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Generalization of Fixed Point Results of Contractive Maps by Altering Distances

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Abstract : We introduce a contractive map involving an altering distance function. The class of such maps need not be continuous and it is larger than the class of contractive maps. We prove the existence of fixed points of contractive maps involving an altering distance fuction in complete metric spaces. Our results generalize the results of Geraghty [M.A. Geraghty, On contractive maps, Proc. Amer. Math. Soc. 40 (1973) 604–608] and improves a result of Sastry et al. [K.P.R. Sastry, G.V.R. Babu, M.V.R. Kameswari, Fixed points of strip φ-contractions, Mathematical Communications 14 (2) (2009) 183–192]. Examples are provided in support of our results.

Keywords : orbitally complete metric space; orbitally continuous map; altering distance function; strip φ-contraction; fixed point.
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1 Introduction

The study of the existence of fixed points of contractive maps in complete metric spaces by using an altering distance function is of present interest. Through

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out this paper $(X,\ d)$ denotes a metric space, we write it simply by X; $\mathbb{N}$, the set of all natural numbers and R^+, the set of all non-negative real numbers.

A mapping $T : X \to X$ is said to be *contractive* if $d(Tx,\ Ty) < d(x,\ y)$ for all $x,\ y$ in X with $x \neq y$. Clearly every contractive map is continuous. Also, if such a map has a fixed point, then it is unique.

Edelstein [1] proved the following theorem for contractive maps.

Theorem 1.1 (Edelstein [1]). *Let X be a metric space and T be a contractive mapping into itself such that there exists a point $x \in X$ whose sequence of iterates $\{T^n x\}$ contains a convergent subsequence $\{T^{n_i}(x)\}$, then $x^* = \lim_{i\to\infty}\ T^{n_i}(x)$ in X is a unique fixed point of T in X.*

Let $T : X \to X$. For $x \in X$, we write $O_T(x) = \{x, Tx, T^2x, ...\}$, *the orbit of x* with respective to T.

In 1980, Park [2] generalized Theorem 1.1 in the following.

Theorem 1.2 (Park [2]). *Let T be a selfmap of a metric space X. Assume that for some integer m, there exists a point $x_0 \in X$ such that $O_{T^m}(x_0)$ has a cluster point z in X and $d(T^m x, T^m y) < d(x, y)$ for all $x, y \in X,\ x \neq y$. Then z is a unique fixed point of T in X.*

In complete metric spaces, Geraghty [3] established a criteria for the sequence of Picard iterates defined by $x_n = Tx_{n-1},\ x_0 \in X$ to be Cauchy for contractive mappings. If it is Cauchy, it is easy to see that it converges to a unique fixed point of T in X.

Theorem 1.3 (Geraghty [3]). (*Geraghty's theorem*) *Let X be a complete metric space. Let $f : X \to X$ with*

$$d(fx,\ fy) < d(x, y) \ \text{ for all } x, y \in X, x \neq y. \tag{1.1}$$

Let $x_0 \in X$ and set $x_n = f(x_{n-1})$ for $n > 0$. Then $x_n \to x_\infty$ in X, with x_∞ is a unique fixed point of f if and only if for any two subsequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ with $x_{h(n)} \neq x_{k(n)}$, we have that $\Delta_n \to 1$ only if $d_n \to 0$; where $\Delta_n = \frac{(d(fx_{h(n)},\ fx_{k(n)}))}{d_n}$ and $d_n = (d(x_{h(n)}, x_{k(n)}))$.

We use the following notation as mentioned in [3].

$$S = \{\alpha : (0, \infty) \to [0, 1)/\alpha(t_n) \to 1 \ \text{ implies } \ t_n \to 0\}.$$

Theorem 1.4 (Geraghty [3]). *Let X be a complete metric space and let $f : X \to X$ be a contractive map. Let $x_0 \in X$ and set $x_n = f(x_{n-1})$ for $n > 0$. Then $x_n \to x_\infty$, where x_∞ is a unique fixed point of f in X if and only if there exists an α in S such that for all $n,\ m$ in $\mathbb{N}$*

$$d(f(x_n),\ f(x_m)) \leq \alpha(d(x_n,\ x_m))d(x_n,\ x_m).$$

Definition 1.5 (Turkoglu et al. [4]). A metric space X is said to be $T-$ *orbitally complete* if every Cauchy sequence which is contained in $O_T(x)$ for all x in X converges to a point of X.

Here we note that every complete metric space is T-orbitally complete for any T, but a T-orbitally complete metric space need not be a complete metric space.

Definition 1.6 (Turkoglu et al. [4]). A mapping $T : X \to X$ is said to be *orbitally continuous* at a point z in X with respective to x in X if for any sequence $\{x_n\} \subset O_T(x)$ with $x_n \to z$ as $n \to \infty$, implies $Tx_n \to Tz$ as $n \to \infty$.

Here we note that any continuous selfmap of a metric space is orbitally continuous, but its converse need not be true. For more details on T-orbitally complete metric spaces and orbitally continuous maps, we refer Turkoglu et al. [4].

Definition 1.7 (Khan et al. [5]). A function $\varphi : R^+ \to R^+$ is said to be an *altering distance function* if

(i) φ is continuous,

(ii) $\varphi(t) = 0$ if and only if $t = 0$.

We denote the class of all altering distance functions $\varphi : R^+ \to R^+$ by Φ. For more literature on the existence of fixed points of different contraction conditions involving altering distance functions, we refer [5–15].

In 1999, Sastry and Babu [13] established the following fixed point theorem in metric spaces.

Theorem 1.8 (Sastry and Babu [13]). *Let T be a self map on a metric space $(X,\ d)$. Suppose that there exists a point $x_0 \in X$ such that the orbit $O_T(x_0) = \{T^n x_0 : n = 0, 1, 2, ...\}$ has a cluster point z in X. If T is orbitally continuous at z and Tz and there exists a φ in Φ such that $\varphi(d(Tx,\ Ty)) < \varphi(d(x, y))$ for each $x, y = Tx \in \overline{O_T(x_0)}$, $x \neq y$, then z is a fixed point of T.*

In 2009, Sastry et al. [14] introduced strip φ-contraction, where φ is an altering distance function and established some fixed point theorems for strip φ-contractions.

Definition 1.9 (Sastry et al. [14]). Let $(X,\ d)$ be a metric space and T be a selfmap on X, let φ in Φ. We say that T is a strip φ-contraction if for a given $\epsilon > 0$, there is a $\delta > 0$, such that

$$\epsilon \leq \varphi(d(x,\ y)) < \epsilon + \delta \text{ implies } \varphi(d(Tx,\ Ty)) < \epsilon \text{ for all } x, y \in X.$$

Theorem 1.10 (Sastry et al. [14]). *Let $(X,\ d)$ be a metric space and T be a selfmap on X. Suppose that for some $x_0 \in X$, $O_T(x_0)$ has a cluster point z in X. Further, assume that given $\epsilon > 0$, there is a φ in Φ and $\delta > 0$, such that*

$$\epsilon \leq \varphi(d(x,\ y)) < \epsilon + \delta \ \textit{implies} \ \varphi(d(Tx,\ Ty)) < \epsilon \tag{1.2}$$

for all x, y in $\overline{O_T(x_0)}, x \neq y,\ y = Tx$. Then z is a fixed point of T in $\overline{O_T(x_0)}$.

We now introduce a contractive map involving an altering distance function.

Definition 1.11. Let $(X,\ d)$ be a metric space and T be a selfmap on X. If there exists a $\varphi \in \Phi$ such that $\varphi(d(Tx, Ty)) < \varphi(d(x, y))$ for all $x, y \in X$, $x \neq y$, then we say that T is a contractive map with respect to an altering distance function φ.

Here we note that the class of all contractive maps with respect to an altering distance function is larger than the class all contractive maps (Example 2.3 of [13]). Also, every contractive map with respect to an altering distance function need not be continuous (Example 2.3).

In Section 2 of this paper, we prove the existence of fixed points of contractive maps in complete metric spaces by using an altering distance function. Our results generalize and improve some of the known results.

2 Main Results

Theorem 2.1. *Let T be a selfmap on a T-orbitally complete metric space X. Let $x_0 \in X$. Assume that there exists a $\varphi \in \Phi$ such that*

$$\varphi(d(Tx,\ Ty)) < \varphi(d(x,\ y)) \text{ for all } x, y = Tx \in \overline{O_T(x_0)}, x \neq y. \tag{2.1}$$

We define $\{x_n\}_{n=1}^{\infty}$ by

$$x_n = T(x_{n-1}) \text{ for } n = 1, 2, 3, \ldots. \tag{2.2}$$

Then $x_n \to z$ with z is a unique fixed point of T in $\overline{O_T(x_0)}$ if and only if for any two subsequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ with $x_{h(n)} \neq x_{k(n)}$, we have that: $\Delta_n \to 1$ implies $d_n \to 0$, provided T is orbitally continuous at z; where

$$\Delta_n = \frac{\varphi(d(Tx_{h(n)},\ Tx_{k(n)}))}{\varphi(d_n)} \text{ and } d_n = d(x_{h(n)}, x_{k(n)}).$$

Proof. First we assume that $x_n \to z$ and z is a unique fixed point of T. Let $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ be any subsequences with $x_{h(n)} \neq x_{k(n)}$. Suppose that $\Delta_n \to 1$ as $n \to \infty$. Now,

$$\begin{aligned}\varphi(d(x_{h(n)}, x_{k(n)})) &= \frac{\varphi(d(x_{h(n)},\ x_{k(n)}))}{\varphi(d(Tx_{h(n)},\ Tx_{k(n)}))}\varphi(d(Tx_{h(n)},\ Tx_{k(n)})) \\ &= \frac{1}{\Delta_n}\varphi(d(Tx_{h(n)},\ Tx_{k(n)})).\end{aligned}$$

On letting $n \to \infty$, we have

$$\lim_{n\to\infty} \varphi(d(x_{h(n)},\ x_{k(n)})) = \lim_{n\to\infty} \frac{1}{\Delta_n}\varphi(d(Tx_{h(n)},\ Tx_{k(n)})).$$

Since T is orbitally continuous at z and $\Delta_n \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \varphi(d(x_{h(n)}, x_{k(n)})) = 0.$$

Since φ is continuous, it follows that $\varphi(\lim_{n\to\infty}(d(x_{h(n)}, x_{k(n)}))) = 0$. Now, from the property (ii) of φ, $\lim_{n\to\infty}(d(x_{h(n)}, x_{k(n)})) = 0$. Hence $\lim\limits_{n\to\infty} d_n = 0$.

Conversely, assume that $\Delta_n \to 1$ implies $d_n \to 0$ as $n \to \infty$. For $x_0 \in X$, we now consider the sequence $\{x_n\}$ defined by (2.2). If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then the conclusion of the theorem trivially holds. Suppose that $x_n \neq x_{n+1}$ for all n. Consider,

$$\begin{aligned} \varphi(d(x_n, x_{n+1})) &= \varphi(d(Tx_{n-1},\ Tx_n)) \\ &< \varphi(d(x_{n-1},\ x_n)) \text{ for all } n. \end{aligned}$$

So $\{\varphi(d(x_n, x_{n+1}))\}$ is a decreasing sequence of non-negative real numbers. Suppose that $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = r$, $r \geq 0$. Now we prove that $r = 0$. Assume that $r > 0$. By choosing $h_n = n$ and $k_n = n+1$, we have $\Delta_n = \frac{\varphi(d(x_n,\ x_{n+1}))}{\varphi(d(x_{n-1},\ x_n))} \to 1$ as $n \to \infty$. Hence, by our assumption, $d_n \to 0$ as $n \to \infty$ so that

$$\lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0. \tag{2.3}$$

We now show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not Cauchy, then there exists some $\epsilon > 0$, we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d(x_{n(k)},\ x_{m(k)}) \geq \epsilon. \tag{2.4}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.4), then

$$d(x_{n(k)},\ x_{m(k)}) \geq \epsilon \text{ and } d(x_{n(k)-1}, x_{m(k)}) < \epsilon. \tag{2.5}$$

Now,

$$\begin{aligned} \epsilon &\leq d(x_{n(k)}, x_{m(k)}) \\ &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\ &< d(x_{n(k)}, x_{n(k)-1}) + \epsilon. \end{aligned} \tag{2.6}$$

On taking limit superior as $k \to \infty$, we get

$$\begin{aligned} \epsilon &\leq \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)}) \\ &\leq \limsup_{k\to\infty} (\epsilon + d(x_{n(k)}, x_{n(k)-1})) = \epsilon. \end{aligned}$$

Hence

$$\limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \epsilon. \tag{2.7}$$

Again, on taking limit inferior as $k \to \infty$ in (2.6) and from (2.7), we have

$$\epsilon \le \liminf_{k\to\infty} d(x_{n(k)}, x_{m(k)}) \le \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)})) = \epsilon.$$

Hence

$$\liminf_{k\to\infty} d(x_{n(k)}, x_{m(k)})) = \epsilon. \tag{2.8}$$

From (2.7) and (2.8), it follows that $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}))$ exists and

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)})) = \epsilon. \tag{2.9}$$

Now, from the triangular inequality,

$$d(x_{n(k)}, x_{m(k)}) \le d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)}). \tag{2.10}$$

On taking limit superior both sides, it follows that

$$\epsilon \le \limsup_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1})). \tag{2.11}$$

Again,

$$d(x_{n(k)+1}, x_{m(k)+1}) \le d(x_{n(k)+1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1}). \tag{2.12}$$

On taking limit superior $k \to \infty$ on both sides, we get

$$\limsup_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1})) \le \epsilon. \tag{2.13}$$

From (2.11) and (2.13) it follows that

$$\limsup_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1})) = \epsilon. \tag{2.14}$$

From (2.10) we write, $d(x_{n(k)+1}, x_{m(k)+1}) \ge d(x_{n(k)}, x_{m(k)}) - d(x_{n(k)}, x_{n(k)+1}) - d(x_{m(k)+1}, x_{m(k)})$. On taking limit inferior as $k \to \infty$ on both sides we get

$$\begin{aligned}
\liminf_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) & \\
&\ge \liminf_{k\to\infty}[d(x_{n(k)}, x_{m(k)})] + \liminf_{k\to\infty}[-d(x_{n(k)}, x_{n(k)+1})] \\
&\quad + \liminf_{k\to\infty}[-d(x_{m(k)+1}, x_{m(k)})] \\
&= \epsilon - \limsup_{k\to\infty}[d(x_{n(k)}, x_{n(k)+1})] - \limsup_{k\to\infty}[d(x_{m(k)+1}, x_{m(k)})] \\
&= \epsilon.
\end{aligned}$$

Now, $\epsilon \le \liminf_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) \le \limsup_{k\to\infty} \ d(x_{n(k)+1}, x_{m(k)+1})) = \epsilon$, hence

$$\liminf_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon. \tag{2.15}$$

From (2.14) and (2.15) it follows that $\lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1})$ exists and

$$\lim_{k\to\infty} d(x_{n(k)+1},\ x_{m(k)+1}) = \epsilon. \tag{2.16}$$

Now, by using (2.9) and (2.16), we get

$$\begin{aligned}\lim_{k\to\infty} \Delta_k &= \lim_{k\to\infty} \frac{\varphi(d(x_{m(k)+1},\ x_{n(k)+1}))}{\varphi(d(x_{m(k)},\ x_{n(k)}))}\\ &= \frac{\varphi\left(\lim_{k\to\infty} d(x_{m(k)+1},\ x_{n(k)+1})\right)}{\varphi\left(\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)})\right)} = 1.\end{aligned}$$

Now, by our assumption we have $d_k \to 0$ as $k \to \infty$. *i.e.,* $\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)}) = 0$, a contradiction to (2.9). This proves that $\{x_n\}$ is a Cauchy sequence. Since X is T-orbitally complete, there exists z in X such that $x_n \to z$ as $n \to \infty$. Now, we prove that $T(z) = z$. We consider

$$\varphi(d(x_n,\ Tx_n)) = \varphi(d(x_n, x_{n+1})) < \varphi(d(x_{n-1}, x_n)).$$

Now, on letting $n \to \infty$, we have $\varphi(d(z,\ Tz)) \le \varphi(d(z, z)) = 0$, since T is orbitally continuous at z. Hence $\varphi(d(z, Tz)) = 0$. Therefore, it follows that $z = Tz$. □

Remark 2.2. *Geraghty's theorem* (*Theorem* 1.3) *follows as a corollary to Theorem* 2.1 *by choosing* $\varphi(t) = t,\ t \ge 0$ *in Theorem* 2.1.

Example 2.3. *Let* $X = [2, 9)$ *with the usual metric. We define* $T : X \to X$ *by*

$$T(x) = \begin{cases} x^2 & if\ x \in [2,\ 3)\\ 2x + \frac{1}{2} & if\ x \in [3,\ 4]\\ 8 & if\ x \in (4,\ 8]\\ 2^3 + \frac{1}{3} & if\ x \in [2^3 + \frac{1}{2},\ 9)\\ 2^3 + \frac{1}{4} & if\ x \in [2^3 + \frac{1}{3},\ 2^3 + \frac{1}{4})\\ 2^3 + \frac{1}{n+2} & if\ x \in [2^3 + \frac{1}{n+1},\ 2^3 + \frac{1}{n+2}), n = 1, 2, 3, \ldots. \end{cases}$$

Clearly, X *is* T*-orbitally complete and* T *is orbitally continuous at* 2^3*. For* $x_0 = 2$, *we define* $x_{n+1} = Tx_n$ *for* $n = 0, 1, 2, \ldots.$ *Then* $O_T(x_0) = \{2,\ 2^2,\ 2^3 + \frac{1}{2},\ 2^3 + \frac{1}{3},\ 2^3 + \frac{1}{4}, \ldots, 2^3 + \frac{1}{n+1}, \ldots\}$ *and* $\overline{O_T(x_0)} = O_T(x_0) \cup \{8\}$*. We define* $\varphi : [0,\ \infty) \to [0,\ \infty)$ *by*

$$\varphi(t) = \begin{cases} t^2 & if\ 0 \le t \le 1\\ \frac{1}{t^2} & if\ t \ge 1. \end{cases}$$

Clearly $\varphi \in \Phi$. *Now*

$$\varphi(d(Tx_0,\ Tx_1)) = \frac{4}{81} < \frac{1}{4} = \varphi(d(x_0,\ x_1))$$

$$\varphi(d(Tx_1,\ Tx_2)) = \frac{1}{36} < \frac{4}{81} = \varphi(d(x_1,\ x_2)), \ldots.$$

In general

$$\varphi(d(Tx_n,\ Tx_{n+1})) = \frac{1}{(n+1)^2(n+2)^2} < \frac{1}{n^2(n+1)^2} = \varphi(d(x_n,\ x_{n+1}))$$

for $n = 0, 1, 2, \ldots$. *And, let* $\{x_{h(n)}\}$ *and* $\{x_{k(n)}\}$ *be any two subsequences of* $\{x_n\}$ *with* $x_{h(n)} \neq x_{k(n)}$. *Now,*

$$\Delta_n = \frac{\varphi(d(Tx_{h(n)},\ Tx_{k(n)}))}{\varphi(d(x_{h(n)},\ x_{k(n)}))} = \frac{\varphi(|2^3 + \frac{1}{h_n+1} - 2^3 - \frac{1}{k_n+1}|)}{\varphi(|2^3 + \frac{1}{h_n} - 2^3 - \frac{1}{k_n}|)}$$
$$= \left(\frac{(k_n - h_n)^2}{(k_n+1)^2(h_n+1)^2}\right)\left(\frac{k_n h_n}{k_n - h_n}\right) \to 1 \ \text{ as } n \to \infty.$$

And $d_n = (d(x_{h(n)},\ x_{k(n)})) = (|\frac{1}{h(n)} - \frac{1}{k(n)}|) \to 0$ *as* $n \to \infty$. *Hence* T *satisfies all the hypotheses of Theorem* 2.1 *and* 2^3 *is the unique fixed point of* T. *But, for* $x_0 = 2$ *and* $x_1 = 2^2$, $|Tx_0 - Tx_1| = \frac{9}{2} \nless 2 = |x_0 - x_1|$ *so that the inequality* (2.1) *fails to hold when* φ *is the identity mapping on* R^+. *Hence* T *is not a contractive map and Geraghty's Theorem is not applicable. Thus Theorem* 2.1 *is a generalization of Theorem* 1.3.

Remark 2.4. *If we relax the condition* T *is orbitally continuous at* z *from the hypotheses of Theorem* 2.1, *the conclusion of Theorem* 2.1 *fails to hold.*

Example 2.5. *Let* $X = \{0, 2\} \cup \{\frac{1}{n}/n = 1, 2, 3, \ldots\}$ *with the usual metric. For* $x_0 = 1$, *we define* T *on* X *by* $T(1) = 2$, $T(2) = \frac{1}{2}$, $T(\frac{1}{2}) = \frac{1}{3}$, $T(\frac{1}{3}) = \frac{1}{4}, \ldots,\ T(\frac{1}{n}) = \frac{1}{n+1}$ *for* $n = 2, 3, 4, \ldots$ *and* $T(0) = 0$. *Here* $O_T(1) = \{1,\ 2,\ \frac{1}{2},\ \frac{1}{3}, \ldots\}$ *and* $\overline{O_T(1)} = O_T(1) \cup \{0\}$. *Clearly* T *is orbitally continuous at* 0. *We define* φ *in* Φ *as in Example* 2.3. *Now*

$$\varphi(d(Tx_0,\ Tx_1)) = \varphi(|\frac{1}{2} - 2|) = \frac{4}{9} < 1 = \varphi(|2 - 1|) = \varphi(d(x_0,\ x_1)).$$

$$\varphi(d(Tx_1,\ Tx_2)) = \frac{1}{36} < \frac{4}{9} = \varphi(d(x_1,\ x_2)), \ldots.$$

In general

$$\varphi(d(Tx_n,\ Tx_{n+1})) = \frac{1}{(n+1)^2(n+2)^2} < \frac{1}{n^2(n+1)^2} = \varphi(d(x_n,\ x_{n+1}))$$

for $n = 0, 1, 2, \ldots$ *And, let* $\{x_{h(n)}\}$ *and* $\{x_{k(n)}\}$ *be any two sequences of* $\{x_n\}$ *with* $x_{h(n)} \neq x_{k(n)}$. *Now*

$$\Delta_n = \frac{\varphi(|\frac{1}{h_n+2} - \frac{1}{k_n+2}|)}{\varphi(|\frac{1}{h_n+1} - \frac{1}{k_n+1}|)} = \frac{(1+\frac{1}{h_n})(1+\frac{1}{k_n})}{(1+\frac{2}{h_n})(1+\frac{2}{k_n})} \to 1 \ \text{as } n \to \infty$$

and $d_n = |\frac{1}{h_n} - \frac{1}{k_n}| \to 0$ *as* $n \to \infty$. *Hence* T *satisfies all the hypotheses of Theorem* 2.1 *and* T *has a unique fixed point '0' in* $\overline{O_T(1)}$.

On the other hand, if we define T *on* X *by* $T(0) = 1,\ T(1) = 2,\ T(2) = \frac{1}{2}$, $T(\frac{1}{n}) = \frac{1}{n+1}$ *for all* $n = 2, 3, 4, \ldots$ *Then* T *is not orbitally continuous at any point of* $O_T(0)$ *even if* T *satisfies all the other hypotheses of Theorem* 2.1 *and* T *has no fixed points in* $\overline{O_T(1)}$.

Theorem 2.6. *Let* T *be a selfmap on a* T*-orbitally complete metric space. Let* x_0 *in* X, *assume that there exists a* φ *in* Φ *such that*

$$\varphi(d(T^m x, T^m y)) < \varphi(d(x, y)) \tag{2.17}$$

for all $x, y = Tx \in \overline{O_T(x_0)}, x \neq y$ *and some positive integer* $m > 1$. *We define* $\{x_n\}_{n=1}^{\infty}$ *by* $x_n = T(x_{n-1})$ *for* $n = 1, 2, 3, \ldots$. *Then* $x_n \to z$ *with* z *is a unique fixed point of* T *in* X *if and only if for any two subsequences* $\{x_{h(n)}\}$ *and* $\{x_{k(n)}\}$ *with* $x_{h(n)} \neq x_{k(n)}$, *we have* $\Delta_n \to 1$ *only if* $d_n \to 0$, *provided* T^m *is orbitally continuous at* z; *where*

$$\Delta_n = \frac{\varphi(d(T^m x_{h(n)},\ T^m x_{k(n)}))}{\varphi(d_n)} \ \text{ and } d_n = d(x_{h(n)}, x_{k(n)}).$$

Proof. By replacing T by T^m in Theorem 2.1, we get T^m has a unique fixed point z in X. *i.e.,* $T^m z = z$. Now $T(z) = T(T^m(z)) = T^{m+1}(z) = T^m(Tz)$. Hence Tz is also a fixed point of T^m. We now show that $Tz = z$. Suppose that $Tz \neq z$. Hence

$$\varphi(d(z,\ Tz)) = \varphi(d(T^m z,\ TT^m z) = \varphi(d(T^m z,\ T^m Tz) < \varphi(d(z,\ Tz)),$$

a contradiction. Thus $Tz = z$. □

Remark 2.7. *Theorem* 1.2 *follows as a corollary to Theorem* 2.6 *by choosing* $\varphi(t) = t,\ t \geq 0$ *in Theorem* 2.6. *In the following, we provide an example for the applicability of Theorem* 2.6, *where the condition* (2.17) *holds for* $m = 4$, *but fails to hold for* $m = 1, 2$ *and* 3 *and hence Theorem* 2.6 *is a generalization of theorem* 2.1.

Example 2.8. *Let* $X = \{1,\ 2,\ 3,\ 4,\ \frac{9}{2}\} \cup \{1 - \frac{1}{n} / n = 1, 2, 3, \ldots\}$ *with the usual metric. We define* T *on* X *by* $T(0) = 2,\ T(2) = 4,\ T(4) = 3,\ T(3) = \frac{9}{2},\ T(\frac{9}{2}) = 1 - \frac{1}{2},\ T(1 - \frac{1}{2}) = 1 - \frac{1}{3}, \ldots,\ T(1 - \frac{1}{n}) = 1 - \frac{1}{n+1}$ *for* $n = 2, 3, 4, \ldots$ *and* $T(1) = 1$. *Thus, for* $x_0 = 0$, *we have* $O_T(0) = \{0,\ 2,\ 4,\ 3,\ \frac{9}{2},\ 1 - \frac{1}{2},\ 1 - \frac{1}{3}, \ldots,\ 1 - \frac{1}{n}, \ldots\}$ *and* $\overline{O_T(0)} = O_T(0) \cup \{1\}$. *We define* φ *in* Φ *as in Example* 2.3. *Now, for the sequence* $x_n = Tx_{n-1},\ n = 1, 2, 3, \ldots$, *we consider the following.*

For $m=1,\ \varphi(d(Tx_0,\ Tx_1))=\frac{1}{4}\not<\frac{1}{4}=\varphi(d(x_0,\ x_1))$.

For $m=2,\ \varphi(d(T^2x_0,\ T^2x_1))=1\not<\frac{1}{4}=\varphi(d(x_0,\ x_1))$.

For $m=3,\ \varphi(d(T^3x_0,\ T^3x_1))=\frac{4}{9}\not<\frac{1}{4}=\varphi(d(x_0,\ x_1))$.

For $m=4,\ \varphi(d(T^4x_0,\ T^4x_1))=\frac{1}{64}<\frac{1}{4}=\varphi(d(x_0,\ x_1))$,

$\varphi(d(T^4x_1,\ T^4x_2))=\frac{1}{36}<\frac{1}{4}=\varphi(d(x_1,\ x_2))$,

$\varphi(d(T^4x_2,\ T^4x_3))=\frac{1}{44}<1=\varphi(d(x_2,\ x_3))$,

$\varphi(d(T^4x_3,\ T^4x_4))=\frac{1}{400}<\frac{4}{9}=\varphi(d(x_3,\ x_4))$

and in general

$$\begin{aligned}\varphi(d(T^4x_{n+3},\ T^4x_{n+4}))&=\varphi(|1-\frac{1}{n+3}-1+\frac{1}{n+2}|)=\frac{1}{(n+3)^2(n+2)^2}\\&<\frac{n^2}{(n+1)^2}=\varphi(d(x_{n+3},\ x_{n+4}))\end{aligned}$$

for $n=0,1,2,\ldots$. *Hence inequality* (2.17) *holds. Moreover* $\Delta_n\to 1$ *implies* $d_n\to 0$. *Hence* T *satisfies all the hypotheses of Theorem* 2.6 *with* $m=4$ *and '*1*' is the unique fixed point of* T.

Corollary 2.9. *Let* T *be a selfmap on a complete metric space* $(X,\ d)$. *Assume that there exists a* $\varphi\in\Phi$ *with*

$$\varphi(d(Tx,\ Ty))<\varphi(d(x,\ y))\ \text{for all}\ x,\ y=Tx\in X\ \text{with}\ x\neq y. \tag{2.18}$$

Let $x_0\in X$, *and set* $x_n=T(x_{n-1})$ *for* $n=1,2,3,\ldots$. *Then* $x_n\to z$ *in* X, *with* z *is a unique fixed point of* T *if and only if for any two subsequences* $\{x_{h(n)}\}$ *and* $\{x_{k(n)}\}$ *with* $x_{h(n)}\neq x_{k(n)}$, *we have that* $\Delta_n\to 1$ *implies* $d_n\to 0$, *where*

$$\Delta_n=\frac{\varphi(d(Tx_{h(n)},\ Tx_{k(n)}))}{\varphi(d_n)}\ \text{ and }\ d_n=\varphi(d(x_{h(n)},x_{k(n)})).$$

Theorem 2.10. *Let* $(X,\ d)$ *be a complete metric space and* T *be a selfmap of* X. *Assume that there exists a* φ *in* Φ *and* k *in* $[0,1)$ *such that*

$$\varphi(d(Tx,\ Ty))\leq k\varphi(d(x,y))\ \text{ for all}\ x,\ y\in X. \tag{2.19}$$

Let $x_0\in X$ *and set* $x_n=Tx_{n-1}$ *for* $n=1,2,3,\ldots$. *Then* $x_n\to z$, z *is a unique fixed point of* T *in* X *if and only if there exists an* α *in* S *such that for all* $n,\ m$ *in* $\mathbb{N}$

$$\varphi(d(Tx_n,\ Tx_m))\leq\alpha((d(x_n,\ x_m)))\varphi(d(x_n,\ x_m)). \tag{2.20}$$

Proof. Since the inequality (2.19) implies (2.18), by Corollary 2.9, it is sufficient to show that there exists an α in S such that (2.20) holds if and only if $\Delta_n\to 1$ implies $d_n\to 0$. Suppose that such an α exists in S. Let $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ be two sequences with $x_{h(n)} \neq x_{k(n)}$. Now, we assume that $\Delta_n \to 1$. From (2.20), we have

$$\frac{\varphi(d(Tx_{h(n)},\ Tx_{k(n)}))}{\varphi(d(x_{h(n)},\ x_{k(n)}))} \leq \alpha(d(x_{h(n)},\ x_{k(n)})) < 1.$$

On letting $n \to \infty$

$$1 = \lim_{n\to\infty} \Delta_n \leq \lim_{n\to\infty} \alpha(d(x_{h(n)},\ x_{k(n)})) \leq 1.$$

Hence $\lim\limits_{n\to\infty} \alpha(d(x_{h(n)},\ x_{k(n)})) = 1$. Since α in S, it follows that $d(x_{h(n)},\ x_{k(n)}) \to 0$ as $n \to \infty$. *i.e.*, $d_n \to 0$ as $n \to \infty$.

Conversely, assume that $\Delta_n \to 1$ implies $d_n \to 0$. We define $\alpha : R^+ \to R$ by $\alpha(t) = sup\{\frac{\varphi(d(Tx_{h(n)},\ Tx_{k(n)}))}{\varphi(d(x_{h(n)},\ x_{k(n)}))}/d(x_n,\ x_m) \geq t\}$. From the inequality (2.19), we have $\frac{\varphi(d(Tx_n,\ Tx_m))}{\varphi(d(x_n,\ x_m))} \leq k$ for all n, m with $x_n \neq x_m$. Hence $\alpha(t) \leq k$, but $k < 1$ implies that $\alpha(t) < 1$ for all $t > 0$. Also, $\alpha(d(x_n,\ x_m)) \geq \frac{\varphi(d(Tx_n,\ Tx_m))}{\varphi(d(x_n,\ x_m))}$. Hence, $\varphi(d(Tx_n,\ Tx_m)) \leq \alpha(d(x_n,\ x_m))\varphi(d(x_n,\ x_m))$. Suppose that $\alpha(t_n) \to 1$ as $n \to \infty$ for a sequence $\{t_n\}$ in $(0, \infty)$. With out loss of generality, we assume that there exists a sequence $\{s_n\}$ in $(0, \infty)$ such that $\lim_{n\to\infty} s_n = 0$ and $1 - s_n < \alpha(t_n) < 1$. Thus for each $n > 0$, there exist two subsequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ with $d(x_{h(n)},\ x_{k(n)}) \geq t_n$ and $1 - s_n < \frac{\varphi(d(Tx_{h(n)},\ Tx_{k(n)}))}{\varphi(d(x_{h(n)},\ x_{k(n)}))} \leq \alpha(t_n) < 1$. Thus it follows that $\Delta_n \to 1$ as $n \to \infty$. Hence by our assumption, $d_n \to 0$ so that $t_n \to 0$. This completes the proof of the theorem. □

We write S^1 for the class of all functions $\alpha : (0,\ \infty) \to [0,\ 1]$ satisfying $\alpha(t_n) \to 1$ implies $t_n \to 0$.

Theorem 2.11. *Let $(X,\ d)$ be a complete metric space and T be a selfmap on X. Assume that there exists a φ in Φ such that*

$$\varphi(d(Tx,\ Ty)) < \varphi(d(x,y)) \ \textit{ for all } x,\ y \in X,\ x \neq y. \tag{2.21}$$

Let $x_0 \in X$ and set $x_n = Tx_{n-1}$ for $n = 1, 2, 3, \ldots$. Then $x_n \to z$, z is a unique fixed point of T in X if and only if there exists an α in S^1 such that for all $n,\ m$ in $\mathbb{N}$

$$\varphi(d(Tx_n,\ Tx_m)) \leq \alpha((d(x_n,\ x_m)))\varphi(d(x_n,\ x_m)). \tag{2.22}$$

Proof. By Corollary 2.9, it is enough to show that there exists an α in S^1 such that (2.22) holds if and only if $\Delta_n \to 1$ implies $d_n \to 0$. Necessary part of the proof follows as that of the proof of Theorem 2.10. For sufficiency, we proceed as follows. We define $\alpha : R^+ \to R$ by

$$\alpha(t) = sup\{\frac{\varphi(d(Tx_{h(n)},\ Tx_{k(n)}))}{\varphi(d(x_{h(n)},\ x_{k(n)}))}/d(x_n,\ x_m) \geq t\}.$$

From the inequality (2.21), it follows that $\frac{\varphi(d(Tx_n,\ Tx_m))}{\varphi(d(x_n,\ x_m))} < 1$ for all $n,\ m$ in $\mathbb{N}$ with $x_n \neq x_m$. Hence $\alpha(t) \leq 1$ for all $t > 0$. The remaining part of the proof runs as on the same lines of the proof of Theorem 2.10. □

Remark 2.12. *Theorem* 1.4 *follows as a corollary to Theorem* 2.11 *by choosing* $\varphi(t) = t,\ t \geq 0$ *in Theorem* 2.11.

The following example suggests that Theorem 2.11 is a generalization of Theorem 1.4.

Example 2.13. *Let* $X = \{0,\ 1,\ 2\}$ *with the usual metric. We define* $T : X \to X$ *by* $T(1) = 0,\ T(0) = 2$ *and* $T(2) = 2$. *We define* $\varphi : [0,\ \infty) \to [0,\ \infty)$ *by*

$$\varphi(t) = \begin{cases} t & if\ 0 \leq t \leq 1 \\ \frac{1}{t} & if\ t \geq 1. \end{cases}$$

Clearly $\varphi \in \Phi$. *And* $\alpha : (0,\ \infty) \to [0,\ 1]$ *defined by* $\alpha(t) = \frac{1}{1+t},\ t > 0$. *If either* $(x = 0,\ y = 2)$ *or* $(x = 2,\ y = 0)$ *then* $d(Tx, Ty) = 0$ *and the inequalities* (2.21) *and* (2.22) *hold trivially. In the remaining cases, i.e., when* $x = 0,\ y = 1;\ x = 1,\ y = 0;\ x = 1,\ y = 2;$ *and* $x = 2,\ y = 1;$ *we have* $d(Tx,\ Ty) = 2,\ d(x,\ y) = 1$ *and* $\alpha(d(x,\ y)) = \frac{1}{2}$. *Hence* $\varphi(d(Tx,\ Ty)) = \frac{1}{2} < 1 = \varphi(d(x,\ y))$ *so that* (2.21) *holds and* $\varphi(d(Tx,\ Ty)) = \frac{1}{2} \leq \frac{1}{2}1 = \frac{1}{1+d(x,\ y)}\varphi(d(x,\ y)) = \alpha(d(x,\ y))\varphi(d(x, y))$ *so that* (2.22) *holds. Hence, all the hypotheses of Theorem* 2.11 *hold and* T *has a unique fixed point* 2. *Moreover, this shows that when* $\varphi(t) = t$ *the inequalities* (2.21) *and* (2.22) *fail to hold at the points* $x = 0,\ y = 1;\ x = 1,\ y = 0;\ x = 1,\ y = 2$ *and* $x = 2,\ y = 1$ *so that Theorem* 1.4 *is not applicable.*

Remark 2.14. *Theorem* 1.10 *follows as a corollary to Theorem* 2.1, *since the inequality* (1.2) *implies the inequality* (2.1).

An open question : Find a selfmap T of X that satisfies the condition (2.1) but fails to be a strip φ-contraction. If this problem is solved in orbitally complete metric spaces, then Theorem 2.1 generalizes Theorem 1.10.

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