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A Generalization of the Fourier Transform of Ultrahyperbolic Operator[1]

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Abstract : In this article, we introduce the Fourier transform of the distributional family $({}_\mu H_\alpha(x,m) \pm_v H_\alpha(x,m))^\lambda$ when ${}_\mu H_\alpha(x,m)$ is defined by (1.1) and ${}_v H_\alpha(x,m)$ is defined by (1.2). As consequence we obtain the Fourier transform of $((x_1^2+\cdots+x_\mu^2)-(x_{\mu+1}^2+\cdots+x_{\mu+v}^2))^\lambda$, $((x_1^2+\cdots+x_\mu^2)-(x_{\mu+1}^2+\cdots+x_{\mu+v}^2)^k$, $\odot^k\delta$, $\diamond^k\delta$ and $\otimes^k\delta$,where $\mu+\nu=n$ dimension of the space,$\odot$ is the operator defined by (2.89) introduced by Satsanit in [1],the operator $\diamond$ is the Diamond operator defined by (2.90) introduced by Kananthai [2] and and the operator $\oplus$ defined by (2.91) introduced by Kananthai et al. in [3].

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1 Introduction

Let ${}_\mu H_\alpha(x,m)$ be quadratic forms with complex coefficients defined by

$$ {}_\mu H_\alpha(x,m) = {}_\mu H_\alpha(x_1,...,x_\mu;m) = \left(\sum_{j=1}^{\mu} \alpha_j x_j^2\right)^m \tag{1.1}$$

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and

$$_vH_\alpha(x,m) =_v H_\alpha(x_{\mu+1},...,x_{\mu+v};m) = \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m \tag{1.2}$$

where $m = 1, 2, ..., \mu + v = n$ dimension of the space.

Associated to $_\mu H_\alpha(x,m)$ and $_vH_\alpha(x,m)$ considering $_{\mu,v}H_\alpha^{\pm}$ defined by

$$_{\mu,v}H_\alpha^{\pm} =_{\mu,v} H_\alpha^{\pm}(x_1,...,x_\mu,x_{\mu+1},...,x_{\mu+v};m) =_\mu H_\alpha(x,m) \pm_v H_\alpha(x,m) \tag{1.3}$$

with complex coefficients, and let $\mathrm{Im}(_\mu H_\alpha(x,m))$ and $\mathrm{Im}(_vH_\alpha(x,m))$be two positive definite quadratic forms,i.e. $\mathrm{Im}\,\alpha_j > 0$ for $j = 1, 2, ..., \mu, \mu + 1, ..., \mu + v$.

The generalized functions $_\mu H_\alpha^\lambda(x,m)$ and $_vH_\alpha^\lambda(x,m)$ and therefore also it's Fourier transform $\mathcal{F}\{_\mu H_\alpha^\lambda(x,m)\}$ and $\mathcal{F}\{_vH_\alpha^\lambda(x,m)\}$ are analytic functions of the α_j in the region $\mathrm{Im}\,\alpha_j > 0$,where λ is a complex number.

We know from [4] that the following formula is valid

$$\begin{aligned}\mathcal{F}\left\{\left(\left(\sum_{j=1}^n\alpha_j x_j^2\right)^m\right)^\lambda\right\} &= \frac{e^{-\frac{n\pi i}{4}}}{\sqrt{-i\alpha_1}\cdots\sqrt{-i\alpha_\mu}\sqrt{-i\alpha_{\mu+1}}\cdots\sqrt{-i\alpha_{\mu+v}}} \\ &\times\frac{2^{m\lambda+n}\pi^{\frac{n}{2}}\Gamma(m\lambda+\frac{n}{2})}{\Gamma(-m\lambda)}\left(\frac{s_1^2}{\alpha_1}+\cdots+\frac{s_n^2}{\alpha_n}\right)^{-m\lambda-\frac{n}{2}}\end{aligned} \tag{1.4}$$

where the Fourier transform is defined by

$$\mathcal{F}\{f(x)\} = \int_{R^n} f(x)e^{-i<x,y>}dx \tag{1.5}$$

with

$$\langle x, y\rangle = x_1y_1 + \cdots + x_\mu y_\mu. \tag{1.6}$$

Using (1.4) we obtain the following formulae

$$\mathcal{F}\{_\mu H_\alpha^\lambda(x,m)\} = \frac{e^{\frac{-\mu\pi i}{4}}}{\sqrt{-i\alpha_1}\cdots\sqrt{-i\alpha_\mu}}\frac{2^{m\lambda+\mu}\pi^{\frac{\mu}{2}}\Gamma(m\lambda+\frac{\mu}{2})}{\Gamma(-m\lambda)}\left(\frac{s_1^2}{\alpha_1}+\cdots+\frac{s_\mu^2}{\alpha_\mu}\right)^{-m\lambda-\frac{\mu}{2}} \tag{1.7}$$

and

$$\mathcal{F}\{_vH_\alpha^\lambda(x,m)\} = \frac{e^{\frac{-v\pi i}{4}}}{\sqrt{-i\alpha_{\mu+1}}\cdots\sqrt{-i\alpha_{\mu+v}}}\frac{2^{m\lambda+v}\pi^{\frac{v}{2}}\Gamma(m\lambda+\frac{v}{2})}{\Gamma(-m\lambda)}\left(\frac{s_{\mu+1}^2}{\alpha_{\mu+1}}+\cdots+\frac{s_{\mu+v}^2}{\alpha_{\mu+v}}\right)^{-m\lambda-\frac{v}{2}}. \tag{1.8}$$

We observe that by putting $\alpha_1 = \alpha_2 = \cdots = \alpha_\mu = 1$ and $\alpha_{\mu+1} = \alpha_{\mu+2} = \cdots = \alpha_{\mu+v} = -1$ in (1.3), we have

$$\begin{aligned}\left(_{\mu,v}H^+\right)^\lambda &= \left(_\mu H(x,m) +_v H(x,m)\right)^\lambda \\ &= \left(\left(\sum_{j=1}^{\mu}x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v}x_j^2\right)^m\right)^\lambda = M^\lambda(x,m)\end{aligned} \tag{1.9}$$

and

$$\left({}_{\mu,v}H^{-}\right)^{\lambda} = ({}_{\mu}H(x,m) -_{v} H(x,m))^{\lambda}$$
$$= \left(\left(\sum_{j=1}^{\mu} x_j^2\right)^m - \left(\sum_{j=\mu+1}^{\mu+v} x_j^2\right)^m\right)^{\lambda} = G^{\lambda}(x,m). \quad (1.10)$$

The distributional family $({}_{\mu,v}H^{-})^{\lambda} = G^{\lambda}(x,m)$ appear in [5] and [6].

On the other hand, from [7] we have,

$$\operatorname*{Res}_{\lambda=-\frac{n}{2m}-k} \left(\left(\sum_{j=1}^{n} x_j^2\right)^m\right)^{\lambda} =$$
$$\frac{e^{\frac{-n\pi i}{4}} 2\pi^{\frac{n}{2}} L_{\alpha}^{km}\{\delta\}}{(2m)^{2km}\Gamma(\frac{n}{2}+mk)(km)!\sqrt{-i\alpha_1}\cdots\sqrt{-i\alpha_{\mu}}\sqrt{-i\alpha_{\mu+1}}\cdots\sqrt{-i\alpha_{\mu+v}}} \frac{\Gamma(m\lambda+\frac{v}{2})}{\Gamma(-m\lambda)} \quad (1.11)$$

where $\alpha_j = ib_j, b_j > 0, j = 1,2,...,n$ and

$$L_{\alpha} = \sum_{j=1}^{n} \frac{1}{\alpha_j} \frac{\partial^2}{\partial x_j^2}. \quad (1.12)$$

2 The Fourier transform of ${}_{\mu}H(x,m) \pm_{v} H(x,m)$

From (1.1),(1.2) and using the definition(1.4) we have,

$$\mathcal{F}\{({}_{\mu}H(x,m) +_{v} H(x,m))^{\lambda}\} = \int_{R^n} ({}_{\mu}H(x,m) +_{v} H(x,m))^{\lambda} e^{-i<s,x>}dx$$
$$\int_{R^n} \left(\left(\sum_{j=1}^{\mu} \alpha_j x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v} \alpha_j x_j^2\right)^m\right)^{\lambda} e^{-i<s,x>}dx \quad (2.1)$$

where $< s,x >$ is defined by (1.6).

In order to obtain the Fourier transform of ${}_{\mu}H(x,m) +_{v} H(x,m)$ we are going to study two cases:

Case 1: $m = 1$. From (2.1) and considering the formula (1.4) we have,

$$\mathcal{F}\{({}_{\mu}H(x,1) +_{v} H(x,1))^{\lambda}\}$$
$$= \mathcal{F}\left\{\left(\sum_{j=1}^{\mu} \alpha_j x_j^2 + \sum_{j=\mu+1}^{\mu+v} \alpha_j x_j^2\right)^{\lambda}\right\} = \mathcal{F}\left\{\left(\sum_{j=1}^{n} \alpha_j x_j^2\right)^{\lambda}\right\}$$
$$= \frac{e^{\frac{-n\pi i}{4}} 2^{2\lambda+n}\pi^{\frac{n}{2}}\Gamma(\lambda+\frac{n}{2})}{\sqrt{-i\alpha_1}\cdots\sqrt{-i\alpha_{\mu}}\sqrt{-i\alpha_{\mu+1}}\cdots\sqrt{-i\alpha_{\mu+v}}\Gamma(-\lambda)}$$
$$\times \left(\frac{s_1^2}{\alpha_1} + \cdots + \frac{s_{\mu}^2}{\alpha_{\mu}} + \frac{s_{\mu+1}^2}{\alpha_{\mu+1}} + \cdots + \frac{s_{\mu+v}^2}{\alpha_{\mu+v}}\right)^{-\lambda-\frac{n}{2}}. \quad (2.2)$$

In particular if $\alpha_1 = \alpha_2 = \cdots = \alpha_\mu = 1$ and $\alpha_{\mu+1} = \alpha_{\mu+2} = \cdots = \alpha_{\mu+v} = -1$, we have

$$\begin{aligned}
&\mathcal{F}\left\{\left((x_1^2+\cdots+x_\mu^2)-(x_{\mu+1}^2+\cdots+x_{\mu+v}^2)\right)^\lambda\right\}\\
&\quad= \frac{e^{-\frac{n\pi i}{4}}}{e^{-\frac{\mu\pi i}{4}}e^{\frac{v\pi i}{4}}}\frac{2^{2\lambda+n}\pi^{\frac{n}{2}}\Gamma(\lambda+\frac{n}{2})}{\Gamma(-\lambda)}(s_1^2+\cdots+s_\mu^2-s_{\mu+1}^2-\cdots-s_{\mu+v}^2)^{-\lambda-\frac{n}{2}}\\
&\quad= \frac{e^{\frac{v\pi i}{4}}2^{2\lambda+n}\pi^{\frac{n}{2}}\Gamma(\lambda+\frac{n}{2})}{\Gamma(-\lambda)}(s_1^2+\cdots+s_\mu^2-s_{\mu+1}^2-\cdots-s_{\mu+v}^2)^{-\lambda-\frac{n}{2}}
\end{aligned} \tag{2.3}$$

if $x_1^2+\cdots+x_\mu^2 \geq x_{\mu+1}^2+\cdots+x_{\mu+v}^2$ and $s_1^2+\cdots+s_\mu^2 \geq s_{\mu+1}^2+\cdots+s_{\mu+v}^2$.

By putting $\sigma_1 = s_1, \sigma_2 = s_2, ..., \sigma_\mu = s_\mu, \sigma_{\mu+1} = s_{\mu+1}, ..., \sigma_{\mu+v} = s_{\mu+v}$ and considering the formulae

$$\begin{aligned}
&\operatorname*{Res}_{\lambda=k} s(\sigma_1^2+\cdots+\sigma_\mu^2-\sigma_{\mu+1}^2-\cdots-\sigma_{\mu+v}^2)^{-\lambda-\frac{n}{2}}\\
&\qquad= \operatorname*{Res}_{\gamma=-\frac{n}{2}-k,k=0,1,2,...} s\,(\sigma_1^2+\cdots+\sigma_\mu^2-\sigma_{\mu+1}^2-\cdots-\sigma_{\mu+v}^2)^{\gamma}\\
&\qquad= \frac{(-1)^{\frac{v}{2}}\pi^{\frac{n}{2}}}{2^{2k}k!\Gamma(\frac{n}{2}+k)}L^k\{\delta(\sigma)\}
\end{aligned} \tag{2.4}$$

if μ is odd v is even (n odd) ([9], page 260, formula 28),

$$\begin{aligned}
&\operatorname*{Res}_{\gamma=-\frac{n}{2}-k,k=0,1,2,...} s\,(\sigma_1^2+\cdots+\sigma_\mu^2-\sigma_{\mu+1}^2-\cdots-\sigma_{\mu+v}^2)^{\gamma}\\
&\qquad= \frac{(-1)(-1)^{\frac{v}{2}}\pi^{\frac{n}{2}}}{2^{2k}k!\Gamma(\frac{n}{2}+k)}L^{k-\frac{n}{2}+1}\{\delta(\sigma)\}
\end{aligned} \tag{2.5}$$

if $k \geq \frac{n}{2}-1$, μ and v are both even (n even) ([10], p. 38–39, formula 59) and

$$\begin{aligned}
&\operatorname*{Res}_{\gamma=-\frac{n}{2}-k,k=0,1,2,...} s\,(\sigma_1^2+\cdots+\sigma_\mu^2-\sigma_{\mu+1}^2-\cdots-\sigma_{\mu+v}^2)^{\gamma}\\
&\qquad= \frac{(-1)(-1)^{\frac{v+1}{2}}\pi^{\frac{n}{2}-1}[\Psi(\frac{\mu}{2})-\Psi(\frac{n}{2})]}{2^{2k}k!\Gamma(\frac{n}{2}+k)}L^{k-\frac{n}{2}+1}\{\delta(\sigma)\}
\end{aligned} \tag{2.6}$$

if $k \geq \frac{n}{2}-1$, μ and v are both odd (n even) ([10], p. 42, formula 60) and the formula

$$\operatorname*{Res}_{z=-l,l=0,1,2,...} s\Gamma(z) = \frac{(-1)^l}{l!} \tag{2.7}$$

we have,

$$\lim_{\lambda\to -k}\frac{(\sigma_1^2+\cdots+\sigma_\mu^2-\sigma_{\mu+1}^2-\cdots-\sigma_{\mu+v}^2)^{-\lambda-\frac{n}{2}}}{\Gamma(-\lambda)}$$
$$= \lim_{\gamma\to -\frac{n}{2}-k}\frac{(\sigma_1^2+\cdots+\sigma_\mu^2-\sigma_{\mu+1}^2-\cdots-\sigma_{\mu+v}^2)^{\gamma}}{\Gamma(\gamma+\frac{n}{2})}$$
$$= \frac{\lim_{\gamma\to -\frac{n}{2}-k}(\gamma+\frac{n}{2}+k)(\sigma_1^2+\cdots+\sigma_\mu^2-\sigma_{\mu+1}^2-\cdots-\sigma_{\mu+v}^2)^{\gamma}}{\lim_{\gamma\to -\frac{n}{2}-k}(\gamma+\frac{n}{2}+k)\Gamma(\gamma+\frac{n}{2})}$$
$$= \frac{\operatorname*{Res}_{\gamma=-\frac{n}{2}-k}(\sigma_1^2+\cdots+\sigma_\mu^2-\sigma_{\mu+1}^2-\cdots-\sigma_{\mu+v}^2)^{\gamma}}{\lim_{z\to -k}(z+k)\Gamma(z)}$$
$$= \frac{\operatorname*{Res}_{\gamma=-\frac{n}{2}-k}(\sigma_1^2+\cdots+\sigma_\mu^2-\sigma_{\mu+1}^2-\cdots-\sigma_{\mu+v}^2)^{\gamma}}{(-1)^k/k!}$$
$$= \frac{(-1)^{\frac{v}{2}}(-1)^k\pi^{\frac{n}{2}}}{2^{2k}\Gamma(\frac{n}{2}+k)}L^k\{\delta\} \tag{2.8}$$

if μ is odd, v is even (n odd),

$$\lim_{\lambda\to -k}\frac{(\sigma_1^2+\cdots+\sigma_\mu^2-\sigma_{\mu+1}^2-\cdots-\sigma_{\mu+v}^2)^{-\lambda-\frac{n}{2}}}{\Gamma(-\lambda)}$$
$$= \frac{(-1)(-1)^{\frac{v}{2}}(-1)^k\pi^{\frac{n}{2}}}{2^{2k}\Gamma(\frac{n}{2}+k)}L^{k-\frac{n}{2}+1}\{\delta\} \tag{2.9}$$

if $k \geq \frac{n}{2}-1$, μ and v are both even (n even), and

$$\lim_{\lambda\to -k}\frac{(\sigma_1^2+\cdots+\sigma_\mu^2-\sigma_{\mu+1}^2-\cdots-\sigma_{\mu+v}^2)^{-\lambda-\frac{n}{2}}}{\Gamma(-\lambda)}$$
$$= \frac{(-1)(-1)^{\frac{v+1}{2}}\pi^{\frac{n}{2}-1}(-1)^k[\Psi(\frac{\mu}{2})-\Psi(\frac{n}{2})]}{2^{2k}\Gamma(\frac{n}{2}+k)}L^{k-\frac{n}{2}+1}\{\delta\} \tag{2.10}$$

if $k \geq \frac{n}{2}-1$, μ and v are both odd (n even), where

$$L = \frac{\partial^2}{\partial\sigma_1^2}+\cdots+\frac{\partial^2}{\partial\sigma_\mu^2}-\frac{\partial^2}{\partial\sigma_{\mu+1}^2}-\cdots-\frac{\partial^2}{\partial\sigma_{\mu+v}^2}, \tag{2.11}$$

$$\Psi(k) = -C+1+\frac{1}{2}+\cdots+\frac{1}{k-1}, k=2,3,... \tag{2.12}$$

$$\Psi\left(k+\frac{1}{2}\right) = -C-2\ln 2+2\left(1+\frac{1}{2}+\cdots+\frac{1}{2k-1}\right) \tag{2.13}$$

and C is Euler's constant.

Therefore from (2.3) and using (2.8),(2.9) and (2.10) we obtain the following formulae

$$\mathcal{F}\{((x_1^2+\cdots+x_\mu^2)-(x_{\mu+1}^2+\cdots+x_{\mu+v}^2))^k\}=(2\pi)^n(-1)^kL^k\delta \text{ if } n \text{ is odd} \tag{2.14}$$

$$\mathcal{F}\{((x_1^2+\cdots+x_\mu^2)-(x_{\mu+1}^2+\cdots+x_{\mu+v}^2))^k\}=-(2\pi)^n(-1)^kL^{k-\frac{n}{2}+1}\delta \tag{2.15}$$

if $k\geq\frac{n}{2}-1$, μ and v are both even (n even), and

$$\begin{aligned}&\mathcal{F}\{((x_1^2+\cdots+x_\mu^2)-(x_{\mu+1}^2+\cdots+x_{\mu+v}^2))^k\}\\&\qquad=e^{\frac{\pi i}{2}}\pi^{n-1}2^n[\Psi(\frac{\mu}{2})-\Psi(\frac{n}{2})](-1)^kL^{k-\frac{n}{2}+1}\delta\end{aligned} \tag{2.16}$$

if $k\geq\frac{n}{2}-1$, μ and v are both odd (n even).

From (2.14),(2.15) and (2.16) we obtain the following formulae

$$\mathcal{F}\{(-1)^kL^k\delta\}=(x_1^2+\cdots+x_\mu^2-x_{\mu+1}^2-\cdots-x_{\mu+v}^2)^k \text{ if } n \text{ is odd}, \tag{2.17}$$

$$\mathcal{F}\{(-1)^kL^{k-\frac{n}{2}+1}\delta\}=-(x_1^2+\cdots+x_\mu^2-x_{\mu+1}^2-\cdots-x_{\mu+v}^2)^k \tag{2.18}$$

if $k\geq\frac{n}{2}-1$, μ and v are both even (n even) and

$$\mathcal{F}\{(-1)^kL^{k-\frac{n}{2}+1}\delta\}=\frac{\pi e^{-\frac{\pi i}{2}}}{[\Psi(\frac{\mu}{2})-\Psi(\frac{n}{2})]}\ (x_1^2+\cdots+x_\mu^2-x_{\mu+1}^2-\cdots-x_{\mu+v}^2)^k \tag{2.19}$$

if $k\geq\frac{n}{2}-1$, μ and v are both odd.

Case 2: $m\geq 2$.

From (1.5) and (1.6) using that $n=\mu+v$ dimension of the space and considering that $\alpha_j=ib_j, j=1,2,...,n$ we have,

$$\begin{aligned}&\mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}\alpha_jx_j^2\right)^m+\left(\sum_{j=\mu+1}^{\mu+v}\alpha_jx_j^2\right)^m\right)^\lambda\right\}\\&\qquad=\int_{R^n}\left\{\left(\left(\sum_{j=1}^{\mu}\alpha_jx_j^2\right)^m+\left(\sum_{j=\mu+1}^{\mu+v}\alpha_jx_j^2\right)^m\right)^\lambda\right.\\&\qquad\qquad\left.\times\ e^{-i(x_1s_1+\cdots+x_\mu s_\mu+x_{\mu+1}s_{\mu+1}+\cdots+x_{\mu+v}s_{\mu+v})}dx\right\}\\&\qquad=\frac{e^{\frac{m\lambda\pi i}{2}}}{\sqrt{-i\alpha_1}.\sqrt{-i\alpha_2}\cdots\sqrt{-i\alpha_{\mu+\nu}}}.\end{aligned}$$

$$\int_{R^v}\left\{\left(\int_{R^\mu}(r^{2m}+s^{2m})^\lambda e^{-i(y_1\frac{s_1}{\sqrt{-i\alpha_1}}+\cdots+y_\mu\frac{s_\mu}{\sqrt{-i\alpha_\mu}})}dy_1\cdots dy_\mu\right)\right.$$
$$\left.\times\ e^{-i(y_{\mu+1}\frac{s_{\mu+1}}{\sqrt{-i\alpha_{\mu+1}}}+\cdots+y_{\mu+\nu}\frac{s_{\mu+\nu}}{\sqrt{-i\alpha_{\mu+\nu}}})}dy_{\mu+1}\cdots dy_{\mu+\nu}\right\}$$
$$=\mathcal{F}_{R^\nu}\left\{\mathcal{F}_{R^\mu}\{(r^{2m}+s^{2m})^\lambda\}(\frac{s_1}{\sqrt{-i\alpha_1}},\frac{s_2}{\sqrt{-i\alpha_2}},...,\frac{s_\mu}{\sqrt{-i\alpha_\mu}})\right\}$$
$$\times\left(\frac{s_{\mu+1}}{\sqrt{-i\alpha_{\mu+1}}},...,\frac{s_{\mu+v}}{\sqrt{-i\alpha_{\mu+v}}}\right). \tag{2.20}$$

First we are going to calculate the integral

$$\mathcal{F}_{R^\mu}\left\{(r^{2m}+s^{2m})^\lambda\right\}(\frac{s_1}{\sqrt{-i\alpha_1}},\frac{s_2}{\sqrt{-i\alpha_2}},...,\frac{s_\mu}{\sqrt{-i\alpha_\mu}})$$
$$=\int_{R^\mu}(r^{2m}+s^{2m})^\lambda e^{-i(\frac{s_1}{\sqrt{-i\alpha_1}}y_1+\cdots+\frac{s_\mu}{\sqrt{-i\alpha_\mu}}y_\mu)}dy_1\cdots dy_\mu \tag{2.21}$$

where

$$r^{2m}=(y_1^2+\cdots+y_\mu^2)^m \tag{2.22}$$

and

$$s^{2m}=(y_{\mu+1}^2+\cdots+y_{\mu+v}^2)^m. \tag{2.23}$$

Without loss of generality we may assume that the components of σ are given by $\sigma=(|\sigma|,0,...,0)$, so that the integral (2.24) becomes

$$\int_{R^\mu}(r^{2m}+s^{2m})^\lambda e^{-i(\frac{s_1}{\sqrt{-i\alpha_1}}y_1+\cdots+\frac{s_\mu}{\sqrt{-i\alpha_\mu}}y_\mu)}dy_1\cdots dy_\mu$$
$$=\int_{R^\mu}(r^{2m}+s^{2m})^\lambda e^{-i|\sigma|y_1}dy_1\cdots dy_\mu \tag{2.24}$$

where

$$\sigma=(\sigma_1,\sigma_2,...,\sigma_\mu)=\left(\frac{s_1}{\sqrt{-i\alpha_1}},\frac{s_2}{\sqrt{-i\alpha_2}},...,\frac{s_\mu}{\sqrt{-i\alpha_\mu}}\right) \tag{2.25}$$

and

$$|\sigma|=\sqrt{\sigma_1^2+\cdots+\sigma_\mu^2}=\sqrt{\frac{s_1^2}{-i\alpha_1}+\cdots+\frac{s_\mu^2}{-i\alpha_\mu}}=e^{\frac{\pi i}{4}}\sqrt{\frac{s_1^2}{\alpha_1}+\cdots+\frac{s_\mu^2}{\alpha_\mu}}. \tag{2.26}$$

We shall perform the integration in (2.24) by going to polar coordinates. After integrating over the angles $\theta_2,\theta_3,...,\theta_{\mu-1}$ and using the fact that

$$\Omega_{\mu-1}=\frac{2\pi^{\frac{\mu-1}{2}}}{\Gamma(\frac{\mu-1}{2})} \tag{2.27}$$

we arrive at

$$\int_{R^\mu}(r^{2m}+s^{2m})^\lambda e^{-i|\sigma|y_1}dy_1\cdots dy_\mu$$
$$=\frac{2\pi^{\frac{\mu-1}{2}}}{\Gamma(\frac{\mu-1}{2})}\int_0^\infty\int_0^\pi(s^{2m}+r^{2m})^\lambda e^{ir|\sigma|\cos\theta_1}\sin^{\mu-2}\theta_1 r^{\mu-1}d\theta_1 dr \quad (2.28)$$

using the formula

$$\int_0^\pi e^{ir|\sigma|\cos\theta_1}\sin^{\mu-2}\theta_1 d\theta_1=\frac{\Gamma(\frac{\mu-1}{2})\sqrt{\pi}}{\left(\frac{r|\sigma|}{2}\right)^{\frac{\mu}{2}-1}}J_{\frac{\mu}{2}-1}(r\,|\sigma|) \quad (2.29)$$

where $J_\beta(z)$ is a Bessel function,

$$J_\beta(z)=\left(\frac{z}{2}\right)^\beta\sum_{t=0}^\infty\frac{(-1)^t\left(\frac{z}{2}\right)^{2t}}{t!\Gamma(\beta+t+1)} \quad (2.30)$$

we have,

$$\int_{R^\mu}(r^{2m}+s^{2m})^\lambda e^{-i|\sigma|y_1}dy_1\cdots dy_\mu$$
$$=\frac{2\pi^{\frac{\mu-\pi}{2}}\sqrt{\pi}}{\Gamma(\frac{\mu-1}{2})}\Gamma(\frac{\mu-1}{2})2^{\frac{\mu}{2}-1}|\sigma|^{-\frac{\mu}{2}+1}\int_0^\infty r^{\frac{\mu}{2}}(r^{2m}+s^{2m})^\lambda J_{\frac{\mu}{2}-1}(r\,|\sigma|)dr, \quad (2.31)$$

using the formula

$$(1+z)^\lambda=\sum_{t=0}^\infty\binom{\lambda}{t}z^t \text{ for } |z|<1 \quad (2.32)$$

where

$$\binom{\lambda}{t}=\frac{\Gamma(\lambda+1)}{t!\Gamma(\lambda+1-t)}=\frac{(-1)^t\Gamma(-\lambda-1+t+1)}{\Gamma(1-\lambda-1)}=\frac{(-1)^t\Gamma(-\lambda+t)}{\Gamma(-\lambda)}, \quad (2.33)$$

$t=0,1,2,...$, we have

$$\int_{R^\mu}(r^{2m}+s^{2m})^\lambda e^{-i|\sigma|y_1}dy_1\cdots dy_\mu$$
$$=2^{\frac{\mu}{2}}\pi^{\frac{\mu}{2}}|\sigma|^{-\frac{\mu}{2}+1}\sum_{j\geq 0}\binom{\lambda}{j}s^{2mj}\int_0^\infty r^{\frac{\mu}{2}+2m\lambda-2mj}J_{\frac{\mu}{2}-1}(r\,|\sigma|)dr. \quad (2.34)$$

Now using the formula

$$\int_0^\infty x^u J_v(ax)dx=\frac{2^u a^{-u-1}\Gamma(\frac{1}{2}+\frac{v}{2}+\frac{u}{2})}{\Gamma(\frac{1}{2}+\frac{v}{2}-\frac{u}{2})} \quad (2.35)$$

if $-\operatorname{Re} v - 1 < \operatorname{Re} u < \frac{1}{2}$ ([8], page 686, formula 4) we have,

$$\int_{R^\mu} (r^{2m} + s^{2m})^\lambda e^{-i|\sigma| y_1} dy_1 \cdots dy_\mu$$
$$= 2^\mu \pi^{\frac{\mu}{2}} 2^{2m\lambda} \sum_{j \geq 0} \binom{\lambda}{j} s^{2mj} 2^{-2mj} \frac{|\sigma|^{-\mu-2m\lambda+2mj} \Gamma(m\lambda + \frac{\mu}{2} - mj)}{\Gamma(-m\lambda + mj)}. \tag{2.36}$$

From (2.20) and (2.36) we have,

$$\mathcal{F}\{((\sum_{j=1}^{\mu} \alpha_j x_j^2)^m + \sum_{j=\mu+1}^{\mu+v} \alpha_j x_j^2)^m)^\lambda\}$$
$$= \frac{e^{\frac{m\lambda\pi i}{2}} 2^\mu \pi^{\frac{\mu}{2}} 2^{2m\lambda}}{\sqrt{-i\alpha_2} \cdots \sqrt{-i\alpha_\mu} \sqrt{-i\alpha_{\mu+1}} \cdots \sqrt{-i\alpha_{\mu+v}}}$$
$$\times \sum_{j \geq 0} \binom{\lambda}{j} \frac{\Gamma(m\lambda + \frac{\mu}{2} - mj)}{\Gamma(-m\lambda + mj) 2^{2mj}} \int_{R^v} s^{2mj} |\sigma|^{-\mu-2m\lambda+2mj}$$
$$\times e^{-i(\frac{s_{\mu+1}}{\sqrt{-i\alpha_{\mu+1}}} y_{\mu+1} + \cdots + \frac{s_{\mu+v}}{\sqrt{-i\alpha_{\mu+v}}} y_{\mu+v})} dy_{\mu+1} \cdots dy_{\mu+v}$$
$$= \frac{e^{\frac{m\lambda\pi i}{2}} 2^\mu \pi^{\frac{\mu}{2}} 2^{2m\lambda}}{\sqrt{-i\alpha_2} \cdots \sqrt{-i\alpha_\mu} \sqrt{-i\alpha_{\mu+1}} \cdots \sqrt{-i\alpha_{\mu+v}}}$$
$$\times \sum_{j \geq 0} \binom{\lambda}{j} \frac{\Gamma(m\lambda + \frac{\mu}{2} - mj)}{\Gamma(-m\lambda + mj) 2^{2mj}} |\sigma|^{-\mu-2m\lambda+2mj}$$
$$\times \int_{R^v} s^{2mj} e^{-i(\frac{s_{\mu+1}}{\sqrt{-i\alpha_{\mu+1}}} y_{\mu+1} + \cdots + \frac{s_{\mu+v}}{\sqrt{-i\alpha_{\mu+v}}} y_{\mu+v})} dy_{\mu+1} \cdots dy_{\mu+v}. \tag{2.37}$$

On the other hand, considering the Fourier transform of

$$r^\lambda = (x_1^2 + \cdots + x_n^2)^{\frac{\lambda}{2}}, \tag{2.38}$$

$$\mathcal{F}\{r^\lambda\} = \int_{R^n} r^\lambda e^{i<y,t>} dy = \frac{\pi^{\frac{n}{2}} 2^{\lambda+n} \Gamma(\frac{\lambda+n}{2})}{\Gamma(-\frac{\lambda}{2})} \rho^{-\lambda-n} \text{ ([9], page 194)} \tag{2.39}$$

and taking into account that r^λ has singularitie at $\lambda = -n - 2k, k = 0, 1, 2, ...,$ we have

$$\begin{aligned}
&\int_{R^v} s^{2mj} e^{-i(\frac{s_{\mu+1}}{\sqrt{-i\alpha_{\mu+1}}} y_{\mu+1}+\cdots+\frac{s_{\mu+v}}{\sqrt{-i\alpha_{\mu+v}}} y_{\mu+v})} dy_{\mu+1}\cdots dy_{\mu+v} \\
&= \int_{R^v} (y_{\mu+1}^2+\cdots+y_{\mu+v}^2)^{mj} e^{-i(\frac{s_{\mu+1}}{\sqrt{-i\alpha_{\mu+1}}} y_{\mu+1}+\cdots+\frac{s_{\mu+v}}{\sqrt{-i\alpha_{\mu+v}}} y_{\mu+v})} dy_{\mu+1}\cdots dy_{\mu+v} \\
&= \mathcal{F}\{s^{2mj}\}\left(\frac{s_{\mu+1}}{\sqrt{-i\alpha_{\mu+1}}},\ldots,\frac{s_{\mu+v}}{\sqrt{-i\alpha_{\mu+v}}}\right) \\
&= \lim_{\lambda\to 2mj}\left[\pi^{\frac{v}{2}} 2^{\lambda+v}\Gamma\left(\frac{\lambda+v}{2}\right)\frac{\rho^{-\lambda-v}}{\Gamma\left(-\frac{\lambda}{2}\right)}\right] \qquad (2.40)
\end{aligned}$$

where

$$\rho=\sqrt{\frac{s_{\mu+1}^2}{b_{\mu+1}}+\cdots+\frac{s_{\mu+v}^2}{b_{\mu+v}}} \qquad (2.41)$$

$$\begin{aligned}
b_{\mu+1} &= -i\alpha_{\mu+1}, \\
b_{\mu+2} &= -i\alpha_{\mu+2} \\
&\vdots \\
b_{\mu+v} &= -i\alpha_{\mu+v}.
\end{aligned} \qquad (2.42)$$

On the other hand, by putting $\beta = m\lambda$ in (1.11) we have,

$$\operatorname*{Re\,s}_{\beta=-\frac{v}{2}-mk,k=0,1,2,\ldots}(\alpha_{\mu+1}x_{\mu+1}^2+\cdots+\alpha_{\mu+v}x_{\mu+v}^2)^{\beta} = \frac{2\pi^{\frac{v}{2}}\ {}_vL_\gamma^{km}\delta}{(2)^{2km}\Gamma(\frac{v}{2}+mk)(km)!\sqrt{-i\alpha_{\mu+1}}\cdots\sqrt{-i\alpha_{\mu+v}}} \qquad (2.43)$$

where

$${}_vL_\gamma=\sum_{j=\mu+1}^{\mu+v}\frac{1}{\alpha_j}\frac{\partial^2}{\partial x_j^2}. \qquad (2.44)$$

Now, using (2.43) and (2.44) we have,

$$\begin{aligned}
&\operatorname*{Re\,s}_{\beta=-\frac{v}{2}-mj}\left(\frac{1}{\alpha_{\mu+1}}s_{\mu+1}^2+\cdots+\frac{1}{\alpha_{\mu+v}}s_{\mu+v}^2\right)^{\beta} \\
&= \frac{2\pi^{\frac{v}{2}}e^{\frac{-v\pi i}{2}}e^{\frac{\pi i}{4}}}{(2m)^{2jm}\Gamma(\frac{v}{2}+mj)(2m)!}\sqrt{\alpha_{\mu+1}}\cdots\sqrt{\alpha_{\mu+v}} \\
&\quad\times\left(\frac{1}{\alpha_{\mu+1}}\frac{\partial^2}{\partial s_{\mu+1}^2}+\cdots+\frac{1}{\alpha_{\mu+v}}\frac{\partial^2}{\partial s_{\mu+v}^2}\right)^{jm}\delta. \qquad (2.45)
\end{aligned}$$

On the other hand, using (2.45) and considering the formula

$$\operatorname*{Re\,s}_{z=-h,h=0,1,2,\ldots}\Gamma(z)=\frac{(-1)^h}{h!} \qquad (2.46)$$

we have

$$
\begin{aligned}
\lim_{\lambda\to 2mj}\frac{\rho^{-\lambda-v}}{\Gamma(-\frac{\lambda}{2})} &= \lim_{\beta\to -2mj-v}\frac{\rho^{\beta}}{\Gamma(\frac{\beta+v}{2})} = \lim_{\beta\to -\frac{v}{2}-2mj}\frac{\rho^{\frac{\beta}{2}}}{\Gamma(\frac{\beta+v}{2})}\\
&= \lim_{\eta\to -\frac{v}{2}-mj}\frac{(\rho^2)^{\eta}}{\Gamma(\eta+\frac{v}{2})} = \lim_{\eta\to -\frac{v}{2}-mj}\frac{(\eta+\frac{v}{2}+mj)(\rho^2)^{\eta}}{(\eta+\frac{v}{2}+mj)\Gamma(\eta+\frac{v}{2})}\\
&= \frac{\operatorname*{Res}_{\eta=-\frac{v}{2}-mj} s\,(\rho^2)^{\eta}}{\lim_{z\to -mj}(z+mj)\Gamma(z)}\\
&= \operatorname*{Res}_{\eta=-\frac{v}{2}-mj} s\left(\frac{s_{\mu+1}^2}{-i\alpha_{\mu+1}}+\cdots+\frac{s_{\mu+v}^2}{-i\alpha_{\mu+v}}\right)^{\eta}\frac{(mj)!}{(-1)^{mj}}\\
&= (e^{\frac{\pi i}{2}})^{-\frac{v}{2}-mj}\operatorname*{Res}_{\eta=-\frac{v}{2}-mj} s\left(\frac{s_{\mu+1}^2}{-i\alpha_{\mu+1}}+\cdots+\frac{s_{\mu+v}^2}{-i\alpha_{\mu+v}}\right)^{\eta}\frac{(m)!}{(-1)^{mj}}\\
&= \frac{e^{-\frac{v\pi i}{4}}e^{-\frac{mji}{2}}2\pi^{\frac{v}{2}}e^{-\frac{v\pi i}{4}}e^{\frac{v\pi i}{4}}\sqrt{\alpha_{\mu+1}}\cdots\sqrt{\alpha_{\mu+v}}}{(2m)^{2jm}\Gamma\left(\frac{v}{2}+mj\right)(jm)!}\frac{(mj)!}{(-1)^{mj}}\\
&\qquad\times\left(\frac{1}{\alpha_{\mu+1}}\frac{\partial^2}{\partial s_{\mu+1}^2}+\cdots+\frac{1}{\alpha_{\mu+v}}\frac{\partial^2}{\partial s_{\mu+v}^2}\right)^{jm}\delta. \qquad (2.47)
\end{aligned}
$$

From (2.40) and using (2.47) we have,

$$
\begin{aligned}
&\mathcal{F}\{s^{2mj}\}\left(\frac{1}{\sqrt{-i\alpha_{\mu+1}}}s_{\mu+1}^2+\cdots+\frac{1}{\sqrt{-i\alpha_{\mu+v}}}s_{\mu+v}^2\right)\\
&= \frac{\pi^{\frac{v}{2}}2^{2mj+v}\Gamma(\frac{v}{2}+mj)e^{-\frac{v\pi i}{4}}e^{-\frac{mji}{2}}2\pi^{\frac{v}{2}}\sqrt{\alpha_{\mu+1}}\cdots\sqrt{\alpha_{\mu+v}}(mj)!}{(2m)^{2jm}\Gamma(\frac{v}{2}+mj)(jm)!(-1)^{mj}}\\
&\qquad\times\left(\frac{1}{\alpha_{\mu+1}}\frac{\partial^2}{\partial s_{\mu+1}^2}+\cdots+\frac{1}{\alpha_{\mu+v}}\frac{\partial^2}{\partial s_{\mu+v}^2}\right)^{jm}\delta\\
&= \frac{(2\pi)^{v}e^{-\frac{v\pi i}{4}}e^{-\frac{mji}{2}}2\sqrt{\alpha_{\mu+1}}\cdots\sqrt{\alpha_{\mu+v}}}{m^{2jm}(-1)^{mj}}\left(\frac{1}{\alpha_{\mu+1}}\frac{\partial^2}{\partial s_{\mu+1}^2}+\cdots+\frac{1}{\alpha_{\mu+v}}\frac{\partial^2}{\partial s_{\mu+v}^2}\right)^{jm}\delta
\end{aligned}
\qquad (2.48)
$$

From (2.1) and(2.37) and using (2.45) we obtain the following formula:

$$
\begin{aligned}
&\mathcal{F}\{({}_\mu H_\alpha(x,m) +_v H_\alpha(x,m))^\lambda\} \\
&\quad = \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^\lambda\right\} \\
&\quad = \frac{e^{\frac{m\lambda\pi i}{2}}\pi^{\frac{n}{2}}2^{2m\lambda}2^\mu}{e^{-\frac{\pi i\mu}{4}}.e^{-\frac{\pi i\mu}{4}}\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}\cdots\sqrt{\alpha_{\mu+v}}} \\
&\qquad \times \sum_{j\geq 0}\binom{\lambda}{j}\frac{\Gamma(m\lambda+\frac{\mu}{2}-mj)}{2^{2mj}\Gamma(-m\lambda+mj)}\,|\sigma|^{-\mu-2m\lambda+2mj} \\
&\qquad \times \frac{(2\pi)^v e^{-\frac{v\pi i}{4}}e^{-\frac{mjj}{2}}.2\sqrt{\alpha_{\mu+1}}\cdots\sqrt{\alpha_{\mu+v}}}{m^{2jm}(-1)^{mj}} \\
&\qquad \times \left(\frac{1}{\alpha_{\mu+1}}\frac{\partial^2}{\partial s_{\mu+1}^2}+\cdots+\frac{1}{\alpha_{\mu+v}}\frac{\partial^2}{\partial s_{\mu+v}^2}\right)^{mj}\delta. \qquad (2.49)
\end{aligned}
$$

From (2.20) and (2.49) and using the formula (2.26) we have,

$$
\begin{aligned}
&\mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^\lambda\right\} \\
&= \frac{\pi^{\frac{\mu}{2}}2^{2m\lambda}2^\mu(2\pi)^v}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}\sum_{j\geq 0}\binom{\lambda}{j}\frac{\Gamma(m\lambda+\frac{\mu}{2}-mj)}{2^{2mj}\Gamma(-m\lambda+mj)}\frac{\left(\frac{s_1^2}{\alpha_1}+\cdots+\frac{s_\mu^2}{\alpha_\mu}\right)^{-\mu-2m\lambda+2mj}}{m^{2jm}(-1)^{mj}} \\
&\quad \times \left(\frac{1}{\alpha_{\mu+1}}\frac{\partial^2}{\partial s_{\mu+1}^2}+\cdots+\frac{1}{\alpha_{\mu+v}}\frac{\partial^2}{\partial s_{\mu+v}^2}\right)^{mj}\delta \qquad (2.50)
\end{aligned}
$$

if m$\geq$ 2.

Similarly, considering the formula (2.38),(2.40),(2.50) and the formula

$$(1-z)^\lambda = \sum_{j=0}^{\infty}(-1)^j\binom{\lambda}{j}z^j \qquad (2.51)$$

for $|z| < 1$,where $\binom{\lambda}{j}$ is defined by (2.33) we obtain the following formula

$$\begin{aligned}
&\mathcal{F}\left\{\left(({}_{\mu}H_{\alpha}(x,m) -_{v} H_{\alpha}(x,m))^{\lambda}\right)\right\} \\
&= \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m - \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^{\lambda}\right\} \\
&= \frac{\pi^{\frac{\mu}{2}}2^{2m\lambda}2^{\mu}(2\pi)^v}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}} \cdot \sum_{j\geq 0}\binom{\lambda}{j}\frac{\Gamma(m\lambda+\frac{\mu}{2}-mj)}{2^{2mj}\Gamma(-m\lambda+mj)(-1)^{mj}} \\
&\quad \times \left(\frac{s_1^2}{\alpha_1}+\cdots+\frac{s_\mu^2}{\alpha_\mu}\right)^{-\mu-2m\lambda+2mj}\left(\frac{1}{\alpha_{\mu+1}}\frac{\partial^2}{\partial s_{\mu+1}^2}+\cdots+\frac{1}{\alpha_{\mu+v}}\frac{\partial^2}{\partial s_{\mu+v}^2}\right)^{mj}\delta.
\end{aligned} \tag{2.52}$$

We observe that the formula (2.52) is valid for the case $m = 1$.

On the other hand, considering the elliptic kernel of Marcel Riesz

$$R_a(x) = \frac{|x|^{\alpha-n}}{D_n(\alpha)} \tag{2.53}$$

where

$$D_n(\alpha) = \frac{\Gamma(\frac{n-\alpha}{2})}{2^{\alpha}\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})} \tag{2.54}$$

and the properties

$$\Delta_n^k R_a(x) = R_{\alpha-2k}(x) \tag{2.55}$$

$$R_{-2k} = (-1)^k\Delta_n^k\delta(x_1...x_n) \tag{2.56}$$

$$R_v(x) = \delta(x) \tag{2.57}$$

where

$$\Delta_n^k = \left\{\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\right\}^k = \left\{\frac{\partial^2}{\partial x_i^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right\}^k \tag{2.58}$$

the formulae (2.51) and (2.52) can be rewritten in the following form

$$
\begin{aligned}
&\mathcal{F}\left\{\left(({}_\mu H_\alpha(x,m) +_v H_\alpha(x,m))^\lambda\right)\right\} \\
&= \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^\lambda\right\} \\
&= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}\cdot\sum_{j\geq 0}\binom{\lambda}{j}\ {}_\mu R_{-2m\lambda+2mj}(\sigma) \\
&\quad\times(-1)^{mj}\left(\frac{1}{\alpha_{\mu+1}}\frac{\partial^2}{\partial s_{\mu+1}^2}+\cdots+\frac{1}{\alpha_{\mu+v}}\frac{\partial^2}{\partial s_{\mu+v}^2}\right)^{mj}\delta \\
&= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}\sum_{j\geq 0}\binom{\lambda}{j}\ {}_\mu R_{-2m\lambda+2mj}(\sigma)(-1)^{mj}\ {}_v\Delta_\alpha^{mj}\delta
\end{aligned}
\tag{2.59}
$$

and

$$
\begin{aligned}
&\mathcal{F}\{\left(({}_\mu H_\alpha(x,m) -_v H_\alpha(x,m))^\lambda\right)\} \\
&= \mathcal{F}\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m - \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^\lambda\} \\
&= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}\cdot\sum_{j\geq 0}\binom{\lambda}{j}\ {}_\mu R_{-2m\lambda+2mj}(\sigma) \\
&\quad\times(-1)^{mj}\left(\frac{1}{\alpha_{\mu+1}}\frac{\partial^2}{\partial s_{\mu+1}^2}+\cdots+\frac{1}{\alpha_{\mu+v}}\frac{\partial^2}{\partial s_{\mu+v}^2}\right)^{mj}\delta \\
&= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}\sum_{j\geq 0}\binom{\lambda}{j}\ {}_\mu R_{-2m\lambda+2mj}(\sigma)(-1)^{mj}\ {}_v\Delta_\alpha^{mj}\delta
\end{aligned}
\tag{2.60}
$$

where

$$
{}_\mu R_{-2m\lambda+2mj}(\sigma) = \frac{|\sigma|^{-2m\lambda+2mj-\mu}\,\Gamma(\frac{\mu+2m\lambda-2mj}{2})}{2^{-2m\lambda+2mj}\pi^{\frac{\mu}{2}}\Gamma(\frac{-2m\lambda+2mj}{2})}
\tag{2.61}
$$

$$
|\sigma| = \sqrt{\sigma_1^2+\cdots+\sigma_\mu^2} = \sqrt{\frac{s_1^2}{\alpha_1^2}+\cdots+\frac{s_\mu^2}{\alpha_\mu^2}}
\tag{2.62}
$$

$$
{}_v\Delta_\alpha = \frac{1}{\alpha_{\mu+1}}\frac{\partial^2}{\partial s_{\mu+1}^2}+\cdots+\frac{1}{\alpha_{\mu+v}}\frac{\partial^2}{\partial s_{\mu+v}^2}.
\tag{2.63}
$$

In particular by putting $\lambda = k$ in (2.59), (2.60) and using the properties

(2.56) we have,

$$\begin{aligned}
&\mathcal{F}\left\{\left(({}_\mu H_\alpha(x,m) +_v H_\alpha(x,m))^k\right)\right\}\\
&\quad= \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^k\right\}\\
&\quad= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}.\sum_{j\geq 0}\binom{k}{j}\ {}_\mu R_{-2m(k-j)}(\sigma)_v\Delta_\alpha^{mj}\delta\\
&\quad= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}\sum_{j\geq 0}\binom{k}{j}\ {}_\mu\Delta_\alpha^{m(k-j)}\delta(-1)^{mj}(-1)^{m(k-j)}\ {}_v\Delta_\alpha^{mj}\delta
\end{aligned} \tag{2.64}$$

and

$$\begin{aligned}
&\mathcal{F}\left\{\left(({}_\mu H_\alpha(x,m) -_v H_\alpha(x,m))^k\right)\right\}\\
&\quad= \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m - \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^k\right\}\\
&\quad= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}.\sum_{j\geq 0}(-1)^j\binom{k}{j}\ {}_\mu R_{-2m(k-j)}(\sigma)_v(-1)^{mj}\Delta_\alpha^{mj}\delta\\
&\quad= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}\sum_{j\geq 0}(-1)^j\binom{k}{j}(-1)^{mj}\ {}_\mu\Delta_\alpha^{m(k-j)}\delta\ {}_v\Delta_\alpha^{mj}\delta
\end{aligned} \tag{2.65}$$

where

$${}_\mu\Delta_\alpha = \frac{1}{\alpha_1}\frac{\partial^2}{\partial s_1^2} + \cdots + \frac{1}{\alpha_\mu}\frac{\partial^2}{\partial s_\mu^2}. \tag{2.66}$$

Developing by into binomial series the second factor of the right-hand member of (2.64) and (2.65) we have,

$$\begin{aligned}
&\mathcal{F}\left\{\left(({}_\mu H_\alpha(x,m) +_v H_\alpha(x,m))^k\right)\right\}\\
&\qquad= \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^k\right\}\\
&\qquad= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}.(-1)^{mk}({}_\mu\Delta_\alpha^m\delta +_v \Delta_\alpha^m\delta)^k
\end{aligned} \tag{2.67}$$

if $m \geq 2$, and

$$\mathcal{F}\left\{\left(({}_{\mu}H_{\alpha}(x,m) -_{v} H_{\alpha}(x,m))^{k}\right)\right\}$$
$$= \mathcal{F}\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m - \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^k\}$$
$$= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}.(-1)^{mk}({}_{\mu}\Delta_{\alpha}^{m}\delta -_{v} \Delta_{\alpha}^{m}\delta)^k \qquad (2.68)$$

$m \geq 2$.

By putting $\alpha_1 = \alpha_1 = \cdots = \alpha_\mu = \alpha_{\mu+1} = \cdots = \alpha_{\mu+v} = 1$ in (2.67) and (2.68) and using (1.9) and (1.10) we have,

$$\mathcal{F}\{(M(x,m))^k\} = \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu} x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v} x_j^2\right)^m\right)^k\right\}$$
$$= (2\pi)^n(-1)^{mk}({}_{\mu}\Delta^m\delta +_{v} \Delta^m\delta)^k \qquad (2.69)$$

and

$$\mathcal{F}\{(G(x,m))^k\} = \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu} x_j^2\right)^m - \left(\sum_{j=\mu+1}^{\mu+v} x_j^2\right)^m\right)^k\right\}$$
$$= (2\pi)^n(-1)^{mk}({}_{\mu}\Delta^m\delta -_{v} \Delta^m\delta)^k. \qquad (2.70)$$

By putting

$${}_{\mu,v}B_{\alpha}^{m} =_{\mu} \Delta_{\alpha}^{m} +_{v} \Delta_{\alpha}^{m} \qquad (2.71)$$

and

$${}_{\mu,v}L_{\alpha}^{m} =_{\mu} \Delta_{\alpha}^{m} -_{v} \Delta_{\alpha}^{m} \qquad (2.72)$$

From (2.67), (2.68), (2.69), (2.70), (2.71) and (2.72) we have,

$$\mathcal{F}\left\{\left(({}_{\mu}H_{\alpha}(x,m) +_{v} H_{\alpha}(x,m))^{k}\right)\right\}$$
$$= \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^k\right\}$$
$$= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}.(-1)^{mk}\ {}_{\mu,v}B_{\alpha}^{mk}\delta, \qquad (2.73)$$

$$\mathcal{F}\left\{\left(({}_{\mu}H_{\alpha}(x,m) -_{v} H_{\alpha}(x,m))^{k}\right)\right\}$$
$$= \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m - \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^k\right\}$$
$$= \frac{(2\pi)^n}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}.(-1)^{mk}{}_{\mu,v}L_{\alpha}^{mk}\delta, \tag{2.74}$$

$$\mathcal{F}\{(M(x,m))^k\} = \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v}x_j^2\right)^m\right)^k\right\}$$
$$= (2\pi)^n(-1)^{mk}{}_{\mu,v}B_{\alpha}^{mk}\delta \tag{2.75}$$

and

$$\mathcal{F}\{(G(x,m))^k\} = \mathcal{F}\left\{\left(\left(\sum_{j=1}^{\mu}x_j^2\right)^m - \left(\sum_{j=\mu+1}^{\mu+v}x_j^2\right)^m\right)^k\right\}$$
$$= (2\pi)^n(-1)^{mk}{}_{\mu,v}L_{\alpha}^{mk}\delta \tag{2.76}$$

where

$${}_{\mu,v}B =_{\mu} \Delta^m +_{v} \Delta^m \tag{2.77}$$

and

$${}_{\mu,v}L =_{\mu} \Delta^m -_{v} \Delta^m. \tag{2.78}$$

From (2.73), (2.74), (2.75) and (2.76) we obtain the following formulae

$$\mathcal{F}\left\{\frac{(-1)^{mk}}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}\;{}_{\mu,v}B_{\alpha}^{mk}\delta\right\}$$
$$= \left\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^k\right\}, \tag{2.79}$$

$$\mathcal{F}\left\{\frac{(-1)^{mk}}{\sqrt{\alpha_1}\sqrt{\alpha_2}\cdots\sqrt{\alpha_\mu}}\;{}_{\mu,v}L_{\alpha}^{mk}\delta\right\}$$
$$= \left\{\left(\left(\sum_{j=1}^{\mu}\alpha_j x_j^2\right)^m - \left(\sum_{j=\mu+1}^{\mu+v}\alpha_j x_j^2\right)^m\right)^k\right\}, \tag{2.80}$$

$$\mathcal{F}\{(-1)^{mk}\;{}_{\mu,v}B_{\alpha}^{mk}\delta\} = \left\{\left(\left(\sum_{j=1}^{\mu}x_j^2\right)^m + \left(\sum_{j=\mu+1}^{\mu+v}x_j^2\right)^m\right)^k\right\} \tag{2.81}$$

and

$$\mathcal{F}\{(-1)^{mk} \quad {}_{\mu,v}L_{\alpha}^{mk}\delta\} = \left\{\left(\left(\sum_{j=1}^{\mu} x_j^2\right)^m - \left(\sum_{j=\mu+1}^{\mu+v} x_j^2\right)^m\right)^k\right\}. \tag{2.82}$$

In particular if $k = 1, m = 2$ and $m = 4$ we have,

$$\mathcal{F}\left\{\left[\left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^2 - \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^2\right]\delta\right\} = \left(\sum_{j=1}^{\mu} x_j^2\right)^2 - \left(\sum_{j=\mu+1}^{\mu+v} x_j^2\right)^2 \tag{2.83}$$

and

$$\mathcal{F}\left\{\left[\left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^4 - \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^4\right]\delta\right\} = \left(\sum_{j=1}^{\mu} x_j^2\right)^4 - \left(\sum_{j=\mu+1}^{\mu+v} x_j^2\right)^4 \tag{2.84}$$

Similarly if $m = 2$ and $m = 4$ for k≥ 1 from (2.81) and (2.82) we have,

$$\mathcal{F}\left\{\left[\left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^2 + \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^2\right]\delta\right\} = \left(\sum_{j=1}^{\mu} x_j^2\right)^2 + \left(\sum_{j=\mu+1}^{\mu+v} x_j^2\right)^2, \tag{2.85}$$

$$\mathcal{F}\left\{\left[\left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^4 + \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^4\right]\delta\right\} = \left(\sum_{j=1}^{\mu} x_j^2\right)^4 + \left(\sum_{j=\mu+1}^{\mu+v} x_j^2\right)^4, \tag{2.86}$$

$$\mathcal{F}\left\{\left[\left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^2 - \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k\delta\right\} = \left(\left(\sum_{j=1}^{\mu} x_j^2\right)^2 - \left(\sum_{j=\mu+1}^{\mu+v} x_j^2\right)^2\right)^k \tag{2.87}$$

and

$$\mathcal{F}\left\{\left[\left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^4 - \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^4\right]^k\delta\right\} = \left(\left(\sum_{j=1}^{\mu} x_j^2\right)^4 - \left(\sum_{j=\mu+1}^{\mu+v} x_j^2\right)^4\right)^k. \tag{2.88}$$

By denoting

$$\odot = \left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^2 + \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^2 \tag{2.89}$$

$$\diamond = \left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^2 - \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^2 \tag{2.90}$$

and

$$\oplus = \left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^4 - \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^4 \tag{2.91}$$

from (2.85), (2.87) and (2.88) we have,

$$\mathcal{F}\{\odot^k \delta\} = \left(\left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^2 + \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^2\right)^k, \tag{2.92}$$

$$\mathcal{F}\{\diamond^k \delta\} = \left(\left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^2 - \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^2\right)^k \tag{2.93}$$

and

$$\mathcal{F}\{\oplus^k \delta\} = \left(\left(\sum_{j=1}^{\mu} \frac{\partial^2}{\partial x_j^2}\right)^4 - \left(\sum_{j=\mu+1}^{\mu+v} \frac{\partial^2}{\partial x_j^2}\right)^4\right)^k \tag{2.94}$$

The operator $\odot$ defined by (2.89) was introduce by Satsanit in [1], the operator $\diamond$ defined by (2.90) was introduced by Kananthai [2] and is named the Diamond operator and the operator $\oplus$ defined by (2.91) has been studied in [3] by Kananthai, Suantai and Longani.

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References

[1] W. Satsanit, On the Green function of the operator $(\odot + m^4)^k$ related to the Klein-Gordon operator and Helmholtz operator, International ofPureand Applied Mathematics 57 (1) (2009) 77–86.

[2] A. Kananthai, On the solution of the n-dimensional Diamond operator, Appl. Math. Comp. 88 (1997) 27–37.

[3] A. Kananthai, S. Suantai, V. Longani, On th operator $\oplus^k$ related to the wave equation and Laplacian, Appl. Math. Comp. 132 (2002) 219–229.

[4] M. Aguirre, The Fourier transform of $\left(\left(\sum_{i=1}^{n} a_i x_i^2\right)^m \pm i0\right)^\lambda$, Internat. J. Pure Appl. Math. 64 (3) (2010) 389–397.

[5] A. Kananthai, K. Nonpaopan, On the residue of generalized function P^λ, Thai J. Math. 1 (2003) 49–97.

[6] M.A. Aguirre, A generalization of residue of the generalized function P^λ, Thai J. Math. 3 (2005) 35–51.

[7] M.A. Aguirre, The residue of generalized functions $(M \pm i0)^\lambda$, to appear.

[8] I.S. Gradshetyn, I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, Inc., 1980.

[9] I.M. Gelfand, G.E. Shilov, *Generalized Functions*, Vol.I, Academic Press, New York, 1964.

[10] M.A. Aguirre, A.L. Barrenechea, A relation between the k-th derivative of the Dirac delta in $(P \pm i0)$ and the residue of distribution $(P \pm i0)^\lambda$, J. Comp. Appl. Math. 108 (1999) 31–40.

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