Thai Journal of Mathematics
Volume 11 (2013) Number 3 : 519–541

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Convergence Analysis for Family of Weak Relatively Nonexpansive Mappings and System of Equilibrium Problems

Yekini Shehu

Department of Mathematics
University of Nigeria, Nsukka, Nigeria
e-mail : deltanougt2006@yahoo.com

Abstract : In this paper, we construct a new iterative scheme by hybrid methods to approximate a common element of the fixed points set of an infinite family of weak relatively nonexpansive mappings and the solutions set of a system of equilibrium problems in a a uniformly smooth and strictly convex real Banach space with Kadec-Klee property using the properties of generalized f-projection operator. Then, we prove strong convergence of the scheme to a common element of the two sets. We give applications of our results in a Banach space. Our results extend many known recent results in the literature.

Keywords : weak relatively nonexpansive mappings; generalized f-projection operator; equilibrium problems; hybrid method; Banach spaces.
2010 Mathematics Subject Classification : 47H06; 47H09; 47J05; 47J25.

---

1 Introduction

Let E be a real Banach space with dual E^* and C be nonempty, closed and convex subset of E. A mapping $T : C \to C$ is called *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \quad \forall x, y \in C. \tag{1.1}$$

A point $x \in C$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^2\}.$$

The following properties of J are well known (The reader can consult [1–3] for more details):

1. If E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E.
2. $J(x) \neq \emptyset, \quad x \in E$.
3. If E is reflexive, then J is a mapping from E onto E^*.
4. If E is smooth, then J is single valued.

Throughout this paper, we denote by ϕ, the functional on $E \times E$ defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, J(y)\rangle + ||y||^2, \quad \forall x, y \in E. \tag{1.2}$$

Let C be a nonempty subset of E and let T be a mapping from C into E. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$. We say that a mapping T is *relatively nonexpansive* (see, for example, [4–13]) if the following conditions are satisfied:

(R1) $F(T) \neq \emptyset$;
(R2) $\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \quad p \in F(T)$;
(R3) $F(T) = \widehat{F}(T)$.

A point $p \in C$ is said to be an *strong asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges strongly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of strong asymptotic fixed points of T is denoted by $\widetilde{F}(T)$. We say that a mapping T is *weak relatively nonexpansive* (see, for example, [14, 15]) if the following conditions are satisfied:

(R1) $F(T) \neq \emptyset$;
(R2) $\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \quad p \in F(T)$;
(R3) $F(T) = \widetilde{F}(T)$.

If $T : E \to E$ is a relatively nonexpansive mapping, then using the definition of ϕ, one can show that $F(T)$ is closed and convex. It is obvious that relatively nonexpansive mapping is weak relatively nonexpansive mapping. In fact, for any mapping $T : C \to C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$. Xu and Su [16] and Kang et al. [14] gave examples of weak relatively nonexpansive mappings which are not relatively nonexpansive.

Remark 1.1. *In [15], the weak relatively nonexpansive mapping is also said to be relatively weak nonexpansive mapping.*

Remark 1.2. *In [17], the authors gave the definition of hemi-relatively nonexpansive mappings as follows: A mapping $T : C \to C$ is said to be hemirelatively nonexpansive if the following conditions are satisfied:*
(1) $F(T) \neq \emptyset$;
(2) $\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \quad p \in F(T)$.

Observe that an operator T in a Banach space E is said to be *closed* if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

The following conclusion is obvious.
Conclusion: A mapping is closed hemi-relatively nonexpansive if and only if it is weak relatively nonexpansive.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$. The equilibrium problem is to find $x \in C$ such that

$$F(x, y) \geq 0, \tag{1.3}$$

for all $y \in C$. We shall denote the set of solutions of this equilibrium problem by $EP(F)$. Thus

$$EP(F) := \{x^* \in C : F(x^*, y) \geq 0, \quad \forall y \in C\}.$$

Numerous problems in Physics, optimization and economics reduce to find a solution of problem (1.3). Some methods have been proposed to solve the equilibrium problem, see for example, [18–32]. The equilibrium problems include fixed point problems, optimization problems and variational inequality problems as special cases (see, for example, [18]).

In [7], Matsushita and Takahashi introduced a hybrid iterative scheme for approximation of fixed points of relatively nonexpansive mapping in a uniformly convex real Banach space which is also uniformly smooth: $x_0 \in C$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{w \in C : \phi(w, y_n) \leq \phi(w, x_n)\}, \\ W_n = \{w \in C : \langle x_n - w, Jx_0 - Jx_n \rangle, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n \geq 0. \end{cases} \tag{1.4}$$

They proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{F(T)} x_0$, where $F(T) \neq \emptyset$.

In [12], Takahashi and Zembayashi introduced the following hybrid iterative scheme for approximation of fixed point of relatively nonexpansive mapping which is also a solution to an equilibrium problem in a uniformly convex real Banach space which is also uniformly smooth: $x_0 \in C, \quad C_1 = C, \quad x_1 = \Pi_{C_1} x_0$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C \\ C_{n+1} = \{w \in C_n : \phi(w, u_n) \leq \phi(w, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1, \end{cases}$$

where J is the duality mapping on E. Then, they proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$, where $F = EP(F) \cap F(T) \neq \emptyset$.

In [33], Plubtieng and Ungchittrakool introduced the following hybrid projection algorithm for a pair of relatively nonexpansive mappings: $x_0 \in C$,

$$\begin{cases} z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n) \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n) \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n) + \alpha_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ Q_n = \{z \in C : \langle x_n - z, Jx_n - Jx_0\rangle \leq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ and $\{\beta_n^{(3)}\}$ are sequences in $(0,1)$ satisfying $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ and T and S are relatively nonexpansive mappings and J is the single-valued duality mapping on E. They proved under the appropriate conditions on the parameters that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a common fixed point of T and S.

Recently, Li et al. [34] introduced the following hybrid iterative scheme for approximation of fixed points of a relatively nonexpansive mapping using the properties of generalized f-projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad n \geq 1, \end{cases}$$

They proved a strong convergence theorem for finding an element in the fixed points set of T. We remark here that the results of Li et al. [34] extended and improved on the results of Matsushita and Takahashi, [7].

Quite recently, motivated by the results of Matsushita and Takahashi [7] and Plubtieng and Ungchittrakool [33], Su et al. [35] proved the following strong convergence theorem by Halpern type hybrid iterative scheme for approximation of common fixed point of two countable families of weak relatively nonexpansive mappings in uniformly convex and uniformly smooth Banach space.

Theorem 1.1 (Su et al. [35])**.** *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $\{T_n\}_{n=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$ are two countable families of weak relatively nonexpansive mappings of C into itself such that $F := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \neq \emptyset$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C$,*

$$\begin{cases} z_n = J^{-1}(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n), \\ y_n = J^{-1}(\alpha_n Jz_n + (1-\alpha_n)Jx_n), \\ C_n = \{w \in C_{n-1} \cap Q_{n-1} : \phi(w, y_n) \\ \qquad\qquad \leq (1-\alpha_n\beta_n^{(1)})\phi(w, x_n) + \alpha_n\beta_n^{(1)}\phi(w, x_0)\}, \\ C_0 = \{w \in C : \phi(w, y_0) \leq \phi(w, x_0)\}, \\ Q_n = \{w \in C_{n-1} \cap Q_{n-1} : \langle x_n - w, Jx_0 - Jx_n\rangle \geq 0\}, \\ Q_0 = C, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n \geq 1, \end{cases} \tag{1.6}$$

with the conditions
(i) $\lim_{n\to\infty} \beta_n^{(1)} = 0$;
(ii) $\limsup_{n\to\infty} \beta_n^{(2)}\beta_n^{(3)} > 0$.
Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_F x_0$.

Motivated by the above mentioned results and the on-going research, we introduce a new iterative scheme by hybrid method and prove strong convergence theorem for an infinite family of weak relatively nonexpansive mappings and a system of equilibrium problems in a uniformly smooth and strictly convex real Banach space which also has Kadec-Klee property using the properties of generalized f-projection operator. Our results extend many recent known results in the literature. Finally, we also apply our results to obtain some applications in a Banach space. Our results extend the results of Matsushita and Takahashi [7], Plubtieng and Ungchittrakool [33], Takahashi and Zembayashi [12], Li et al. [34] and other recent results in the literature.

2 Preliminaries

Let E be a real Banach space. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{1}{2}(||x+y|| + ||x-y||) - 1 : ||x|| \leq 1, ||y|| \leq \tau\right\}.$$

E is uniformly smooth if and only if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

Let $\dim E \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left|\left|\frac{x+y}{2}\right|\right| : ||x|| = ||y|| = 1; \epsilon = ||x-y||\right\}.$$

E is *uniformly convex* if for any $\epsilon \in (0, 2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $||x|| \leq 1$, $||y|| \leq 1$ and $||x-y|| \geq \epsilon$, then $||\frac{1}{2}(x+y)|| \leq 1 - \delta$.

Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. A normed space E is called *strictly convex* if for all $x, y \in E$, $x \neq y$, $||x|| = ||y|| = 1$, we have $||\lambda x + (1-\lambda)y|| < 1$, $\forall \lambda \in (0, 1)$.$E$ is said to be *2-uniformly convex* if there exists constant $c > 0$ such that $\delta_E(\epsilon) > c\epsilon^2$ for all $\epsilon \in (0, 2]$. The constant $\frac{1}{c}$ is called the *2-uniformly convexity a constant* of E. We know that a 2-uniformly convex Banach space is uniformly convex.

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E. Following Alber [36], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) := arg\min_{y \in C} \phi(y, x), \quad \forall x \in E.$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [3, 36–39]). If E is a Hilbert space, then Π_C is the metric projection of H onto C. From [39], in uniformly convex and uniformly smooth Banach spaces, we have

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \quad \forall x, y \in E. \tag{2.1}$$

The fixed points set $F(T)$ of a weak relatively nonexpansive mapping is closed convex as given in the following lemma.

Lemma 2.1 (Su et al. [35])**.** *Let C be a nonempty, closed and convex subset of a smooth, strictly convex Banach space E. Let T be a weak relatively nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Next, we recall the concept of generalized f-projector operator, together with its properties. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = ||\xi||^2 - 2\langle \xi, \varphi \rangle + ||\varphi||^2 + 2\rho f(\xi), \tag{2.2}$$

where $\xi \in C$, $\varphi \in E^*$, ρ is a positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. From the definitions of G and f, it is easy to see the following properties:
(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed;
(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed.

Definition 2.2. Let E be a real Banach space with its dual E^*. Let C be a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E^* \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f \varphi = \Big\{ u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi) \Big\}, \quad \forall \varphi \in E^*.$$

For the generalized f-projection operator, Wu and Huang [40] proved the following theorem basic properties:

Lemma 2.3 (Wu and Huang [40]). *Let E be a real reflexive Banach space with its dual E^*. Let C be a nonempty, closed and convex subset of E. Then the following statements hold:*
(i) Π_C^f is a nonempty closed convex subset of C for all $\varphi \in E^*$;
(ii) If E is smooth, then for all $\varphi \in E^*, x \in \Pi_C^f$ if and only if

$$\langle x - y, \varphi - Jx\rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C;$$

(iii) If E is strictly convex and $f : C \to \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$), then Π_C^f is a single valued mapping.

Fan et al. [41] showed that the condition f is positive homogeneous which appeared in Lemma 2.3 can be removed.

Lemma 2.4 (Fan et al. [41]). *Let E be a real reflexive Banach space with its dual E^* and C a nonempty, closed and convex subset of E. Then if E is strictly convex, then Π_C^f is a single valued mapping.*

Recall that J is a single valued mapping when E is a smooth Banach space. There exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$ for each $x \in E$. This substitution in (2.2) gives

$$G(\xi, Jx) = ||\xi||^2 - 2\langle \xi, Jx\rangle + ||x||^2 + 2\rho f(\xi). \tag{2.3}$$

Now, we consider the second generalized f-projection operator in a Banach space.

Definition 2.5. Let E be a real Banach space and C a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f x = \left\{ u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx)\right\}, \quad \forall x \in E.$$

Obviously, the definition of T is a weak relatively nonexpansive mapping is equivalent to
$(R'1)$ $F(T) \neq \emptyset$;
$(R'2)$ $G(p, JTx) \leq G(p, Jx), \quad \forall x \in C, \quad p \in F(T)$.
$(R'3)$ $F(T) = \widetilde{F}(T)$.

Lemma 2.6 (Deimling [42]). *Let E be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous convex functional. Then there exists $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^*\rangle + \alpha, \quad \forall x \in E.$$

We know that the following lemmas hold for operator Π_C^f.

Lemma 2.7 (Li et al. [34]). *Let C be a nonempty, closed and convex subset of a smooth and reflexive Banach space E. Then the following statements hold:*
(i) $\Pi_C^f x$ *is a nonempty closed and convex subset of C for all $x \in E$;*
(ii) *for all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x} + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C;$$

(iii) *if E is strictly convex, then $\Pi_C x^f$ is a single valued mapping.*

Lemma 2.8 (Li et al. [34]). *Let C be a nonempty, closed and convex subset of a smooth and reflexive Banach space E. Let $x \in E$ and $\hat{x} \in \Pi_C^f$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \quad \forall y \in C.$$

Also, this following lemma will be used in the sequel.

Lemma 2.9 (Chang et al. [43]). *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, for any given sequence $\{x_n\}_{n=1}^\infty \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^\infty$ of positive numbers such that $\sum_{i=1}^\infty \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R}, \quad g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\Big|\Big| \sum_{n=1}^\infty \lambda_n x_n \Big|\Big|^2 \leq \sum_{n=1}^\infty \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:
(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y, \in C$;
(A3) for each $x, y \in C$, $\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.10 (Blum and Oettli [18]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \quad \text{for all} \quad y \in K.$$

Lemma 2.11 (Takahashi and Zembayashi [44]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r^F : E \to C$ as follows:*

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}$$

for all $z \in E$. Then, the following hold:
1. T_r^F *is single-valued;*
2. T_r^F *is firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,*

$$\langle T_r^F x - T_r^F y, JT_r^F x - JT_r^F y\rangle \leq \langle T_r^F x - T_r^F y, Jx - Jy\rangle;$$

3. $F(T_r^F) = EP(F)$;
4. $EP(F)$ *is closed and convex.*

Lemma 2.12 (Takahashi and Zembayashi [44]). *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4) and let $r > 0$. Then for each $x \in E$ and $q \in F(T_r^F)$,*

$$\phi(q, T_r^F x) + \phi(T_r^F x, x) \leq \phi(q, x).$$

We recall that a Banach space E has *Kadec-Klee property* if for any sequence $\{x_n\}_{n=0}^\infty \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, then $x_n \to x$ as $n \to \infty$. We note that every uniformly convex Banach space has the Kadec-Klee property. For more details on Kadec-Klee property, the reader is referred to [2, 38].

Lemma 2.13 (Li et al. [34]). *Let E be a Banach space and $y \in E$. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous mapping with convex domain $D(f)$. If $\{x_n\}$ is a sequence in $D(f)$ such that $x_n \rightharpoonup x \in \mathrm{int}(D(f))$ and $\lim_{n\to\infty} G(x_n, Jy) = G(x, Jy)$, then $\lim_{n\to\infty} ||x_n|| = ||x||$.*

3 Main Results

We now prove the following strong convergence theorem.

Theorem 3.1. *Let E be a uniformly smooth and strictly convex real Banach space which also has Kadec-Klee property. Let C be a nonempty, closed and convex subset of E. For each $k = 1, 2, ..., m$, let F_k be a bifunction from $C \times C$ satisfying $(A1)-(A4)$ and suppose $\{T_i\}_{i=1}^\infty$ is an infinite family of weak relatively nonexpansive mappings of C into itself such that $\Omega := \Big(\cap_{k=1}^m EP(F_k)\Big) \cap \Big(\cap_{i=1}^\infty F(T_i)\Big) \neq \emptyset$. Let Let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous mapping with $C \subset \mathrm{int}(D(f))$ and $\{x_n\}_{n=0}^\infty$ be iteratively generated by $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}^f x_0$,*

$$\begin{cases} y_n = J^{-1}(\alpha_{n0} Jx_n + \sum_{i=1}^\infty \alpha_{ni} JT_i x_n) \\ u_n = T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n \\ C_{n+1} = \{w \in C_n : G(w, Ju_n) \leq G(w, Jx_n)\} \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad n \geq 1, \end{cases} \tag{3.1}$$

where J is the duality mapping on E. Suppose $\{\alpha_{ni}\}_{n=1}^\infty$ for each $i = 0, 1, 2, ...$ is a sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0$, $i = 1, 2, 3, ...$, $\sum_{i=0}^\infty \alpha_{ni} = 1$ and $\{r_{k,n}\}_{n=1}^\infty \subset (0, \infty)$, $(k = 1, 2, ..., m)$ satisfying $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1, 2, ..., m)$. Then, $\{x_n\}_{n=0}^\infty$ converges strongly to $\Pi_\Omega^f x_0$.

Proof. We first show that C_n, $\forall n \geq 1$ is closed and convex. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n > 1$. From the definition of C_{n+1}, we have that $z \in C_{n+1}$ implies $G(z, Ju_n) \leq G(z, Jx_n)$. This is equivalent to

$$2\Big(\langle z, Jx_n\rangle - \langle z, Ju_n\rangle\Big) \leq ||x_n||^2 - ||u_n||^2.$$

This implies that C_{n+1} is closed and convex for the same $n > 1$. Hence, C_n is closed and convex $\forall n \geq 1$. This shows that $\Pi^f_{C_{n+1}} x_0$ is well defined for all $n \geq 0$. By taking $\theta^k_n = T^{F_k}_{r_{k,n}} T^{F_{k-1}}_{r_{k-1,n}} \cdots T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n}}$, $k = 1, 2, ..., m$ and $\theta^0_n = I$ for all $n \geq 1$, we obtain $u_n = \theta^m_n y_n$.

Since $f : E \to \mathbb{R}$ is a convex and lower semi-continuous, applying Lemma 2.6, we see that there exists $u^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, u^*\rangle + \alpha, \quad \forall y \in E.$$

It follows that

$$\begin{aligned} G(x_n, Jx_0) &= ||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho f(x_n) \\ &\geq ||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho\langle x_n, u^*\rangle + 2\rho\alpha \\ &= ||x_n||^2 - 2\langle x_n, Jx_0 - \rho u^*\rangle + ||x_0||^2 + 2\rho\alpha \\ &\geq ||x_n||^2 - 2||x_n||||Jx_0 - \rho u^*|| + ||x_0||^2 + 2\rho\alpha \\ &= (||x_n|| - ||Jx_0 - \rho u^*||)^2 + ||x_0||^2 - ||Jx_0 - \rho u^*||^2 + 2\rho\alpha. \end{aligned} \tag{3.2}$$

Since $x_n = \Pi^f_{C_n} x_0$, it follows from (3.2) that

$$G(x^*, Jx_0) \geq G(x_n, Jx_0) \geq (||x_n|| - ||Jx_0 - \rho u^*||)^2 + ||x_0||^2 - ||Jx_0 - \rho u^*||^2 + 2\rho\alpha$$

for each $x^* \in \Omega$. This implies that $\{x_n\}^\infty_{n=0}$ is bounded and so is $\{G(x_n, Jx_0)\}^\infty_{n=0}$. This implies that $\{T_i x_n\}^\infty_{n=1}$ is bounded for each $i = 1, 2, \ldots$. We next show that $\Omega \subset C_n$, $\forall n \geq 1$. For $n = 1$, we have $\Omega \subset C = C_1$. Since E is uniformly smooth, we know that E^* is uniformly convex. Then from Lemma 2.9, we have for any positive integer $j \geq 1$ that

$$\begin{aligned} G(x^*, Ju_n) &= G(x^*, J\theta^m_n y_n) \leq G(x^*, Jy_n) \\ &= G(x^*, (\alpha_{n0} Jx_n + \sum_{i=1}^{\infty} \alpha_{ni} JT_i x_n)) \\ &= ||x^*||^2 - 2\alpha_{n0}\langle x^*, Jx_n\rangle - 2\sum_{i=1}^{\infty} \alpha_{ni}\langle x^*, JT_i x_n\rangle \\ &\quad + \Big|\Big|\alpha_{n0} Jx_n + \sum_{i=1}^{\infty} \alpha_{ni} JT_i x_n\Big|\Big|^2 + 2\rho f(x^*). \end{aligned}$$

This furthermore implies that

$$\begin{aligned}
G(x^*, Ju_n) &\leq ||x^*||^2 - 2\alpha_{n0}\langle x^*, Jx_n\rangle - 2\sum_{i=1}^{\infty}\alpha_{ni}\langle x^*, JT_ix_n\rangle \\
&\quad + \alpha_{n0}||Jx_n||^2 + \sum_{i=1}^{\infty}\alpha_{ni}\Big|\Big|JT_ix_n\Big|\Big|^2 - \alpha_{n0}\alpha_{nj}g(||Jx_n - JT_jx_n||) \\
&\quad + 2\rho f(x^*) \\
&= ||x^*||^2 - 2\alpha_{n0}\langle x^*, Jx_n\rangle - 2\sum_{i=1}^{\infty}\alpha_{ni}\langle x^*, JT_ix_n\rangle \\
&\quad + \alpha_{n0}||Jx_n||^2 + \sum_{i=1}^{\infty}\alpha_{ni}\Big|\Big|JT_ix_n\Big|\Big|^2 - \alpha_{n0}\alpha_{nj}g(||Jx_n - JT_jx_n||) \\
&\quad + 2\rho f(x^*) \\
&\leq G(x^*, Jx_n) - \alpha_{n0}\alpha_{nj}g(||Jx_n - JT_jx_n||) \qquad (3.3) \\
&\leq G(x^*, Jx_n).
\end{aligned}$$

So, $x^* \in C_n$. This implies that $\Omega \subset C_n, \quad \forall n \geq 1$.

We now show that $\lim_{n\to\infty} G(x_n, Jx_0)$ exists. By the construction of C_n, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m}^f x_0 \in C_n$ for any positive integer $m \geq n$. It then follows Lemma 2.8 that

$$\phi(x_m, x_n) + G(x_n, Jx_0) \leq G(x_m, Jx_0). \tag{3.4}$$

It is obvious that

$$\phi(x_m, x_n) \geq (||x_m|| - ||x_n||)^2 \geq 0.$$

In particular,

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0)$$

and

$$\phi(x_{n+1}, x_n) \geq (||x_{n+1}|| - ||x_n||)^2 \geq 0,$$

and so $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ is nondecreasing. It follows that the limit of $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ exists.

Now since $\{x_n\}_{n=0}^{\infty}$ is bounded in C and E is reflexive, we may assume that $x_n \rightharpoonup p$ and since C_n is closed and convex for each $n \geq 0$, it is easy to see that $p \in C_n$ for each $n \geq 0$. Again since $x_n = \Pi_{C_n}^f x_0$, from the definition of $\Pi_{C_n}^f$, we obtain

$$G(x_n, Jx_0) \leq G(p, Jx_0), \quad \forall n \geq 0.$$

Since

$$\begin{aligned}
\liminf_{n\to\infty} G(x_n, Jx_0) &= \liminf_{n\to\infty}\Big\{||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho f(x_n)\Big\} \\
&\geq ||p||^2 - 2\langle p, Jx_0\rangle + ||x_0||^2 + 2\rho f(p) = G(p, Jx_0)
\end{aligned}$$

then, we obtain

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(p, Jx_0).$$

This implies that $\lim_{n\to\infty} G(x_n, Jx_0) = G(p, Jx_0)$. By Lemma 2.13, we obtain $\lim_{n\to\infty} ||x_n|| = ||p||$. In view of Kadec-Klee property of E, we have that $\lim_{n\to\infty} x_n = p$.

By the fact that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_n$, we obtain

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n).$$

Now, (3.4) implies that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \leq G(x_{n+1}, Jx_0) - G(x_n, Jx_0). \tag{3.5}$$

Taking the limit as $n \to \infty$ in (3.5), we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0.$$

Therefore,

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

It then yields that $\lim_{n\to\infty}(||x_{n+1}|| - ||u_n||) = 0$. Since $\lim_{n\to\infty} ||x_{n+1}|| = ||p||$, we have

$$\lim_{n\to\infty} ||u_n|| = ||p||.$$

Hence,

$$\lim_{n\to\infty} ||Ju_n|| = ||Jp||.$$

This implies that $\{||Ju_n||\}_{n=0}^{\infty}$ is bounded in E^*. Since E is reflexive, and so E^* is reflexive, we can then assume that $Ju_n \rightharpoonup f_0 \in E^*$. In view of reflexivity of E, we see that $J(E) = E^*$. Hence, there exists $x \in E$ such that $Jx = f_0$. Since

$$\begin{aligned} \phi(x_{n+1}, u_n) &= ||x_{n+1}||^2 - 2\langle x_{n+1}, Ju_n\rangle + ||u_n||^2 \\ &= ||x_{n+1}||^2 - 2\langle x_{n+1}, Ju_n\rangle + ||Ju_n||^2. \end{aligned} \tag{3.6}$$

Taking the limit inferior of both sides of (3.6) and in view of weak lower semicontinuity of $||.||$, we have

$$\begin{aligned} 0 &\geq ||p||^2 - 2\langle p, f_0\rangle + ||f_0||^2 = ||p||^2 - 2\langle p, Jx\rangle + ||Jx||^2 \\ &= ||p||^2 - 2\langle p, Jx\rangle + ||x||^2 = \phi(p, x), \end{aligned}$$

that is, $p = x$. This implies that $f_0 = Jp$ and so $Ju_n \rightharpoonup Jp$. It follows from $\lim_{n\to\infty} ||Ju_n|| = ||Jp||$ and Kadec-Klee property of E^* that $Ju_n \to Jp$. Note that $J^{-1} : E^* \to E$ is hemi-continuous, it yields that $u_n \rightharpoonup p$. It then follows from $\lim_{n\to\infty} ||u_n|| = ||p||$ and Kadec-Klee property of E that $\lim_{n\to\infty} u_n = p$. Therefore, Hence,

$$\lim_{n\to\infty} ||x_n - u_n|| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n\to\infty}||x_n - u_n|| = 0$, we obtain

$$\lim_{n\to\infty}||Jx_n - Ju_n|| = 0.$$

It then follows from (3.3) that

$$\alpha_{n0}\alpha_{nj}g(||Jx_n - JT_jx_n||) \leq G(x^*, Jx_n) - G(x^*, Ju_n).$$

On the other hand,

$$\begin{aligned} G(x^*, Jx_n) - G(x^*, Ju_n) &= ||x_n||^2 - ||u_n||^2 - 2\langle x^*, Jx_n - Ju_n\rangle \\ &\leq \Big| ||x_n||^2 - ||u_n||^2 \Big| + 2\Big|\langle x^*, Jx_n - Ju_n\rangle\Big| \\ &\leq \Big| ||x_n|| - ||u_n|| \Big|(||x_n|| + ||u_n||) + 2||x^*||||Jx_n - Ju_n|| \\ &\leq ||x_n - u_n||(||x_n|| + ||u_n||) + 2||x^*||||Jx_n - Ju_n||. \end{aligned}$$

From $\lim_{n\to\infty}||x_n - u_n|| = 0$ and $\lim_{n\to\infty}||Jx_n - Ju_n|| = 0$, we obtain

$$G(x^*, Jx_n) - G(x^*, Ju_n) \to 0, \quad n \to \infty. \tag{3.7}$$

Using the condition $\liminf_{n\to\infty}\alpha_{n0}\alpha_{nj} > 0$, we have for any $j \geq 1$ that

$$\lim_{n\to\infty}g(||Jx_n - JT_jx_n||) = 0.$$

By property of g, we have $\lim_{n\to\infty}||Jx_n - JT_jx_n|| = 0,\ \ j \geq 1$. Since $x_n \to p$ and J is uniformly norm-to-norm continuous on bounded sets, we have $Jx_n \to Jp$. Now, from $\lim_{n\to\infty}||Jx_n - JT_jx_n|| = 0$ and $Jx_n \to Jp$, we get $\lim_{n\to\infty}||JT_jx_n - Jp|| = 0$. Since J^{-1} is also hemi-continuous on bounded sets, we have $T_jx_n \rightharpoonup p$. On the other hand,

$$\begin{aligned} |||T_jx_n - ||p||| &= |||JT_jx_n - ||Jp||| \\ &\leq ||JT_jx_n - Jp|| \to 0, \quad n \to \infty. \end{aligned}$$

Since E has the Kadec-Klee property, we get $T_jx_n \to p,\ \ n \to \infty$. This further implies that

$$\lim_{n\to\infty}||Jx_n - JT_jx_n|| = 0, \quad j \geq 1. \tag{3.8}$$

Since $x_n \to p$, we obtain that $x_n \to p$ as $n \to \infty$. Now, from the definition of weak relatively nonexpansive mappings, $x_n \to p$ as $n \to \infty$ and $\lim_{n\to\infty}||Jx_n - JT_ix_n|| = 0,\ \ i \geq 1$ imply that $p \in \widetilde{F}(T_i) = F(T_i)$ for each $i \geq 1$. Hence, $p \in \cap_{i=1}^{\infty}F(T_i)$.

Next, we show that $p \in \cap_{k=1}^{m}EP(F_k)$. From (3.3), we obtain

$$\begin{aligned} \phi(x^*, u_n) &= \phi(x^*, \theta_n^m y_n) \\ &= \phi(x^*, T_{r_{m,n}}^{F_m}\theta_n^{m-1}y_n) \\ &\leq \phi(x^*, \theta_n^{m-1}y_n) \leq \ldots \leq \phi(x^*, x_n). \end{aligned} \tag{3.9}$$

Since $x^* \in EP(F_m) = F(T^{F_m}_{r_{m,n}})$ for all $n \geq 1$, it follows from (3.9) and Lemma 2.12 that

$$\begin{aligned}\phi(u_n, \theta_n^{m-1}y_n) &= \phi(T^{F_m}_{r_{m,n}}\theta_n^{m-1}y_n, \theta_n^{m-1}y_n)\\ &\leq \phi(x^*, \theta_n^{m-1}y_n) - \phi(x^*, u_n)\\ &\leq \phi(x^*, x_n) - \phi(x^*, u_n).\end{aligned}$$

From (3.7), we obtain $\lim_{n\to\infty}\phi(\theta_n^m y_n, \theta_n^{m-1}y_n) = \lim_{n\to\infty}\phi(u_n, \theta_n^{m-1}y_n) = 0$. It then yields that $\lim_{n\to\infty}(||u_n|| - ||\theta_n^{m-1}y_n||) = 0$. Since $\lim_{n\to\infty}||u_n|| = ||p||$, we have

$$\lim_{n\to\infty}||\theta_n^{m-1}y_n|| = ||p||.$$

Hence,

$$\lim_{n\to\infty}||J\theta_n^{m-1}y_n|| = ||Jp||.$$

This implies that $\{||J\theta_n^{m-1}y_n||\}_{n=1}^{\infty}$ is bounded in E^*. Since E^* is reflexive, we can then assume that $J\theta_n^{m-1}y_n \rightharpoonup f_1 \in E^*$. In view of reflexivity of E, we see that $J(E) = E^*$. Hence, there exists $z \in E$ such that $Jz = f_1$. Since

$$\begin{aligned}\phi(u_n, \theta_n^{m-1}y_n) &= ||u_n||^2 - 2\langle u_n, J\theta_n^{m-1}y_n\rangle + ||\theta_n^{m-1}y_n||^2\\ &= ||u_n||^2 - 2\langle u_n, J\theta_n^{m-1}y_n\rangle + ||J\theta_n^{m-1}y_n||^2. \qquad (3.10)\end{aligned}$$

Taking the limit inferior of both sides of (3.10) and in view of weak lower semi-continuity of $||.||$, we have

$$\begin{aligned}0 &\geq ||p||^2 - 2\langle p, f_1\rangle + ||f_1||^2 = ||p||^2 - 2\langle p, Jz\rangle + ||Jz||^2\\ &= ||p||^2 - 2\langle p, Jz\rangle + ||z||^2 = \phi(p, z),\end{aligned}$$

that is, $p = z$. This implies that $f_1 = Jp$ and so $J\theta_n^{m-1}y_n \rightharpoonup Jp$. It follows from $\lim_{n\to\infty}||J\theta_n^{m-1}y_n|| = ||Jp||$ and Kadec-Klee property of E^* that $J\theta_n^{m-1}y_n \to Jp$. Note that $J^{-1} : E^* \to E$ is hemi-continuous, it yields that $\theta_n^{m-1}y_n \rightharpoonup p$. It then follows from $\lim_{n\to\infty}||\theta_n^{m-1}y_n|| = ||p||$ and Kadec-Klee property of E that $\lim_{n\to\infty}\theta_n^{m-1}y_n = p$. Therefore,

$$\lim_{n\to\infty}||\theta_n^m y_n - \theta_n^{m-1}y_n|| = \lim_{n\to\infty}||u_n - \theta_n^{m-1}y_n|| = 0.$$

Furthermore, using Lemma 2.12 again, we have that

$$\begin{aligned}\phi(\theta_n^{m-1}y_n, \theta_n^{m-2}y_n) &= \phi(T^{F_{m-1}}_{r_{m-1,n}}\theta_n^{m-2}y_n, \theta_n^{m-2}y_n)\\ &\leq \phi(x^*, \theta_n^{m-2}y_n) - \phi(x^*, \theta_n^{m-1}y_n)\\ &\leq \phi(x^*, x_n) - \phi(x^*, u_n) \to 0, \quad n \to \infty,\end{aligned}$$

which yields that $\lim_{n\to\infty}(||\theta_n^{m-1}y_n|| - ||\theta_n^{m-2}y_n||) = 0$. Since $\lim_{n\to\infty}||\theta_n^{m-1}y_n|| = ||p||$, we have

$$\lim_{n\to\infty}||\theta_n^{m-2}y_n|| = ||p||.$$

Hence,

$$\lim_{n\to\infty}||J\theta_n^{m-2}y_n|| = ||Jp||.$$

This implies that $\{||J\theta_n^{m-2}y_n||\}_{n=1}^{\infty}$ is bounded in E^*. Since E^* is reflexive, we can then assume that $J\theta_n^{m-2}y_n \rightharpoonup f_2 \in E^*$. In view of reflexivity of E, we see that $J(E) = E^*$. Hence, there exists $w \in E$ such that $Jw = f_2$. Since

$$\begin{aligned}\phi(\theta_n^{m-1}y_n, \theta_n^{m-2}y_n) &= ||\theta_n^{m-1}y_n||^2 - 2\langle\theta_n^{m-1}y_n, J\theta_n^{m-2}y_n\rangle + ||\theta_n^{m-2}y_n||^2\\ &= ||\theta_n^{m-1}y_n||^2 - 2\langle\theta_n^{m-1}y_n, J\theta_n^{m-2}y_n\rangle + ||J\theta_n^{m-2}y_n||^2. \quad (3.11)\end{aligned}$$

Taking the limit inferior of both sides of (3.11) and in view of weak lower semicontinuity of $||.||$, we have

$$\begin{aligned}0 &\geq ||p||^2 - 2\langle p, f_2\rangle + ||f_2||^2 = ||p||^2 - 2\langle p, Jw\rangle + ||Jw||^2\\ &= ||p||^2 - 2\langle p, Jw\rangle + ||w||^2 = \phi(p, w),\end{aligned}$$

that is, $p = w$. This implies that $f_2 = Jp$ and so $J\theta_n^{m-2}y_n \rightharpoonup Jp$. It follows from $\lim_{n\to\infty}||J\theta_n^{m-2}y_n|| = ||Jp||$ and Kadec-Klee property of E^* that $J\theta_n^{m-2}y_n \to Jp$. Note that $J^{-1} : E^* \to E$ is hemi-continuous, it yields that $\theta_n^{m-2}y_n \rightharpoonup p$. It then follows from $\lim_{n\to\infty}||\theta_n^{m-2}y_n|| = ||p||$ and Kadec-Klee property of E that $\lim_{n\to\infty}\theta_n^{m-2}y_n = p$. Therefore,

$$\lim_{n\to\infty}||\theta_n^{m-1}y_n - \theta_n^{m-2}y_n|| = 0$$

In a similar way, we can verify that

$$\lim_{n\to\infty}\theta_n^{m-3}y_n = \cdots = \lim_{n\to\infty}y_n = p$$

and

$$\lim_{n\to\infty}||\theta_n^{m-2}y_n - \theta_n^{m-3}y_n|| = \cdots = \lim_{n\to\infty}||\theta_n^1 y_n - y_n|| = 0.$$

Hence, we conclude that $\theta_n^k y_n \to p$, we have for each $k = 1, 2, ..., m$ and

$$\lim_{n\to\infty}||\theta_n^k y_n - \theta_n^{k-1}y_n|| = 0, \quad k = 1, 2, ..., m. \tag{3.12}$$

Also, since J is uniformly norm-to-norm continuous on bounded sets and using (3.12), we obtain

$$\lim_{n\to\infty}||J\theta_n^k y_n - J\theta_n^{k-1}y_n|| = 0, \quad k = 1, 2, ..., m.$$

Since $\liminf\limits_{n\to\infty} r_{k,n} > 0, \quad k = 1, 2, ..., m$, then

$$\lim_{n\to\infty}\frac{||J\theta_n^k y_n - J\theta_n^{k-1}y_n||}{r_{k,n}} = 0. \tag{3.13}$$

By Lemma 2.11, we have that for each $k = 1, 2, ..., m$

$$F_k(\theta_n^k y_n, y) + \frac{1}{r_{k,n}}\langle y - \theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1}y_n\rangle \geq 0, \quad \forall y \in C.$$

Furthermore, using $(A2)$ we obtain

$$\frac{1}{r_{k,n}}\langle y-\theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1}y_n\rangle \geq F_k(y,\theta_n^k y_n). \tag{3.14}$$

By $(A4)$, (3.13) and $\theta_n^k y_n \to p$, we have for each $k=1,2,...,m$

$$F_k(y,p) \leq 0, \quad \forall y \in C.$$

For fixed $y \in C$, let $z_{t,y} := ty + (1-t)p$ for all $t \in (0,1]$. This implies that $z_t \in C$. This yields that $F_k(z_t,p) \leq 0$. It follows from $(A1)$ and $(A4)$ that

$$\begin{aligned} 0 = F_k(z_t,z_t) &\leq tF_k(z_t,y) + (1-t)F_k(z_t,p) \\ &\leq tF_k(z_t,y) \end{aligned}$$

and hence

$$0 \leq F_k(z_t,y).$$

From condition $(A3)$, we obtain

$$F_k(p,y) \geq 0, \quad \forall y \in C.$$

This implies that $p \in EP(F_k), \quad k=1,2,...,m$. Thus, $p \in \cap_{k=1}^m EP(F_k)$. Hence, we have $p \in \Omega := \Big(\cap_{k=1}^m EP(F_k)\Big) \cap \Big(\cap_{i=1}^\infty F(T_i)\Big)$.

Finally, we show that $p = \Pi_\Omega^f x_0$. Since $\Omega = \Big(\cap_{k=1}^m EP(F_k)\Big) \cap \Big(\cap_{i=1}^\infty F(T_i)\Big)$ is a closed and convex set, from Lemma 2.7, we know that $\Pi_\Omega^f x_0$ is single valued and denote $w = \Pi_\Omega^f x_0$. Since $x_n = \Pi_{C_n}^f x_0$ and $w \in \Omega \subset C_n$, we have

$$G(x_n, Jx_0) \leq G(w, Jx_0), \quad \forall n \geq 1.$$

We know that $G(\xi, J\varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed. This implies that

$$G(p,Jx_0) \leq \liminf_{n\to\infty} G(x_n,Jx_0) \leq \limsup_{n\to\infty} G(x_n,Jx_0) \leq G(w,Jx_0).$$

From the definition of $\Pi_\Omega^f x_0$ and $p \in \Omega$, we see that $p = w$. This completes the proof. $\square$

Corollary 3.2. *Let E be a uniformly smooth and strictly convex real Banach space which also has Kadec-Klee property. Let C be a nonempty, closed and convex subset of E. For each $k=1,2,...,m$, let F_k be a bifunction from $C \times C$ satisfying $(A1)-(A4)$ and suppose $\{T_i\}_{i=1}^\infty$ is an infinite family of weak relatively nonexpansive mappings of C into itself such that $\Omega := \Big(\cap_{k=1}^m EP(F_k)\Big) \cap \Big(\cap_{i=1}^\infty F(T_i)\Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^\infty$ be iteratively generated by $x_0 \in C, \quad C_1 = C, \quad x_1 = \Pi_{C_1}x_0$,*

$$\begin{cases} y_n = J^{-1}(\alpha_{n0}Jx_n + \sum_{i=1}^\infty \alpha_{ni}JT_ix_n) \\ u_n = T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n \\ C_{n+1} = \{w \in C_n : \phi(w,u_n) \leq \phi(w,x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad n \geq 1, \end{cases} \tag{3.15}$$

where J is the duality mapping on E. Suppose $\{\alpha_{ni}\}_{n=1}^{\infty}$ for each $i = 0, 1, 2, ...$ is a sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0$, $\quad i = 1, 2, 3, ...,\quad \sum_{i=0}^{\infty} \alpha_{ni} = 1$ *and* $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $\quad (k = 1, 2, ..., m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0$, $\quad (k = 1, 2, ..., m)$. *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_\Omega x_0$.

Proof. Take $f(x) = 0$ for all $x \in E$ in Theorem 3.1, then $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi_C^f x_0 = \Pi_C x_0$. Then, the desired conclusion follows. □

Corollary 3.3 (Li et al. [34]). *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $T : C \to C$ is a relatively nonexpansive mappings of C into itself such that $F(T) \neq \emptyset$ and $f : E \to \mathbb{R}$ is a convex and lower semicontinuous mapping with $C \subset\ \mathrm{int}(D(f))$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C$, $\quad C_1 = C$, $\quad x_1 = \Pi_{C_1} x_0$,*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n) \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\} \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad n \geq 1, \end{cases}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$. *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_{F(T)}^f x_0$.

Corollary 3.4 (Takahashi and Zembayashi [12]). *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Let F be a bifunction from $C \times C$ satisfying $(A1)-(A4)$. Suppose T is a relatively nonexpansive mapping of C into itself such that $\Omega := EP(F) \cap F(T) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in C$, $\quad C_1 = C$, $\quad x_1 = \Pi_{C_1} x_0$,*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n) \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C \\ C_{n+1} = \{w \in C_n : \phi(w, u_n) \leq \phi(w, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1, \end{cases} \tag{3.16}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $(k = 1, 2, ..., m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1, 2, ..., m)$. *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_\Omega x_0$.

4 Applications

Let A be a monotone operator from C into E^*, the classical variational inequality is to find $x^* \in C$ such that

$$\langle y - x, Ax^*\rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of solutions of (4.1) is denoted by $VI(C, A)$.

Let $\varphi : C \to \mathbb{R}$ be a real-valued function. The convex minimization problem is to find $x^* \in C$ such that

$$\varphi(x^*) \le \varphi(y), \quad \forall y \in C. \tag{4.2}$$

The set of solutions of (4.2) is denoted by $CMP(\varphi)$.

The following lemmas are special cases of Lemma 2.8 and Lemma 2.9 of [44].

Lemma 4.1. *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $A : C \to E^*$ is a continuous and monotone operator. For $r > 0$ and $x \in E$, define a mapping $T_r^A : E \to C$ as follows:*

$$T_r^A(x) = \{z \in C : \langle Az, y - z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \forall y \in C\}.$$

Then, the following hold:
1. T_r^A is single-valued;
2. $F(T_r^A) = VI(C, A)$;
3. $VI(C, A)$ is closed and convex;
4. $\phi(q, T_r^A x) + \phi(T_r^A x, x) \le \phi(q, x), \quad \forall q \in F(T_r^A)$.

Lemma 4.2. *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $\varphi : C \to \mathbb{R}$ is a lower semicontinuous and convex. For $r > 0$ and $x \in E$, define a mapping $T_r^\varphi : E \to C$ as follows:*

$$T_r^\varphi(x) = \{z \in C : \varphi(y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge \varphi(z), \forall y \in C\}.$$

Then, the following hold:
1. T_r^φ is single-valued;
2. $F(T_r^\varphi) = CMP(\varphi)$;
3. $CMP(\varphi)$ is closed and convex;
4. $\phi(q, T_r^\varphi x) + \phi(T_r^\varphi x, x) \le \phi(q, x), \quad \forall q \in F(T_r^\varphi)$.

Then we obtain the following theorems from Theorem 3.1.

Theorem 4.3. *Let E be a uniformly smooth and strictly convex real Banach space which also has Kadec-Klee property. Let C be a nonempty, closed and convex subset of E. For each $k = 1, 2, ..., m$, let A_k be a continuous and monotone operator from C into E^* and suppose $\{T_i\}_{i=1}^\infty$ is an infinite family of weak relatively nonexpansive mappings of C into itself such that $\Omega := \Big(\cap_{k=1}^m VI(C, A_k)\Big) \cap \Big(\cap_{i=1}^\infty F(T_i)\Big) \ne \emptyset$. Let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous mapping with $C \subset \operatorname{int}(D(f))$ and $\{x_n\}_{n=0}^\infty$ be iteratively generated by $x_0 \in C, \quad C_1 = C, \quad x_1 = \Pi_{C_1}^f x_0$,*

$$\begin{cases} y_n = J^{-1}(\alpha_{n0}Jx_n + \sum_{i=1}^\infty \alpha_{ni}JT_i x_n) \\ u_n = T_{r_m,n}^{A_m} T_{r_{m-1},n}^{A_{m-1}} \cdots T_{r_2,n}^{A_2} T_{r_1,n}^{A_1} y_n \\ C_{n+1} = \{w \in C_n : G(w, Ju_n) \le G(w, Jx_n)\} \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad n \ge 1, \end{cases} \tag{4.3}$$

where J is the duality mapping on E. Suppose $\{\alpha_{ni}\}_{n=1}^{\infty}$ for each $i = 0, 1, 2, ...$ is a sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0$, $i = 1, 2, 3, ...,$ $\sum_{i=0}^{\infty} \alpha_{ni} = 1$ *and* $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $(k = 1, 2, ..., m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1, 2, ..., m)$. *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_{\Omega}^{f} x_0$.

Theorem 4.4. *Let E be a uniformly smooth and strictly convex real Banach space which also has Kadec-Klee property. Let C be a nonempty, closed and convex subset of E. For each $k = 1, 2, ..., m$, let $\varphi_k : C \to \mathbb{R}$ be a lower semi-continuous and convex and suppose $\{T_i\}_{i=1}^{\infty}$ is an infinite family of weak relatively nonexpansive mappings of C into itself such that* $\Omega := \Big(\cap_{k=1}^{m} CMP(\varphi_k)\Big) \cap \Big(\cap_{i=1}^{\infty} F(T_i)\Big) \neq \emptyset$. *Let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous mapping with* $C \subset \operatorname{int}(D(f))$ *and* $\{x_n\}_{n=0}^{\infty}$ *be iteratively generated by* $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}^{f} x_0$,

$$\begin{cases} y_n = J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{\infty} \alpha_{ni} J T_i x_n) \\ u_n = T_{r_{m,n}}^{\varphi_m} T_{r_{m-1,n}}^{\varphi_{m-1}} \cdots T_{r_{2,n}}^{\varphi_2} T_{r_{1,n}}^{\varphi_1} y_n \\ C_{n+1} = \{w \in C_n : G(w, J u_n) \leq G(w, J x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}}^{f} x_0, \quad n \geq 1, \end{cases} \tag{4.4}$$

where J is the duality mapping on E. Suppose $\{\alpha_{ni}\}_{n=1}^{\infty}$ for each $i = 0, 1, 2, ...$ is a sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0$, $i = 1, 2, 3, ...,$ $\sum_{i=0}^{\infty} \alpha_{ni} = 1$ *and* $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $(k = 1, 2, ..., m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1, 2, ..., m)$. *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_{\Omega}^{f} x_0$.

Example 4.5. *The following are examples of parameters appearing in the iterative schemes:*

$$\alpha_{ni} := \frac{1}{2^{i+1}}, \quad i = 0, 1, 2, ..., \quad n = 1, 2, ...$$

$$r_{k,n} := \frac{n+1}{n}, \quad k = 1, 2, ..., m, ..., \quad n = 1, 2, ...$$

Example 4.6. *Let C be a closed, convex and nonempty subset of a real Hilbert space H. If we define $F_k(x, y) := \langle A_k x, y - x\rangle$ for all $x, y \in C$ and $k = 1, 2, \ldots, m$. We see that $F(T_{r_k}^{F_k}) = EP(F_k) = VI(C, A_k) = F(T_{r_k}^{A_k})$ for each $k = 1, 2, \ldots, m$. Also, we can easily show that $\{F_k\}_{k=1}^{m}$ satisfy conditions $(A1) - (A4)$. Let T be a nonexpansive mapping of C into itself. Let I denote the identity mapping on H. For each $k = 1, 2, ..., m$, let $A_k = I - T$. Then A_k, $k = 1, 2, ..., m$ is continuous and monotone operator, with $D(A_k) = C$.*

Example 4.7. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. If we define $F_k(x, y) := \varphi_k(y) - \varphi_k(x)$ for all $x, y \in C$ and $k = 1, 2, \ldots, m$. We see that $F(T_{r_k}^{F_k}) = EP(F_k) = CMP(\varphi_k) = F(T_{r_k}^{\varphi_k})$ for each $k = 1, 2, \ldots, m$. Furthermore, we can easily show that $\{F_k\}_{k=1}^{m}$ satisfy conditions $(A1) - (A4)$. For each $k = 1, 2, ..., m$, let $\varphi_k = \delta_C$, where δ_C is the indicator function defined by*

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{otherwise.} \end{cases}$$

Then $\varphi_k, \ \ k = 1, 2, ..., m$ *is a proper convex function which is lower semi-continuous.*

Example 4.8. *Next, we give an explicit example of nonexpansive mappings considered. Let* C *be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space* E. *Now take* $T_i = \Pi_C, \ \ i = 1, 2, 3, ...,$ *where* Π_C *is the generalized projection map from a uniformly smooth and uniformly convex real Banach space* E. *It is known (see, e.g., [45]) that* Π_C *is a closed relatively-quasi-nonexpansive mapping from* E *onto* C *and* $\cap_{i=1}^{\infty} F(T_i) = F(\Pi_C) = C \neq \emptyset$.

Remark 4.1. *Our result improves on the result of Matsushita and Takahashi [7]. In fact, our iterative procedure* (3.1) *is simpler than* (1.4) *in the following two aspects:* (a) *the process of computing* $W_n = \{w \in C : \langle x_n - w, Jx_0 - Jx_n\rangle \geq 0\}$ *is removed;* (b) *the process of computing* $\Pi_{H_n \cap W_n} x_0$ *is replaced by computing* Π_{C_n}.

References

[1] C.E. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations*, Springer Verlag Series: Lecture Notes in Mathematics, Vol. 1965 (2009), XVII, 326p, ISBN 978-1-84882-189-7.

[2] W. Takahashi, *Nonlinear Functional Analysis-Fixed Point Theory and Applications*, Yokohama Publishers Inc., Yokohama, 2000 (in Japanese).

[3] W. Takahashi, *Nonlinear functional analysis*, Yokohama Publishers, Yokohama, 2000.

[4] D. Butnariu, S. Reich, A.J. Zaslavski, Asymptotic behaviour of relatively nonexpansive operators in Banach spaces, J. Appl. Anal. 7 (2001) 151–174.

[5] D. Butnariu, S. Reich, A.J. Zaslavski, Weak convergence of orbits of nonlinear operator in reflexive Banach spaces, Numer. Funct. Anal. Optim. 24 (2003) 489–508.

[6] Y. Censor, S. Reich, Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, Optimization 37 (1996) 323–339.

[7] S. Matsushita, W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in Banach spaces, J. Approx. Theory 134 (2005) 257–266.

[8] N. Onjai-uea, P. Kumam, Existence and Convergence Theorems for the new system of generalized mixed variational inequalities in Banach spaces, J. Inequalities and Applications 2012, 2012:9.

[9] S. Plubtieng, T. Thammathiwat, Shrinking projection methods for a family of relatively nonexpansive mappings, equilibrium problems and variational inequality problems in Banach spaces, J. Nonlinear Analysis and Optimization 1 (1) (2010) 97–110.

[10] S. Saewan, P. Kumam, A Generalized f-ProjectionMethod for Countable Families of Weak Relatively Nonexpansive Mappings and the System of Generalized Ky Fan Inequalities, J. Global Optim. DOI: 10.1007/s10898-012-9922-3.

[11] S. Saewan, P. Kumam, A strong convergence theorem concerning a hybrid projection method for finding common fixed points of a countable family of relatively quasi-nonexpansive mappings, J. Nonlinear Convex Anal. 13 (2) (2012) 313–330.

[12] W. Takahashi, K. Zembayashi, Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, Fixed Point Theory and Applications, Vol. 2008 (2008), Article ID 528476, 11 pages.

[13] K. Wattanawitoon, P. Kumam, Generalized Mixed Equilibrium Problems for-Maximal Monotone Operators and Two Relatively Quasi-Nonexpansive Mappings, Thai J. Math. 9 (2011) 171–195.

[14] J. Kang, Y. Su, X. Zhang, Hybrid algorithm for fixed points of weak relatively nonexpansive mappings and applications, Nonlinear Analysis: Hybrid Systems 4 (2010) 755–765.

[15] H. Zegeye, N. Shahzad, Strong convergence theorems for monotone mappings and weak relatively nonexpansive mappings, Nonlinear Anal. 70 (2009) 2707–2716.

[16] Y. Xu, Y. Su, On the weak relatively nonexpansive mappings in Banach spaces, Fixed Points Theory and Applications, Vol. 2010 (2010), Article ID 189751, 7 pages.

[17] Y. Su, D. Wang, M. Shang, Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings, Fixed Point Theory and Applications, Vol. 2008 (2008), Article ID 284613, 8 pages.

[18] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 123–145.

[19] A. Chinchuluun, P. Pardalos, A. Migdalas, L. Pitsoulis, *Pareto Optimality, Game Theory and Equilibria*, Springer, 2008.

[20] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117–136.

[21] C.A. Floudas, A. Christodoulos, P.M. Pardalos, *Encyclopedia of Optimization* - 2ND EDITION, Springer, (2009), XXXIV, 4626 p., 613 illus., ISBN 978-0-387-74760-6.

[22] F. Giannessi, A. Maugeri, P.M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*, Springer, Vol. 58 (2002).

[23] L.-J. Lin, System of Generalized Vector Quasi-Equilibrium Problems with Applications to Fixed Point Theorems for a Family of Nonexpansive Multi-valued Mappings, J. Global Optim. 34 (1) (2006) 15–32.

[24] Y. Liu, A general iterative method for equilibrium problems and strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 71 (2009) 4852–4861.

[25] A. Moudafi, Weak convergence theorems for nonexpansive mappings and equilibrium problems, J. Nonlinear Convex Anal. 9 (2008) 37–43.

[26] P.M. Pardalos, T.M. Rassias, A.A. Khan, *Nonlinear Analysis and Variational Problems*, Springer, 2010.

[27] S. Plubtieng, R. Punpaeng, A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings, Appl. Math. Comput. 197 (2008) 548–558.

[28] X. Qin, M. Shang, Y. Su, Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, Math. Comp. Model. 48 (2008) 1033–1046.

[29] Y. Su, M. Shang, X. Qin, An iterative method of solution for equilibrium and optimization problems, Nonlinear Anal. 69 (2008) 2709–2719.

[30] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–518.

[31] R. Wangkeeree, An extragradient approximation method for equilibrium problems and fixed point problems of a countable family of nonexpansive mappings, Fixed Point Theory and Applications, Vol. 2008 (2008), Article ID 134148, 17 pages.

[32] R. Wangkeeree, U. Kamraksa, The shrinking projection method for Generalized mixed Equilibrium Problems and Fixed Point Problems in Banach Spaces, J. Nonlinear Analysis and Optimization 1 (1) (2010) 111–129.

[33] S. Plubtieng, K. Ungchittrakool, Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space, J. Approx. Theory 149 (2007) 103–115.

[34] X. Li, N. Huang, D. O'Regan, Strong convergence theorems for relatively nonexpansive mappings in Banach spaces with applications, Comp. Math. Appl. 60 (2010) 1322–1331.

[35] Y. Su, H.K. Xu, X. Zhang, Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications, Nonlinear Anal. 73 (2010) 3890–3906.

[36] Y.I. Alber, Metric and generalized projection operator in Banach spaces: properties and applications, *in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type* vol 178 of Lecture Notes in Pure and Applied Mathematics, pp 15-50, Dekker, New York, NY, USA, 1996.

[37] Y. I. Alber, S. Reich, An iterative method for solving a class of nonlinear operator equations in Banach spaces, Pan. Amer. Math. J. 4 (1994) 39–54.

[38] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic, Dordrecht, 1990.

[39] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2002) 938–945.

[40] K.Q. Wu, N.J. Huang, The generalized f-projection operator with application, Bull. Aust. Math. Soc. 73 (2006) 307–317.

[41] J.H. Fan, X. Liu, J.L. Li, Iterative schemes for approximating solutions of generalized variational inequalities in Banach spaces, Nonlinear Anal. 70 (2009) 3997–4007.

[42] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[43] S. Chang, J.K. Kim, X.R. Wang, Modified Block iterative algorithm for solving convex feasibility problems in Banach spaces, J. Ineq. Appl., Vol. 2010 (2010), Article ID 869684, 14 pages.

[44] W. Takahashi, K. Zembayashi, Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, Nonlinear Anal. 70 (2009) 45–57.

[45] X. Qin, Y.J. Cho, S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math. 225 (2009) 20–30.

(Received 18 January 2012)
(Accepted 31 August 2012)

Thai J. Math. Online @ http://thaijmath.in.cmu.ac.th