1_{Z>x+c(b_0)-h(b_0,Y_1)(\omega)}\right] dP(\omega)$$
$$= \Phi_1(x,\pi) + \int\limits_{\{\omega\in\Omega: h(b_0,Y_1)(\omega)\in[0,x+c(b_0)]\}} P\left(Z > x+c(b_0)-h(b_0,Y_1)(\omega)\right) dP(\omega)$$

$$= \Phi_1(x,\pi) + \int_{\{y\in R: 0\le h(b_0,y)\le x+c(b_0)\}} P\left(Z > x + c(b_0) - h(b_0,y)\right) dF_{Y_1}(y)$$

$$= \Phi_1(x,\pi) + \int_{\{y: 0\le h(b_0,y)\le x+c(b_0)\}} \Phi_{N-1}(x + c(b_0) - h(b_0,y),\pi) dF_{Y_1}(y).$$

This proves equation (3.8).

Remark 3.5. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N,[\underline{b},\overline{b}])$ be stationary, $\alpha \in (0,1)$. Assume that $\{Y_n, n \ge 1\}$ is an iid sequence of exponential distribution with intensity $\lambda > 0$, i.e., Y_1 has the probability density function*

$$f(y) = \lambda e^{-\lambda y}.$$

By Lemma 3.4, the ruin probability can be written in a recursive form as follows:

Case 1: For an excess of loss reinsurance, we get

$$\Phi_n(x,\pi) = \Phi_{n-1}(x,\pi) + \frac{[\lambda(x+nc(b_0))]^{n-1}}{(n-1)!} e^{-\lambda[x+nc(b_0)]} \frac{x+c(b_0)}{x+nc(b_0)} \tag{3.10}$$

for $b_0 \ge x + c(b_0)$ and $n = 1,2,3,\ldots,N$.

Case 2: For a proportional reinsurance, we get

$$\Phi_0(x,\pi) = 0 \text{ and}$$

$$\Phi_n(x,\pi) = \Phi_{n-1}(x,\pi) + \frac{1}{(n-1)!}\left[\frac{\lambda}{b_0}(x+nc(b_0))\right]^{n-1} e^{-\frac{\lambda}{b_0}(x+nc(b_0))} \frac{x+c(b_0)}{x+nc(b_0)} \tag{3.11}$$

for all $n = 1,2,3,\ldots,N$. Further, for $b_0 = \overline{b}_0 = 1$, we also obtained the recursive form as follows :

$$\Phi_0(x,\pi) = 0 \text{ and}$$

$$\Phi_n(x,\pi) = \Phi_{n-1}(x,\pi) + \frac{1}{(n-1)!}[\lambda(x+nc_0)]^{n-1} e^{-\lambda(x+nc_0)} \frac{x+c_0}{x+nc_0}$$

for all $n = 1,2,3,\ldots,N$.

Definition 3.6. *(Sub-adjustment coefficient). Let $s > 0$ and Y be a non-negative random variable. If there exists $d_0 > 0$ such that*

$$E\left[e^{d_0 Y}\right] \le e^{d_0 s}, \tag{3.12}$$

then d_0 is called a sub-adjustment coefficient of (s,Y). Specifically, if (3.12) is an equality then d_0 is called an adjustment coefficient of (s,Y).

Theorem 3.7. *Let* $N \in \{1, 2, 3, \ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ *be stationary, and let* $c(b_0) > 0$ *be a net income rate. If* $d_0 > 0$ *is a sub-adjustment coefficient of* $(c(b_0), h(b_0, Y_1))$, *then*

$$\Phi_n(x, \pi) \leq e^{-d_0 x}, \tag{3.13}$$

for all $x \geq 0$ *and all* $n = 1, 2, 3, \ldots, N$.

Proof : Let $x \geq 0$ and $d_0 > 0$ be a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$, i.e.,

$$E\left[e^{d_0 h(b_0, Y_1)}\right] \leq e^{d_0 c(b_0)}.$$

We will prove this theorem by induction. We start with $n = 1$,

$$\begin{aligned}
\Phi_1(x, \pi) &= P(h(b_0, Y_1) > x + c(b_0)) \\
&= P(d_0 h(b_0, Y_1) > d_0(x + c(b_0))) \\
&= P(e^{d_0 h(b_0, Y_1)} > e^{d_0(x + c(b_0))}) \\
&\leq \frac{E\left[e^{d_0 h(b_0, Y_1)}\right]}{e^{d_0(x + c(b_0))}} \qquad \text{(By Markov's inequality)} \\
&\leq \frac{e^{d_0 c(b_0)}}{e^{d_0(x + c(b_0))}} = e^{-d_0 x}.
\end{aligned}$$

Let $k \leq N - 1$. Assume that inequality (3.13) holds for $1 < n \leq k$. Next, we shall show that inequality (3.13) holds for $n = k + 1$. By Lemma 3.4 and inductive assumption, we get

$$\begin{aligned}
&\Phi_{k+1}(x, \pi) \\
&= \Phi_1(x, \pi) + \int_{\{y: 0 \leq h(b_0, y) \leq x + c(b_0)\}} \Phi_k(x + c(b_0) - h(b_0, y), \pi) dF_{Y_1}(y) \\
&\leq \Phi_1(x, \pi) + \int_{\{y: 0 \leq h(b_0, y) \leq x + c(b_0)\}} e^{-d_0(x + c(b_0) - h(b_0, y))} dF_{Y_1}(y). \qquad (3.14)
\end{aligned}$$

Next, we will calculate the first term of right-hand side of inequality (3.14).

$$\begin{aligned}
&\Phi_1(x, \pi) \\
&= P(h(b_0, Y_1) > x + c(b_0)) \\
&= P\left(e^{d_0 h(b_0, Y_1)} 1_{(x + c(b_0), \infty)}(h(b_0, Y_1)) > e^{d_0(x + c(b_0))}\right) \\
&\leq \frac{E\left[e^{d_0 h(b_0, Y_1)} 1_{(x + c(b_0), \infty)}(h(b_0, Y_1))\right]}{e^{d_0(x + c(b_0))}} \qquad \text{(By Markov's inequality)}
\end{aligned}$$

$$= \frac{\int\limits_R e^{d_0 h(b_0,y)} 1_{(x+c(b_0),\infty)}(h(b_0,y))dF_{Y_1}(y)}{e^{d_0(x+c(b_0))}}$$

$$= \frac{\int\limits_{\{y:x+c(b_0)<h(b_0,y)<\infty\}} e^{d_0 h(b_0,y)} dF_{Y_1}(y)}{e^{d_0(x+c(b_0))}}$$

$$= \int\limits_{\{y:x+c(b_0)<h(b_0,y)<\infty\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y).$$

Thus inequality (3.14) can be modified to be the following

$$\Phi_{k+1}(x,\pi)$$

$$\le \int\limits_{\{y:x+c(b_0)<h(b_0,y)<\infty\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y)$$

$$+ \int\limits_{\{y:0\le h(b_0,y)\le x+c(b_0)\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y)$$

$$= \int\limits_{\{y:0\le h(b_0,y)<\infty\}} e^{-d_0(x+c(b_0)-h(b_0,y))} dF_{Y_1}(y)$$

$$= \frac{e^{-d_0 x}}{e^{d_0 c(b_0)}} \int\limits_{\{y:0\le h(b_0,y)<\infty\}} e^{d_0 h(b_0,y)} dF_{Y_1}(y)$$

$$= \frac{e^{-d_0 x}}{e^{d_0 c(b_0)}} E\left[e^{d_0 h(b_0,Y_1)}\right]$$

$$\le \frac{e^{-d_0 x}}{e^{d_0 c(b_0)}} e^{d_0 c(b_0)} = e^{-d_0 x}.$$

This proves equation (3.13) for $n = k+1$ and concludes the proof.

Corollary 3.8. *Let* $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N,[\underline{b},\overline{b}])$ *be stationary,* $\alpha \in (0,1)$, *and let* $c(b_0) > 0$ *be a net income rate. Assume that* $d_0 > 0$ *is a sub-adjustment coefficient of* $(c(b_0), h(b_0,Y_1))$, *then there exists an acceptable initial capital* $x (x \ge 0)$ *corresponding to* $(\alpha, N, \{c(b_{n-1}) = c(b_0)\}_{n\ge1}, \{h(b_0,Y_n)\}_{n\ge1})$ *such that*

$$0 \le x \le -\frac{ln\alpha}{d_0} \;\; or \;\; \alpha \le e^{-d_0 x}.$$

Proof : Let $d_0 > 0$ be a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$. By Theorem 3.7, we have

$$\Phi_N(u, \pi) \leq e^{-d_0 u},$$

for all $u \geq 0$. Let $\alpha \in (0, 1)$. By Corollary 3.3, there exists $v \geq 0$ which is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq 1}, \{h(b_0, Y_n)\}_{n\geq 1})$. By Definition 3.1, we have

$$\Phi_N(v, \pi) \leq \alpha.$$

Since $\Phi_N(v, \pi)$ is non-increasing in v for each π, then there exists $0 \leq x \leq v$ such that $\alpha = \Phi_N(x, \pi) \leq e^{-d_0 x}$. Hence x is an acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq 1}, \{h(b_0, Y_n)\}_{n\geq 1})$. The proof is now complete.

Note:It's known that a large initial capital results in a small ruin probability. However, an insurance company usually does not posses unlimited initial capital, but only a small initial capital, that must be sufficient for a predetermined solvency (not ruin) condition for the firm is preferable. If an acceptable ruin probability is fixed, the firm can find an interval of acceptable initial capital by virtue of Corollary (3.8).

Example 3.9. *(Exponential claims under the proportional reinsurance). We assume that $\{Y_n\}_{n\geq 1}$ is a sequence of claims with iid exponential $Exp(1)$, and $\{X_n\}_{n\geq 0}$ is a sequence of surplus which satisfies the model (2.3). Let $N \in \{1, 2, 3, \ldots\}$, and $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ be stationary. Suppose that $h(b_0, y)$ is the proportional reinsurance with retention level b_0, and $c(b_0) > 0$ is a net income rate which is calculated by the expected value principle, i.e.,*

$$c(b_0) = c_0 - (1 + \theta_1)E[Y_1 - h(b_0, Y_1)] = \theta_0 - \theta_1 + b_0(1 + \theta_1). \qquad (3.15)$$

Assume that $\alpha = 0.05$, $\theta_0 = \theta_1 = 0.1$, and $b_0 = 0.6$. Then there exists an adjustment coefficient $d_0 = 0.2935569060$ of $(c(b_0), b_0 Y_1)$ such that

$$0 \leq x \leq \frac{-ln 0.05}{0.2935569060} = 10.20494566$$

which is an interval of acceptable initial capital with corresponding to $(1, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq 1}, \{b_0 Y_n\}_{n\geq 1})$

Let

$$f(d) := E\left[e^{d b_0 Y_1}\right] - e^{d c(b_0)}.$$

Note that

$$E\left[e^{db_0Y_1}\right] = \int_0^\infty e^{db_0y} f_{Y_1}(y)dy = \int_0^\infty e^{db_0y} e^{-y} dy = \frac{1}{1-db_0} \quad \text{and}$$

$$e^{dc(b_0)} = e^{db_0(1+\theta_1)}. \tag{3.16}$$

By Definition 3.6, d_0 is an adjustment coefficient of $(c(b_0), b_0Y_1)$ if $f(d_0) = 0$. Hence $E\left[e^{d_0b_0Y_1}\right] = e^{d_0c(b_0)}$. By substitute b_0 and θ_1 into equation (3.16), we get

$$\frac{1}{1-0.6d_0} = e^{0.66d_0}.$$

Solving for d_0, we get $d_0 = 0.2935569060$. By Corollary 3.8, we get

$$0 \leq x \leq \frac{-ln0.05}{0.2935569060} = 10.20494566$$

which is an interval of acceptable initial capital with corresponding to $(0.05, N, \{c(b_{n-1}) = 0.66\}_{n\geq1}, \{0.6Y_n\}_{n\geq1})$. This means that $\Phi_N(x,\pi) \leq 0.05$ for all $0 \leq x \leq 10.20494566$.

Example 3.10. *(Exponential claims under the excess of loss reinsurance). We assume that $\{Y_n\}_{n\geq1}$ and $\{X_n\}_{n\geq0}$ are the sequences given in example 3.9. Let $N \in \{1,2,3,\ldots\}$, and $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ be stationary. Suppose that $h(b_0, y)$ is the excess of loss reinsurance with retention level b_0. By expected value principle, the net income rate $c(b_0)$ satisfies the following equation*

$$c(b_0) = c_0 - (1+\theta_1)E[Y_1 - h(b_0, Y_1)] = \theta_0 - \theta_1 + (1+\theta_1)[1-e^{-b_0}]. \tag{3.17}$$

Assume that $\alpha = 0.05$, $\theta_0 = \theta_1 = 0.1$ and $b_0 = 100$. Then there exists a sub-adjustment coefficient $d_0 = 0.17$ of $(c(b_0), h(b_0, Y_1))$ such that

$$0 \leq x \leq -\frac{ln0.05}{0.17} = 17.6220$$

which is an interval of acceptable initial capital with corresponding to $(0.05, N, \{c(b_{n-1}) = c(b_0)\}_{n\geq1}, \{h(b_0, Y_n)\}_{n\geq1})$

Let

$$f(d) := E\left[e^{dh(b_0,Y_1)}\right] - e^{dc(b_0)}.$$

Note that

$$E\left[e^{dh(b_0,Y_1)}\right] = \int_0^\infty e^{dh(b_0,y)}e^{-y}dy = \int_0^{b_0} e^{dy}e^{-y}dy + \int_{b_0}^\infty e^{b_0 d}e^{-y}dy = \frac{de^{b_0(d-1)}-1}{d-1},$$

$$\text{and} \quad e^{dc(b_0)} = e^{d(1+\theta_1)\left[1-e^{-b_0}\right]}. \tag{3.18}$$

By Definition 3.6, d_0 is a sub-adjustment coefficient of $(c(b_0), h(b_0, Y_1))$ if $f(d_0) \leq 0$. Hence $E\left[e^{d_0 h(b_0,Y_1)}\right] \leq e^{d_0 c(b_0)}$. By substitute b_0, θ_0 and θ_1 into equation (3.18), we get

$$\frac{d_0 e^{100(d_0-1)}-1}{d_0-1} \leq e^{1.1d_0\left[1-e^{-100}\right]}.$$

Solving for d_0, we get $d_0 = 0.17$. By Corollary (3.8), we get

$$0 \leq x \leq -\frac{ln 0.05}{0.17} = 17.6220$$

which is an interval of acceptable initial capital with corresponding to $(0.05, N, \{c(b_{n-1}) = 1.1\}_{n\geq1}, \{h(100, Y_n)\}_{n\geq1})$. This means that $\Phi_N(x,\pi) \leq 0.05$ for all $0 \leq x \leq 17.6220$.

3.3 Existence of Minimal Capital

Let $\alpha \in (0,1)$. As a result of Corollary (3.8) that $\{x \geq 0 : \Phi_N(x,\pi) \leq \alpha\}$ is a non-empty set. Since the set $\{x \geq 0 : \Phi_N(x,\pi) \leq \alpha\}$ is an infinite set, then there are many acceptable initial capital corresponding to $(\alpha, N, \{c(b_{n-1})\}_{n\geq1}, \{h(b_{n-1}, Y_n)\}_{n\geq1})$. In this section, we will prove the existence of a minimum initial capital that correspond to $(\alpha, N, \{c(b_{n-1})\}_{n\geq1}, \{h(b_{n-1}, Y_n)\}_{n\geq1})$.

Theorem 3.11. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ and let $\alpha \in (0,1)$. Then there exists $x^* \geq 0$ such that*

$$x^* = \min_{x\geq0}\{x : \Phi_N(x,\pi) \leq \alpha\}.$$

Proof : Let $\pi \in \mathcal{P}(N, [\underline{b}, \overline{b}])$ be fixed. We consider by case.

Case 1: For $\Phi_N(0,\pi) \leq \alpha$. We get $\min_{x\geq0}\{x : \Phi_N(x,\pi) \leq \alpha\} = 0$.

Case 2: For $\Phi_N(0,\pi) > \alpha$. Since $\Phi_N(x,\pi)$ is non-increasing in x, by Corollary 3.3, there exists $\tilde{x} > 0$ such that $\Phi_N(\tilde{x},\pi) \le \alpha$. Obviously, there exists non-empty set A such that $A = \{x \in [0,\tilde{x}] : \Phi_N(x,\pi) \le \alpha\}$. We claim that there exists $x^* \in [0,\tilde{x}]$ such that

$$x^* = \min_{x\in[0,\tilde{x}]}\{x : \Phi_N(x,\pi) \le \alpha\} = \min_{x\ge 0}\{x : \Phi_N(x,\pi) \le \alpha\}.$$

Next, we prove claim. Since $A \subseteq [0,\tilde{x}]$, then there exists $x^* \in [0,\tilde{x}]$ such that $x^* = \inf A$. For $x^* = \tilde{x}$, we obtain

$$x^* = \min A = \min_{x\in[0,\tilde{x}]}\{x : \Phi_N(x,\pi) \le \alpha\} = \min_{x\ge 0}\{x : \Phi_N(x,\pi) \le \alpha\}.$$

For $0 < x^* < \tilde{x}$. We assume that $x^* \in [0,\tilde{x}] - A$. Hence $\Phi_N(x^*,\pi) > \alpha$. Since $x^* = \inf A$, then there exists $x_n^* \in A$ such that $x^* < x_n^* \le x^* + \frac{1}{n}$ for $n \in \{1,2,...\}$. Since $\Phi_N(\cdot,\pi)$ is right continuous at x^*, then

$$\alpha < \Phi_N(x^*,\pi) = \lim_{n\to\infty}\Phi_N(x_n^*,\pi) \le \alpha.$$

Contradiction. Hence $x^* \in A$ and $x^* = \min A = \min\limits_{x\in[0,\tilde{x}]}\{x : \Phi_N(x,\pi) \le \alpha\}$ $= \min\limits_{x\ge 0}\{x : \Phi_N(x,\pi) \le \alpha\}$.

This proves case 1 and 2, and conclude the proof.

The next theorem is proved similarly to Theorem 2.8 (Sattayatham et al. [13]).

Theorem 3.12. *Let $N \in \{1,2,3,\ldots\}$, $\pi \in \mathcal{P}(N,[\underline{b},\overline{b}])$ and let $\alpha \in (0,1)$. Assume that $v_0, x_0 \ge 0$ such that $v_0 < x_0$. Let $\{v_m\}_{m\ge 1}$ and $\{x_m\}_{m\ge 1}$ be two real sequences defined by*

$$\begin{cases} v_m = v_{m-1} \quad \text{and} \quad x_m = \dfrac{x_{m-1}+v_{m-1}}{2}, & \text{if} \ \ \Phi_N\left(\dfrac{x_{m-1}+v_{m-1}}{2},\pi\right) \le \alpha \\ v_m = \dfrac{v_{m-1}+x_{m-1}}{2} \quad \text{and} \quad x_m = x_{m-1}, & \text{if} \ \ \Phi_N\left(\dfrac{x_{m-1}+v_{m-1}}{2},\pi\right) > \alpha \end{cases}$$

for all $m = 1,2,3,\ldots$. If $\Phi_N(x_0,\pi) \le \alpha < \Phi_N(v_0,\pi)$, then

$$\lim_{m\to\infty} x_m = \min_{x\ge 0}\{x : \Phi_N(x,\pi) \le \alpha\} = x^*.$$

4 Numerical Results

In this section, we provide numerical illustration of main results. We approximate the minimal initial capital of the discrete-time surplus process (2.3) by using Theorem 3.12 according to the following cases:

(a). **Proportional Reinsurance**.
We assume that $\{Y_n\}_{n\geq 1}$ is a sequence of claims with iid exponential $Exp(1)$ and $h(b_0, y)$ is the proportional reinsurance with retention level b_0. Let $N \in \{1, 2, 3, \ldots\}$ be the time horizon and $\pi = \{b_{n-1} = 0.6\}_{n=1}^{N}$ be stationary. We choose model parameters as follows: $\theta_0 = \theta_1 = 0.10$ which give $c(b_0) = 0.66$ and $\theta_0 = \theta_1 = 0.25$ which give $c(b_0) = 0.75$. Moreover, we choose $\alpha = 0.05, \alpha = 0.1$ and $\alpha = 0.2$. As a result, we get the table of the minimum initial capital as below :

	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.2$
N	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$
10	3.3909 : 2.7854	2.5919 : 2.0384	1.7358 : 1.2562
20	4.4983 : 3.3728	3.4846 : 2.4796	2.3918 : 1.5524
30	5.2438 : 3.6605	4.0747 : 2.6854	2.8148 : 1.6829
40	5.8067 : 3.8215	4.5137 : 2.7963	3.1233 : 1.7504
50	6.2558 : 3.9175	4.8593 : 2.8605	3.3619 : 1.7884
100	7.6364 : 4.0664	5.8902 : 2.9559	4.0471 : 1.8426
200	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
300	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
400	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
500	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
1,000	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
5,000	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497
10,000	8.5466 : 4.0881	6.5345 : 2.9690	4.4496 : 1.8497

Table 1: Minimum initial capital in the case proportional reinsurance.

Table 1 shows an approximation of $\min_{x\geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\}$ with $m = 25$, $v_0 = 0, x_0 = 20$ as mentioned in Theorem 3.12 and $\Phi_N(x, \pi)$ is computed by using the recursive form as mentioned in equation (3.11). The numerical results in Table 1 show a minimum initial capital $x = 3.3909$ for $\alpha = 0.05$, $N = 10$ and $\theta_0 = \theta_1 = 0.1$ etc.

(b). **Excess of Loss Reinsurance**.
Again we assume that $\{Y_n\}_{n\geq 1}$ is a sequence of claims with iid exponential $Exp(1)$ and $h(b_0, y)$ is the excess of loss reinsurance with retention level $b_0 = 100$. Let $N \in \{1, 2, 3, \ldots\}$ be the time horizon and $\pi = \{b_{n-1} = 100\}_{n=1}^{N}$ be stationary. We choose model parameters as follows: $\theta_0 = \theta_1 = 0.10$ which give $c(b_0) = 1.1$ and $\theta_0 = \theta_1 = 0.25$ which give $c(b_0) = 1.25$. Moreover, we choose $\alpha = 0.05, \alpha = 0.1$ and $\alpha = 0.2$. As a result, we get the table of the minimum initial capital as below :

	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.2$
N	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$	$\theta_0 = 0.1 : \theta_0 = 0.25$
10	5.6515 : 4.6424	4.3198 : 3.3973	2.8930 : 2.0936
20	7.4972 : 5.6213	5.8076 : 4.1327	3.9863 : 2.5874
30	8.7396 : 6.1009	6.7911 : 4.4756	4.6913 : 2.8048
40	9.6779 : 6.3692	7.5229 : 4.6605	5.2054 : 2.9174
50	10.4264 : 6.5291	8.0989 : 4.7675	5.6031 : 2.9806
100	12.7273 : 6.7773	9.8169 : 4.9265	6.7452 : 3.0709
200	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
300	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
400	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
500	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
1,000	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
5,000	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828
10,000	14.2443 : 6.8135	10.8909 : 4.9484	7.4160 : 3.0828

Table 2: Minimum initial capital in the case excess of loss reinsurance.

Table 2 shows an approximation of $\min_{x\geq 0}\{x : \Phi_N(x, \pi) \leq \alpha\}$ with $m = 25$, $v_0 = 0, x_0 = 20$ as mentioned in Theorem 3.12 and $\Phi_N(x, \pi)$ is computed by using the recursive form as mentioned in equation (3.10). The numerical results in Table 2 show a minimum initial capital $x = 5.6515$ for $\alpha = 0.05$, $N = 10$ and $\theta_0 = \theta_1 = 0.1$ etc.

References

[1] S. Browne, Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin, Mathematics of Operations Research 20(1995),937-958.

[2] D. Bersekas, S.E. Shreve, Stochastic Optimal Control: the Discrete-Time Case, Academic Press, New York, 1978.

[3] W. S. Chan, L. Zhang, Direct Derivative of Finite-Time Ruin Probabilities in the Discrete Risk Model with Exponential or Geometric Claims, North American Actuarial Journal 10(2006), 269-279.

[4] D. C. M. Dickson, Insurance Risk and Ruin, Cambridge University Press, New York, 2005.

[5] R. M. Dudley, Real Analysis and Probability, Cambridge University Press, New York, 2002.

[6] F. Jone, Lebesgue Integration on Euclidean Space,Jones and Bartlett Publishers International, Boston, 1993.

[7] F. C. Klebaner, Introduction Stochastis Calculus with Application, Imperial College Press, London, 1998.

[8] S. Li, Distributions of the Surplus before Ruin, the Deficit at Ruin and the Claim Causing Ruin in a Class of Direct Time Renewal Risk Models, Scandinavian Actuarial Journal 4(2005b), 271-284.

[9] S. Li, On a Class of Direct Time Renewal Risk Models, Scandinavian Actuarial Journal 4(2005a), 241-260.

[10] M. Mikosch, Non-life insurance mathematics: An introduction with the Poisson process, Springer-Verlag, Berlin, 2004.

[11] K. P. Pavlova, G. E. Willmot, The Discrete Stationary Renewal Risk Model and the Gerber-Shui Discounted Penalty Function, Insurance : Mathematics and Economics 35(2004), 267-277.

[12] W. Rongming, L. Haifeng, On the Ruin Probability under a Class of Risk Process, Astin Bulletin, 32(2002), 81-90.

[13] P. Sattayatham, K Sangaroon, W. Klongdee, Ruin Probability-Based Initial Capital of the Discrete-Time Surplus Process, Variance, 7(2013), 74-81.

[14] M. Schăl, On discrete-time dynamic programming in insurance: Exponential utility and maximizing the ruin probability, Scandinavian Actuarial Journal 3(2004), 189-210.

[15] M. Schäl, On piecewise deterministic Markov control processes control of jumps and if risk processes in insurance, Insurance: Mathematics and Economics 22(1998), 75-91.

[16] M. Schäl, Portfolio optimization and martingale measures. Mathematical Finance 10(2000), 289-303.

(Received 16 August 2012)
(Accepted 3 July 2013)