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On a Half-Discrete Mulholland's Inequality and Its Extension[1]

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Abstract : By using the way of weight functions and technique of real analysis, a half-discrete Mulholland's inequality with a best constant factor is given. The extension with multi-parameters, equivalent forms and operator expressions are also considered.

Keywords : Mulholland's inequality; weight function; equivalent form.
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1 Introduction

Assuming that $f, g(\geq 0) \in L^2(0,\infty), ||f|| = \{\int_0^\infty f^2(x)dx\}^{\frac{1}{2}} > 0$, $||g|| > 0$, we have the following Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi ||f|| ||g||, \tag{1.1}$$

where the constant factor π is the best possible. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^2, b = \{b_n\}_{n=1}^\infty \in l^2, ||a|| = \{\sum_{m=1}^\infty a_m^2\}^{\frac{1}{2}} > 0, ||b|| > 0$, then we still have the following discrete Hilbert's inequality with the same best possible constant factor π:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \pi ||a|| ||b||. \tag{1.2}$$

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Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [2–4]). Also we have the following Mulholland's inequality with the same best possible constant factor π (cf. [1, 5]):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn} < \pi\left\{\sum_{m=2}^{\infty} m a_m^2 \sum_{n=2}^{\infty} n b_n^2\right\}^{\frac{1}{2}}. \tag{1.3}$$

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [6] gave an extension of (1.1). Refining the corresponding results from paper [6], Yang [7] gave some best extensions of (1.1) and (1.2) as follows: If $p > 1, \frac{1}{p}+\frac{1}{q} = 1, \lambda, \lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1+\lambda_2 = \lambda, k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, $k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}dt \in R_+$, $\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$, $f(\geq 0) \in L_{p,\phi}(0,\infty) = \{f|||f||_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty\}, g(\geq 0) \in L_{q,\psi}(0,\infty), ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, then we have

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{1.4}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover if $k_\lambda(x,y)$ is also finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing for $x > 0 (y > 0)$, then for $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} = \{a|||a||_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty\}, b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $||a||_{p,\phi}, ||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{1.5}$$

where the constant factor $k(\lambda_1)$ is still the best possible. For $p = q = 2, \lambda = 1, k_1(x,y) = \frac{1}{x+y}, \lambda_1 = \lambda_2 = \frac{1}{2}$, (1.3) reduces to (1.1), and (1.4) reduces to (1.2).

On half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the the constant factors in inequalities are the best possible. In addition, Yang [8] gave a result by introducing an interval variable and proved that the constant factor is the best possible. Recently, Yang [9] gave the following half-discrete Hilbert's inequality with the best possible constant factor $B(\lambda_1, \lambda_2) (\lambda, \lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda)$:

$$\int_0^\infty f(x)\sum_{n=1}^{\infty}\frac{a_n}{(x+n)^\lambda}dx < B(\lambda_1,\lambda_2)||f||_{p,\phi}||a||_{q,\psi}, \tag{1.6}$$

where $B(\cdot,\cdot)$ is the usual Beta function.

In this paper, by using the way of weight functions and technique of real analysis, a half-discrete Mulholland's inequality with a best constant factor is given as follows:

$$\int_1^\infty f(x)\sum_{n=2}^{\infty}\frac{a_n}{\ln xn}dx < \pi\left\{\int_1^\infty x f^2(x)dx \sum_{n=2}^{\infty} n a_n^2\right\}^{\frac{1}{2}}. \tag{1.7}$$

A best extension of (1.7) with multi-parameters as (1.6), some equivalent forms, the operator expressions are considered.

2 Some Lemmas

Lemma 2.1. *Let $\lambda, \lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda$, and let the weight functions $\omega(n)$ and $\varpi(x)$ be define as follows:*

$$\omega(n) := (\ln n)^{\lambda_2} \int_1^{\infty} \frac{1}{x(\ln xn)^{\lambda}} (\ln x)^{\lambda_1 - 1} dx, n \in \mathbf{N}\backslash\{1\}, \tag{2.1}$$

$$\varpi(x) := (\ln x)^{\lambda_1} \sum_{n=2}^{\infty} \frac{1}{n(\ln xn)^{\lambda}} (\ln n)^{\lambda_2 - 1}, x \in [1, \infty). \tag{2.2}$$

Then

$$\varpi(x) < \omega(n) = B(\lambda_1, \lambda_2). \tag{2.3}$$

Proof. Setting $t = \frac{\ln x}{\ln n}$ in (2.1), by calculation, we have

$$\omega(n) = \int_0^{\infty} \frac{1}{(1+t)^{\lambda}} t^{\lambda_1 - 1} dt = B(\lambda_1, \lambda_2).$$

Since for fixed $x \geq 1$, the function

$$h(x, y) := \frac{1}{y(\ln xy)^{\lambda}} (\ln y)^{\lambda_2 - 1} = \frac{1}{y(\ln x + \ln y)^{\lambda} (\ln y)^{1 - \lambda_2}}$$

is strictly decreasing for $y \in (1, \infty)$, it follows that

$$\varpi(x) < (\ln x)^{\lambda_1} \int_1^{\infty} \frac{1}{y(\ln xy)^{\lambda}} (\ln y)^{\lambda_2 - 1} dy$$

$$\overset{t=(\ln y)/(\ln x)}{=} \int_0^{\infty} \frac{t^{\lambda_2 - 1}}{(1+t)^{\lambda}} dt = B(\lambda_2, \lambda_1) = B(\lambda_1, \lambda_2),$$

so (2.3) holds. □

Lemma 2.2. *Let the assumptions of Lemma 2.1 be fulfilled and additionally, let $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n \geq 0, n \in \mathbf{N}\backslash\{1\}, f(x)$ be a non-negative measurable function in $[1, \infty)$. Then we have the following inequalities:*

$$J := \left\{ \sum_{n=2}^{\infty} \frac{(\ln n)^{p\lambda_2 - 1}}{n} \left[\int_1^{\infty} \frac{f(x)}{(\ln xn)^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}}$$

$$\leq [B(\lambda_1, \lambda_2)]^{\frac{1}{q}} \left\{ \int_1^{\infty} \varpi(x) x^{p-1} (\ln x)^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \tag{2.4}$$

$$L_1 := \left\{ \int_1^\infty \frac{(\ln x)^{q\lambda_1 - 1}}{x[\varpi(x)]^{q-1}} \left[\sum_{n=2}^\infty \frac{a_n}{(\ln xn)^\lambda} \right]^q dx \right\}^{\frac{1}{q}}$$
$$\leq \left\{ B(\lambda_1, \lambda_2) \sum_{n=2}^\infty n^{q-1} (\ln n)^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}. \tag{2.5}$$

Proof. (i) By Hölder's inequality with weight (cf. [10]) and (2.3), it follows

$$\left[\int_1^\infty \frac{f(x)}{(\ln xn)^\lambda} dx \right]^p = \left\{ \int_1^\infty \frac{1}{(\ln xn)^\lambda} \left[\frac{(\ln x)^{(1-\lambda_1)/q}}{(\ln n)^{(1-\lambda_2)/p}} \frac{x^{1/q}}{n^{1/p}} f(x) \right] \right.$$
$$\left. \times \left[\frac{(\ln n)^{(1-\lambda_2)/p}}{(\ln x)^{(1-\lambda_1)/q}} \frac{n^{1/p}}{x^{1/q}} \right] dx \right\}^p$$
$$\leq \int_1^\infty \frac{1}{(\ln xn)^\lambda} \frac{x^{p-1} (\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln n)^{1-\lambda_2}} f^p(x) dx$$
$$\times \left\{ \int_1^\infty \frac{1}{(\ln xn)^\lambda} \frac{n^{q-1} (\ln n)^{(1-\lambda_2)(q-1)}}{x(\ln x)^{1-\lambda_1}} dx \right\}^{p-1}$$
$$= \left\{ \omega(n) \frac{(\ln n)^{q(1-\lambda_2)-1}}{n^{1-q}} \right\}^{p-1} \int_1^\infty \frac{x^{p-1} (\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda (\ln n)^{1-\lambda_2}} f^p(x) dx$$
$$= \frac{[B(\lambda_1, \lambda_2)]^{p-1} n}{(\ln n)^{p\lambda_2 - 1}} \int_1^\infty \frac{x^{p-1} (\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda (\ln n)^{1-\lambda_2}} f^p(x) dx.$$

Then, by Lebesgue term by term integration theorem (cf. [11]), we have

$$J \leq [B(\lambda_1, \lambda_2)]^{\frac{1}{q}} \left\{ \sum_{n=2}^\infty \int_1^\infty \frac{x^{p-1} (\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda (\ln n)^{1-\lambda_2}} f^p(x) dx \right\}^{\frac{1}{p}}$$
$$= [B(\lambda_1, \lambda_2)]^{\frac{1}{q}} \left\{ \int_1^\infty \sum_{n=2}^\infty \frac{x^{p-1} (\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda (\ln n)^{1-\lambda_2}} f^p(x) dx \right\}^{\frac{1}{p}}$$
$$= [B(\lambda_1, \lambda_2)]^{\frac{1}{q}} \left\{ \int_1^\infty \varpi(x) x^{p-1} (\ln x)^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}},$$

and (2.4) follows. Another application of Hölder's inequality with weight yields

$$\left[\sum_{n=2}^\infty \frac{a_n}{(\ln xn)^\lambda} \right]^q = \left\{ \sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda} \left[\frac{(\ln x)^{(1-\lambda_1)/q}}{(\ln n)^{(1-\lambda_2)/p}} \frac{x^{1/q}}{n^{1/p}} \right] \left[\frac{(\ln n)^{(1-\lambda_2)/p}}{(\ln x)^{(1-\lambda_1)/q}} \frac{n^{1/p}}{x^{1/q}} a_n \right] \right\}^q$$
$$\leq \left\{ \sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda} \frac{x^{p-1} (\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln n)^{1-\lambda_2}} \right\}^{q-1}$$
$$\times \sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda} \frac{n^{q-1} (\ln n)^{(1-\lambda_2)(q-1)}}{x(\ln x)^{1-\lambda_1}} a_n^q$$

$$= \frac{x[\varpi(x)]^{q-1}}{(\ln x)^{q\lambda_1 - 1}} \sum_{n=2}^{\infty} \frac{1}{(\ln xn)^{\lambda}} \frac{(\ln x)^{\lambda_1 - 1}}{x} n^{q-1} (\ln n)^{(q-1)(1-\lambda_2)} a_n^q.$$

Moreover, by Lebesgue term by term integration theorem, we have

$$\begin{aligned} L_1 &\leq \left\{ \int_1^{\infty} \sum_{n=2}^{\infty} \frac{1}{(\ln xn)^{\lambda}} \frac{(\ln x)^{\lambda_1 - 1}}{x} n^{q-1} (\ln n)^{(q-1)(1-\lambda_2)} a_n^q dx \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{n=2}^{\infty} \left[(\ln n)^{\lambda_2} \int_1^{\infty} \frac{(\ln x)^{\lambda_1 - 1}}{x (\ln xn)^{\lambda}} dx \right] n^{q-1} (\ln n)^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{n=2}^{\infty} \omega(n) n^{q-1} (\ln n)^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}, \end{aligned}$$

and then, in view of (2.3), inequality (2.5) follows. □

3 Main Results

In the following, we set two functions $\Phi(x)$ and $\Psi(n)$ as:

$$\Phi(x) := x^{p-1} (\ln x)^{p(1-\lambda_1)-1} (x \in (1, \infty)),$$

$$\Psi(n) := n^{q-1} (\ln n)^{q(1-\lambda_2)-1} (n \in \mathbf{N} \backslash \{1\}),$$

wherefrom $[\Phi(x)]^{1-q} = \frac{(\ln x)^{q\lambda_1 - 1}}{x}$, $[\Psi(n)]^{1-p} = \frac{(\ln n)^{p\lambda_2 - 1}}{n}$, and consider the following main result:

Theorem 3.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda$, $f(x), a_n \geq 0$, $f \in L_{p,\Phi}(1, \infty), a = \{a_n\}_{n=2}^{\infty} \in l_{q,\Psi}$, $||f||_{p,\Phi} > 0$ *and* $||a||_{q,\Psi} > 0$, *then we have the following equivalent inequalities:*

$$\begin{aligned} I &:= \sum_{n=2}^{\infty} a_n \int_1^{\infty} \frac{f(x)}{(\ln xn)^{\lambda}} dx \\ &= \int_1^{\infty} f(x) \sum_{n=2}^{\infty} \frac{a_n}{(\ln xn)^{\lambda}} dx < B(\lambda_1, \lambda_2) ||f||_{p,\Phi} ||a||_{q,\Psi}, \end{aligned} \tag{3.1}$$

$$J = \left\{ \sum_{n=2}^{\infty} \frac{(\ln n)^{p\lambda_2 - 1}}{n} \left[\int_1^{\infty} \frac{f(x)}{(\ln xn)^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}} < B(\lambda_1, \lambda_2) ||f||_{p,\Phi}, \tag{3.2}$$

$$L := \left\{ \int_1^{\infty} \frac{(\ln x)^{q\lambda_1 - 1}}{x} \left[\sum_{n=2}^{\infty} \frac{a_n}{(\ln xn)^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} < B(\lambda_1, \lambda_2) ||a||_{q,\Psi}, \tag{3.3}$$

where the same constant factor $B(\lambda_1, \lambda_2)$ *in the above inequalities is the best possible.*

Proof. By Lebesgue term by term integration theorem, there are two expressions for I in (3.1). In view of (2.4), for $\varpi(x) < B(\lambda_1, \lambda_2)$, we have (3.2).

By Hölder's inequality, we have

$$I = \sum_{n=2}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_1^{\infty} \frac{1}{(\ln xn)^{\lambda}} f(x)dx \right] [\Psi^{\frac{1}{q}}(n) a_n] \leq J||a||_{q,\Psi}. \tag{3.4}$$

Then by (3.2), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

$$a_n := [\Psi(n)]^{1-p} \left[\int_1^{\infty} \frac{1}{(\ln xn)^{\lambda}} f(x)dx \right]^{p-1}, \, n \in \mathbf{N}\backslash\{1\},$$

we have $J^{p-1} = ||a||_{q,\Psi}$. By (2.4), we find $J < \infty$. If $J = 0$, then (3.2) is naturally valid; if $J > 0$, then by (3.1), we have

$$||a||_{q,\Psi}^q = J^p = I < B(\lambda_1, \lambda_2)||f||_{p,\Phi}||a||_{q,\Psi},$$
$$||a||_{q,\Psi}^{q-1} = J < B(\lambda_1, \lambda_2)||f||_{p,\Phi},$$

and we have (3.2), which is equivalent to (3.1).

In view of (2.5), since $[\varpi(x)]^{1-q} > [B(\lambda_1, \lambda_2)]^{1-q}$, we have (3.3). By Hölder's inequality, we find

$$I = \int_1^{\infty} [\Phi^{\frac{1}{p}}(x) f(x)] \left[\Phi^{\frac{-1}{p}}(x) \sum_{n=2}^{\infty} \frac{1}{(\ln xn)^{\lambda}} a_n \right] dx \leq ||f||_{p,\Phi} L. \tag{3.5}$$

Then by (3.3), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

$$f(x) := [\Phi(x)]^{1-q} \left[\sum_{n=2}^{\infty} \frac{1}{(\ln xn)^{\lambda}} a_n \right]^{q-1}, x \in (1, \infty),$$

we have $L^{q-1} = ||f||_{p,\Phi}$. By (2.5), it follows that $L < \infty$. If $L = 0$, then (3.3) is naturally valid; if $L > 0$, then by (3.1), we have

$$||f||_{p,\Phi}^p = L^q = I < B(\lambda_1, \lambda_2)||f||_{p,\Phi}||a||_{q,\Psi},$$
$$||f||_{p,\Phi}^{p-1} = L < B(\lambda_1, \lambda_2)||a||_{q,\Psi},$$

and we have (3.3) which is equivalent to (3.1).

Hence inequalities (3.1), (3.2) and (3.3) are equivalent.

For $0 < \varepsilon < p\lambda_1$, setting $\widetilde{f}(x) = 0, x \in (1, e); \widetilde{f}(x) = \frac{1}{x}(\ln x)^{\lambda_1 - \frac{\varepsilon}{p} - 1}, x \in [e, \infty)$, and $\widetilde{a}_n = \frac{1}{n}(\ln n)^{\lambda_2 - \frac{\varepsilon}{q} - 1}, n \in \mathbf{N}\backslash\{1\}$, if there exists a positive number $k(\leq B(\lambda_1, \lambda_2))$, such that (3.1) is valid as we replace $B(\lambda_1, \lambda_2)$ by k, then in particular, it follows

$$\widetilde{I} := \sum_{n=2}^{\infty} \int_1^{\infty} \frac{1}{(\ln xn)^{\lambda}} \widetilde{a}_n \widetilde{f}(x)dx < k||\widetilde{f}||_{p,\Phi}||\widetilde{a}||_{q,\Psi}$$
$$= k \left\{ \int_e^{\infty} \frac{dx}{x(\ln x)^{\varepsilon+1}} \right\}^{\frac{1}{p}} \left\{ \frac{1}{2(\ln 2)^{\varepsilon+1}} + \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^{\varepsilon+1}} \right\}^{\frac{1}{q}}$$

$$< k(\frac{1}{\varepsilon})^{\frac{1}{p}} \left\{ \frac{1}{2(\ln 2)^{\varepsilon+1}} + \int_2^\infty \frac{1}{x(\ln x)^{\varepsilon+1}} dx \right\}^{\frac{1}{q}}$$
$$= \frac{k}{\varepsilon} \left\{ \frac{\varepsilon}{2(\ln 2)^{\varepsilon+1}} + \frac{1}{(\ln 2)^{\varepsilon}} \right\}^{\frac{1}{q}}, \tag{3.6}$$

$$\widetilde{I} = \sum_{n=2}^\infty (\ln n)^{\lambda_2 - \frac{\varepsilon}{q} - 1} \frac{1}{n} \int_e^\infty \frac{1}{x(\ln xn)^\lambda} (\ln x)^{\lambda_1 - \frac{\varepsilon}{p} - 1} dx$$
$$\overset{t=(\ln x)/(\ln n)}{=} \sum_{n=2}^\infty \frac{1}{n(\ln n)^{\varepsilon+1}} \int_{1/\ln n}^\infty \frac{1}{(t+1)^\lambda} t^{\lambda_1 - \frac{\varepsilon}{p} - 1} dt$$
$$= B(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}) \sum_{n=2}^\infty \frac{1}{n(\ln n)^{\varepsilon+1}} - A(\varepsilon)$$
$$> B(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}) \int_e^\infty \frac{1}{y(\ln y)^{\varepsilon+1}} dy - A(\varepsilon)$$
$$= \frac{1}{\varepsilon} B(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}) - A(\varepsilon),$$
$$A(\varepsilon) := \sum_{n=2}^\infty \frac{1}{n(\ln n)^{\varepsilon+1}} \int_0^{1/\ln n} \frac{1}{(t+1)^\lambda} t^{\lambda_1 - \frac{\varepsilon}{p} - 1} dt. \tag{3.7}$$

We find

$$0 < A(\varepsilon) \le \sum_{n=2}^\infty \frac{1}{n(\ln n)^{\varepsilon+1}} \int_0^{1/\ln n} t^{\lambda_1 - \frac{\varepsilon}{p} - 1} dt$$
$$= \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} \sum_{n=2}^\infty \frac{1}{n(\ln n)^{\lambda_1 + \frac{\varepsilon}{q} + 1}} < \infty,$$

and then $A(\varepsilon) = O(1)(\varepsilon \to 0^+)$. Hence by (3.6) and (3.7), it follows

$$B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right) - \varepsilon O(1) < k \left\{ \frac{\varepsilon}{2(\ln 2)^{\varepsilon+1}} + \frac{1}{(\ln 2)^\varepsilon} \right\}^{\frac{1}{q}}, \tag{3.8}$$

and $B(\lambda_1, \lambda_2) \le k(\varepsilon \to 0^+)$. Hence $k = B(\lambda_1, \lambda_2)$ is the best value of (3.1).

We confirm that the constant factor $B(\lambda_1, \lambda_2)$ in (3.2) ((3.3)) is the best possible. Otherwise we can came to a contradiction by (3.4) ((3.5)) that the constant factor in (3.1) is not the best possible. □

Remark 3.2.

(i) Define the first type half-discrete Mulholland's operator $T : L_{p,\Phi}(1, \infty) \to l_{p,\Psi^{1-p}}$ *as: For* $f \in L_{p,\Phi}(1, \infty)$, *there exists an unified representation* $Tf \in l_{p,\Psi^{1-p}}$, *satisfying*

$$Tf(n) = \int_1^\infty \frac{1}{(\ln xn)^\lambda} f(x) dx, n \in \mathbf{N} \backslash \{1\}.$$

Then by (3.2), it follows

$$||Tf||_{p.\Psi^{1-p}} \leq B(\lambda_1, \lambda_2)||f||_{p,\Phi},$$

and then T is bounded with $||T|| \leq B(\lambda_1, \lambda_2)$. Since by Theorem 1, the constant factor in (3.2) is the best possible, we have $||T|| = B(\lambda_1, \lambda_2)$.

(ii) Define the second type half-discrete Mulholland's operator $\widetilde{T} : l_{q,\Psi} \rightarrow L_{q,\Phi^{1-q}}(1,\infty)$ as: for $a \in l_{q,\Psi}$, there exists an unified representation $\widetilde{T}a \in L_{q,\Phi^{1-q}}(1,\infty)$, satisfying

$$\widetilde{T}a(x) = \sum_{n=2}^{\infty} \frac{1}{(\ln xn)^{\lambda}} a_n, x \in (1,\infty).$$

Then by (3.3), it follows

$$||\widetilde{T}a||_{q.\Phi^{1-q}} \leq B(\lambda_1, \lambda_2)||a||_{q,\Psi},$$

and then $\widetilde{T}$ is bounded with $||\widetilde{T}|| \leq B(\lambda_1, \lambda_2)$. Since by Theorem 1, the constant factor in (3.3) is the best possible, we have $||\widetilde{T}|| = B(\lambda_1, \lambda_2)$.

Remark 3.3. *For $p = q = 2, \lambda = 1, \lambda_1 = \lambda_2 = \frac{1}{2}$ in (3.1), (3.2) and (3.3), we deduce (1.7) and the following equivalent inequalities:*

$$\left\{\sum_{n=2}^{\infty} \frac{1}{n} \left[\int_1^{\infty} \frac{f(x)}{\ln xn} dx\right]^2\right\}^{\frac{1}{2}} < \pi \left\{\int_1^{\infty} x f^2(x) dx\right\}^{\frac{1}{2}}, \tag{3.9}$$

$$\left\{\int_1^{\infty} \frac{1}{x} \left[\sum_{n=2}^{\infty} \frac{a_n}{\ln xn}\right]^2 dx\right\}^{\frac{1}{2}} < \pi \left\{\sum_{n=2}^{\infty} n a_n^2\right\}^{\frac{1}{2}}. \tag{3.10}$$

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