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Approximation and Generalized Lower Order of Entire Functions of Several Complex Variables Having Slow Growth

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Abstract : In the present paper, we study the growth properties of entire functions of several complex variables having slow growth. The characterizations of generalized lower order of entire functions of several complex variables have been obtained in terms of their Taylor's series coefficients. Also we have obtained the characterization of generalized lower order of entire functions of several complex variables in terms of approximation and interpolation errors.

Keywords : entire function; maximum term; generalized lower order; approximation error; interpolation error.
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1 Introduction

We denote complex $N-$ space by C^N. Thus, $z \in C^N$ means that $z = (z_1, z_2, ..., z_N)$, where $z_1, z_2, ..., z_N$ are complex numbers. A function $g(z)\,,\ z \in C^N$ is said to be analytic at a point $\xi \in C^N$ if it can be expanded in some neighborhood of ξ as an absolutely convergent power series. If we assume $\xi = (0, 0, ..., 0)$,

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then $g(z)$ has representation

$$g(z) = \sum_{|k|=0}^{\infty} a_{k_1,k_2,...,k_N} z_1^{k_1} z_2^{k_2} \cdots z_N^{k_N} = \sum_{n=0}^{\infty} a_k z^k, \tag{1.1}$$

where $k = (k_1, k_2, ..., k_N) \in \mathbb{N}_0^N$ and $n = |k| = k_1 + k_2 + \cdots + k_N$. For $r > 0$, the maximum modulus $S(r, g)$ of entire function $g(z)$ is given by (see [1])

$$S(r, g) = \sup\{|g(z)| \ : \ |z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 = r^2\}.$$

For $r > 0$, the maximum term $\mu(r)$ of entire function $g(z)$ is defined as (see [2, 3])

$$\mu(r) = \mu(r, g) = \max_{n\geq 0}\{||a_k||r^n\}.$$

Also the index k with maximal length n for which maximum term is achieved is called the central index and is denoted by $\nu(r) = \nu(r, g) = k$.

For generalization of the classical characterizations of growth of entire functions, Seremeta [4], Kapoor and Nautiyal [5] introduced the concept of the generalized order with the help of general growth functions as follows:

Let L^0 denote the class of functions $h(x)$ satisfying the following conditions:

(i) $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii) $\lim_{x\to\infty} \frac{h[\{1+1/\psi(x)\}x]}{h(x)} = 1$, for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let Λ denote the class of functions $h(x)$ satisfying conditions (i) and

(iii) $\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$, for every $c>0$, that is $h(x)$ is slowly increasing.

Let Ω be the class of functions $h(x)$ satisfying conditions (i) and

(iv) there exist a function $\delta(x) \in \Lambda$ and constants x_0, K_1 and K_2 such that $0 < K_1 \leq \frac{d\{h(x)\}}{d\{\delta(\log x)\}} \leq K_2 < \infty$, for all $x>x_0$.

Let $\overline{\Omega}$ be the class of functions $h(x)$ satisfying (i) and

(v) $\lim_{x\to\infty} \frac{d\{h(x)\}}{d(\log x)} = K,\ 0 < K < \infty$.

Kapoor and Nautiyal [5] showed that classes Ω and $\overline{\Omega}$ are contained in Λ and $\Omega \bigcap \overline{\Omega} = \phi$.

For an entire function $g(z)$ and functions $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$, we define the generalized lower order $\lambda(\alpha, g)$ of $g(z)$ as

$$\lambda(\alpha, g) = \lim_{r\to\infty} \inf \frac{\alpha[\log S(r, g)]}{\alpha(\log r)}. \tag{1.2}$$

Also we define the generalized lower order $\lambda(\alpha, g)$ of $g(z)$ in terms of central index as

$$\lambda(\alpha, g) = \lim_{r\to\infty} \inf \frac{\alpha\left\{|\nu(r)|\right\}}{\alpha(\log r)}. \tag{1.3}$$

Let K be a compact set in C^N and let $||.||_K$ denote the *supremum* norm on K. The function

$$\Phi_{\mathrm{K}}(\mathrm{z}) = \sup\left[|p(z)|^{1/\mathrm{n}}:\ p\text{−polynomial},\ \deg p \le n\,,\ \ ||p||_K \ \le \ 1\right],$$

where $n = 1, 2, ...$ and $z \in C^N$, is called the Siciak extremal function of the compact set K (see [1, 6]). Given a function f defined and bounded on K, for $n = 1, 2, ...$, we put

$$\begin{aligned} E_n^1(f, K) &= \ ||f - t_n||_K; \\ E_n^2(f, K) &= \ ||f - l_n||_K; \\ E_{n+1}^3(f, K) &= \ ||l_{n+1} - l_n||_K; \end{aligned}$$

where t_n denotes the n^{th} Chebyshev polynomial of the best approximation to f on K and l_n denotes the n^{th} Lagrange interpolation for f with nodes at extremal points of K (see [1, 6]).

Kumar and Srivastava [7] have obtained the characterizations of lower order of entire functions of several complex variables in terms of their Taylor's series coefficients. In the present paper we have obtained the characterizations of generalized lower order of entire functions of several complex variables having slow growth in terms of their Taylor's series coefficients. Also we have obtained the characterization of generalized lower order of entire functions of several complex variables having slow growth in terms of approximation and interpolation errors.

2 Main Results

Now we prove

Theorem 2.1. *Let $g(z)$ be an entire function whose Taylor's series representation is given by (1.1). If $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$, then the generalized lower order $\lambda(\alpha, g)$ of this entire function $g(z)$ satisfies*

$$\lambda(\alpha, g) - 1 \ge \lim_{n\to\infty} \inf \frac{\alpha(n)}{\alpha\left\{\log ||a_k||^{-1/n}\right\}}. \tag{2.1}$$

Also if

$$\psi(n) \ = \ \max_{|k|=n} \left\{\frac{||a_k||}{||a_{k'}||}\,,\ |k^{'}| = \ |k| \ + 1\right\}$$

is a non-decreasing function of n, *then equality holds in* (2.1).

Proof. Write $\lambda = \lambda(\alpha, g)$ and

$$\Phi = \lim_{n\to\infty} \inf \frac{\alpha(n)}{\alpha\left\{\log ||a_k||^{-1/n}\right\}}.$$

First we prove that $\Phi \leq \lambda - 1$. The coefficients of an entire Taylor's series satisfy Cauchy's inequality, that is

$$||a_k|| \;\leq\; r^{-n}\, S(r, g), \tag{2.2}$$

Also from (1.2), for arbitrary $\varepsilon > 0$ and a sequence $r = r_s \to \infty$ as $s \to \infty$, we have

$$S(r, g) \leq \exp[\alpha^{-1}\{\overline{\lambda}\, \alpha(\log r)\}],$$

where $\overline{\lambda} = \lambda + \varepsilon$.

Now from (2.2), we get

$$||a_k|| \leq r^{-n} \exp[\alpha^{-1}\{\overline{\lambda}\, \alpha(\log r)\}]$$

or

$$||a_k|| \leq \exp[-n \log r + \alpha^{-1}\{\overline{\lambda}\alpha(\log r)\}]. \tag{2.3}$$

Since $\alpha(x)$ is an increasing function of x, we define $r = r(n)$ as the unique root of the equation

$$\alpha\left[\frac{n\, \log r}{\overline{\lambda}}\right] = \overline{\lambda}\, \alpha(\log r). \tag{2.4}$$

Then for large values of n, we have

$$\log r \simeq \alpha^{-1}\left\{\frac{1}{\overline{\lambda} - 1}\alpha(n)\right\} = F\left(n, \frac{1}{\overline{\lambda} - 1}\right). \tag{2.5}$$

Using (2.4) and (2.5) in (2.3), we get

$$||a_k|| \leq \exp[-n\, F \;+\; (n/\overline{\lambda})\; F]$$

or

$$\frac{\overline{\lambda}}{\overline{\lambda} - 1} \log\{||a_k||\}^{-1/n} \geq \alpha^{-1}\left\{\frac{1}{\overline{\lambda} - 1}\alpha(n)\right\}$$

or

$$\frac{\alpha(n)}{\alpha\left[\frac{\overline{\lambda}}{\overline{\lambda}-1} \log\{||a_k||\}^{-1/n}\right]} \leq \overline{\lambda} - 1$$

or

$$\frac{\alpha(n)}{\alpha\left[\log\{||a_k||\}^{-1/n}\right]} \leq \left(\overline{\lambda} - 1\right) \frac{\alpha\left[\frac{\overline{\lambda}}{\overline{\lambda}-1} \log\{||a_k||\}^{-1/n}\right]}{\alpha\left[\log\{||a_k||\}^{-1/n}\right]}.$$

Since $\alpha(cx) \simeq \alpha(x)$ as $x \to \infty$, proceeding to limits as $n \to \infty$ we get

$$\Phi \le \lambda - 1.$$

Now we prove that $\lambda - 1 \le \Phi$. From the assumption on ψ, $\psi(n) \to \infty$ as $n \to \infty$. By the definition given in section 1, if $||a_k||\, r^{|k|}$ is the maximum term for r, then for $|k_1| \le |k| < |k_2|$,

$$||a_{k_1}||\, r^{|k_1|} \le ||a_k||\, r^{|k|} > ||a_{k_2}||\, r^{|k_2|}$$

and for $|k| = n$

$$\psi(n-1) \le r < \psi(n).$$

Now suppose that $||a_{k^1}||r^{|k^1|}$ and $||a_{k^2}||r^{|k^2|}$ are two consecutive maximum terms. Then

$$|k^1| \le |k^2| - 1.$$

Let

$$|k^1| \le n \le |k^2|.$$

Then

$$|\nu(r)| = |k^1|$$

for

$$\psi(|k^{1^*}|) \le r < \psi(|k^1|)\,,$$

where $|k^{1^*}| = |k^1| - 1$.

Hence from (1.3), for arbitrary $\varepsilon > 0$ and all $r > r_0(\varepsilon)$, we have

$$|k^1| = |\nu(r)| > \alpha^{-1}\left\{\lambda' \alpha(\log r)\right\}, \lambda' = \lambda - \varepsilon$$

or

$$|k^1| = |\nu(r)| \ge \alpha^{-1}\left\{\lambda' \alpha[\log\{\psi(|k^1|) - q\}]\right\},$$

where q is a constant such that

$$0 < q < \min\left\{1, [\psi(|k^1|) - \psi(|k^{1^*}|)]/2\right\}$$

or

$$\log\psi(|k^1|) \le O(1) + \alpha^{-1}\{\alpha(|k^1|)/\lambda'\}.$$

Further we have

$$\psi(|k^1|) = \psi(|k^1| + 1) = \cdots = \psi(n-1).$$

Now we can write

$$\psi(|k^0|)\cdots\psi(|k^*|) = \frac{||a_{k^0}||}{||a_k||} \le [\psi(|k^*|)]^{n-|k^0|},$$

where $|k^*| = n-1$ and $n \gg |k^0|$ or

$$\begin{aligned}\log ||a_k||^{-1} &\le n \log \psi(|k^1|) + O(1)\\ &\le n\alpha^{-1}\{\alpha(|k^1|)/\lambda'\} + O(1)\end{aligned}$$

or

$$-\frac{1}{n}\log ||a_k|| \le [\alpha^{-1}\{\alpha(|k^1|)/\lambda'\}][1+o(1)]$$

or

$$-\frac{1}{n}\log ||a_k|| \le [\alpha^{-1}\{\alpha(n)/\lambda'\}][1+o(1)]$$

or

$$\lambda' \le \frac{\alpha(n)}{\alpha\left\{\log ||a_k||^{-1/n}\right\}}[1+o(1)].$$

Now taking limits as $n \to \infty$, we get $\lambda \le \Phi$ or $\lambda - 1 \le \Phi$. Hence the Theorem 2.1 is proved. □

Next we prove

Theorem 2.2. *Let $K \subseteq C^N$ be a compact set such that Φ_K is locally bounded in C^N. If $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$ then the function f, defined and bounded on K, is a restriction to K of an entire function g of generalized lower order $\lambda(\alpha, g)$ if and only if*

$$\lambda(\alpha, g) - 1 \ge \lim_{n\to\infty} \sup \frac{\alpha(n)}{\alpha\left[\log\left\{E_n^s(f,K)\right\}^{-1/n}\right]};\ s = 1,2,3. \tag{2.6}$$

Also if $E^s_{|k|}(f,K)/E^s_{|k'|}(f,K)$, where $|k'| = n+1$, is a non-decreasing function of n, then equality holds in (2.6).

Proof. First we assume that f has an entire function extension g which is of generalized lower order $\lambda = \lambda(\alpha, g)$. We write

$$\theta_s = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\alpha\left[\log\left\{E_n^s\right\}^{-1/n}\right]};\ s = 1,2,3.$$

Here E_n^s stands for $E_n^s\left(g|_K, K\right),\ s = 1,2,3$. Following Winiarski [8], we have

$$E_n^1 \le E_n^2 \le (n_* + 2)E_n^1,\ n \ge 0 \tag{2.7}$$

and

$$E_n^3 \le 2(n_*+2)\, E_{n-1}^1,\ n \ge 1, \tag{2.8}$$

where $n_* = \begin{pmatrix} n+N \\ n \end{pmatrix}$. Using Stirling formula for the approximate value of $n!$ we get $n_* \approx \frac{n^N}{N!}$ for all large values of n. Hence for all large values of n, we have

$$E_n^1 \le E_n^2 \le \frac{n^N}{N!}\{1+o(1)\}E_n^1$$

and

$$E_n^3 \leq 2\frac{n^N}{N!}\{1+o(1)\}\, E_n^1.$$

Thus $\theta_3 \leq \theta_2 = \theta_1$. First we prove that $\theta_s \leq \lambda - 1$. Without any loss of generality, we may suppose that

$$K \subset B = \left\{z \in C^N : |z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 \leq 1\right\}.$$

Then

$$E_n^1 \leq E_n^1(g, B).$$

Now following Janik ([1, p. 324]), we have

$$E_n^1(g, B) \leq r^{-n}\, S(r, g),\ r \geq 2,\ n \geq 0$$

or

$$E_n^1 \leq r^{-n} \exp\left\{\alpha^{-1}\left[\overline{\lambda}\alpha\,(\log r)\right]\right\}.$$

or

$$E_n^1 \leq \exp[-n \log r + \alpha^{-1}\{\overline{\lambda}\alpha(\log r)\}. \tag{2.9}$$

Using (2.4) and (2.5) in (2.9), we get

$$E_n^1 \leq \exp[-n\, F\ +\ (n/\overline{\lambda}\,)\, F]$$

or

$$\frac{\overline{\lambda}}{\overline{\lambda}-1} \log\left\{E_n^1\right\}^{-1/n} \geq \alpha^{-1}\left\{\frac{1}{\overline{\lambda}-1}\alpha(n)\right\}$$

or

$$\frac{\alpha(n)}{\alpha\left[\frac{\overline{\lambda}}{\overline{\lambda}-1}\log\{E_n^1\}^{-1/n}\right]} \leq \overline{\lambda} - 1$$

or

$$\frac{\alpha(n)}{\alpha\left[\log\{E_n^1\}^{-1/n}\right]} \leq \left(\overline{\lambda}-1\right)\frac{\alpha\left[\frac{\overline{\lambda}}{\overline{\lambda}-1}\log\left\{E_n^1\right\}^{-1/n}\right]}{\alpha\left[\log\{E_n^1\}^{-1/n}\right]}.$$

Since $\alpha(cx) \simeq \alpha(x)$as $x \to \infty$, proceeding to limits as $n \to \infty$ we get

$$\theta_1 \leq \lambda - 1$$

or

$$\theta_s \leq \lambda - 1.$$

Now we will prove that $\lambda - 1 \leq \theta_s$. Let

$$\psi(n) = E^s_{|k|}/E^s_{|k'|}.$$

Then

$$\psi(n) \to \infty \ \text{ as } \ n \to \infty.$$

Now as in the proof of Theorem 2.1, here we have

$$\begin{aligned}\log{[E_n^s]}^{-1} &\leq n\log\psi(|k^1|)+O(1)\\ &\leq n\alpha^{-1}\{\alpha(|k^1|)/\lambda^{'}\}+O(1)\end{aligned}$$

or

$$-\frac{1}{n}\log E_n^s \leq [\alpha^{-1}\{\alpha(|k^1|)/\lambda^{'}\}][1+o(1)]$$

or

$$-\frac{1}{n}\log E_n^s \leq [\alpha^{-1}\{\alpha(n)/\lambda^{'}\}][1+o(1)]$$

or

$$\lambda^{'} \leq \frac{\alpha(n)}{\alpha\left\{\log{[E_n^s]}^{-1/n}\right\}}[1+o(1)].$$

Now taking limits as $n\to\infty$, we get

$$\lambda \leq \theta_s$$

or

$$\lambda - 1 \leq \theta_s.$$

Now let f be a bounded function defined on K and such that for $s=1,2,3$

$$\theta_s = \lim_{n\to\infty}\sup\frac{\alpha(n)}{\alpha\left[\log{\{E_n^s\}}^{-1/n}\right]}.$$

Then for every $d_1{>}\theta_s$ and for sufficiently large value of n, we have

$$\frac{\alpha(n)}{\alpha\left[\log{\{E_n^s\}}^{-1/n}\right]} \leq d_1$$

or

$$0 \leq E_n^s \leq \exp\left[-n\alpha^{-1}\left\{\frac{1}{d_1}\alpha(n)\right\}\right].$$

Proceeding to limits as $n\to\infty$, we get

$$\lim_{n\to\infty}{[E_n^s]}^{1/n}=0.$$

So by Janik ([6, Prop. 3.1]), we claim that the function f can be continuously extended to an entire function. Let us put

$$g = l_0 + \sum_{n=1}^{\infty}(l_n - l_{n-1}),$$

where $\{l_n\}$ is the sequence of Lagrange interpolation polynomials of f as defined earlier. Now we claim that g is the required continuation of f and $\lambda(\alpha, g) - 1 = \theta_s$. As in the proof of this theorem given above, we have

$$\lambda - 1 \leq \theta_s.$$

Now using the inequalities (2.7), (2.8) and the proof of first part given above, we have $\lambda(\alpha, g) - 1 = \theta_s$, as claimed. This completes the proof of the Theorem 2.2. □

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