Thai Journal of Mathematics
Volume 11 (2013) Number 2 : 411–427
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

$H(\cdot,\cdot)$-Co-Accretive Mapping with an Application for Solving a System of Variational Inclusions[1]

Rais Ahmad[†,2], **B.S. Lee**[‡] **and Mohd. Akram**[†]

[†]Department of Mathematics
Aligarh Muslim University, Aligarh-202002, India
e-mail : raisain_123@rediffmail.com (R. Ahmad)
akramkhan_20@rediffmail.com (Mohd. Akram)

[‡]Department of Mathematics
Kyungsung University, Busan 608-736, Republic of Korea
e-mail : bslee@ks.ac.kr

Abstract : In this paper, we define a new class of $H(\cdot,\cdot)$-co-accretive mappings which are the sum of symmetric cocoercive mappings and symmetric accretive mappings. We prove that the resolvent operator associated with $H(\cdot,\cdot)$-co-accretive mapping is single-valued and Lipschitz continuous. Furthermore, we apply these new results to solve a system of variational inclusions in real q-uniformly smooth Banach spaces. Our results are extensions and improvements of some known results existing in the literature. An illustrative example is also given.

Keywords : $H(\cdot,\cdot)$-co-accretive mapping; system; cocoercive; variational inclusion.

2010 Mathematics Subject Classification : 47H09; 49J40; 47H17.

---

[1]The first author is supported by Department of Science and Technology, Government of India under grant no.SR/S4/MS:577/09.
[2]Corresponding author.

1 Introduction

Variational inequalities and variational inclusions are interesting and important mathematical problems and have been studied intensively in the recent past, since they have wide applications in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences, etc. (see, for example [1–7]). The resolvent operator technique for solving variational inequalities and variational inclusions is interesting and important. For more details of this area, we refer to [8–12].

In 2008, Zou and Huang [13] introduced and studied $H(\cdot,\cdot)$-accretive mapping and its resolvent operator in Banach spaces. Very recently, Ahmad et al. [14] introduced and studied $H(\cdot,\cdot)$-cocoercive mapping and its resolvent operator in real Hilbert spaces. They also gave some examples to illustrate their results.

Keeping in view the recent interesting developments of this area, we define a new mapping called $H(\cdot,\cdot)$-co-accretive mapping in Banach spaces. We define the resolvent operator associated with the $H(\cdot,\cdot)$-co-accretive mapping and prove that it is single-valued and Lipschitz continuous. Finally, we apply these new concepts to solve a system of variational inclusions and an example is given.

2 Preliminaries

Let E be a real Banach space with its norm $\|\cdot\|$, E^* be the topological dual of E, d is the metric induced by the norm $\|\cdot\|$. Let $\langle\cdot,\cdot\rangle$ be the duality pairing between E and $E^*, CB(E)$ (respectively, 2^E) be the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of E and $\mathcal{D}(\cdot,\cdot)$ be the Häusdorff metric on $CB(E)$ defined by

$$\mathcal{D}(A,B)=\max\left\{\sup_{x\in A}\ d(x,B),\ \sup_{y\in B}\ d(A,y)\right\},$$

where $A,B\in CB(E)$ and $d(x,B)=\inf_{y\in B}d(x,y)$ and $d(A,y)=\inf_{x\in A}d(x,y)$.

Definition 2.1 ([15])**.** For $q>1$, the mapping $J_q:E\to 2^{E^*}$ defined by

$$J_q(x)=\{f\in E^*:\langle x,f\rangle=\|x\|^q,\|x\|^{q-1}=\|f\|\},\ \forall\ x\in E,$$

is called a generalized duality mapping.

In particular, J_2 is the usual normalized duality mapping on E. It is well known that $J_q(x)=\|x\|^{q-2}J_2(x),\forall\ x(\neq 0)\in E$. Also if $E\equiv X$, a real Hilbert space, then J_2 becomes the identity mapping on X.

Definition 2.2 ([15])**.** A Banach space E is called *smooth* if, for every $x\in E$ with $\|x\|=1$, there exists a unique $f\in E^*$ such that $\|f\|=f(x)=1$.

The modulus of smoothness of E is the function $\rho_E:[0,\infty)\to[0,\infty)$, defined by

$$\rho_E(t)=\sup\left\{\frac{\|x+y\|+\|x-y\|}{2}-1:x,y\in E,\ \|x\|=1,\ \|y\|=t\right\}.$$

Definition 2.3 ([15])**.** A Banach space E is said to be

(i) *uniformly smooth*, if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t}=0;$$

(ii) *q-uniformly smooth, for* $q>1$, if there exists a constant $C>0$ such that

$$\rho_E(t)\le Ct^q, t\in[0,\infty).$$

It is well known (see e.g., [16]) that

$$L_q(or\ l_q) \text{ is} \begin{cases} \text{q}-\text{uniformly smooth, if } 1<q\le 2,\\ 2-\text{uniformly smooth, if } q\ge 2.\end{cases}$$

Note that if E is uniformly smooth then J_q is single-valued. Xu [15] proved the following important lemma.

Lemma 2.4. *Let $q>1$ be a real number and E be a smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $C_q>0$ such that for every $x,y\in E$,*

$$\|x+y\|^q\le\|x\|^q+q\langle y,J_q(x)\rangle+C_q\|y\|^q.$$

3 $H(\cdot,\cdot)$-Co-Accretive Mapping

Throughout the paper, unless otherwise specified, we take E to be a q-uniformly smooth Banach space. First, we recall the following definitions and results.

Definition 3.1 ([13, 14])**.** A mapping $g:E\to E$ is said to be

(i) *accretive*, if

$$\langle g(x)-g(y),J_q(x-y)\rangle\ge 0,\ \forall\ x,y\in E;$$

(ii) *strictly accretive*, if

$$\langle g(x)-g(y),J_q(x-y)\rangle> 0,\ \forall\ x,y\in E,$$

and the equality holds if and only if $x=y$;

(iii) δ_g*-strongly accretive*, if there exists a constant $\delta_g>0$ such that

$$\langle g(x)-g(y),J_q(x-y)\rangle\ge \delta_g\|x-y\|^q,\ \forall\ x,y\in E;$$

(iv) *relaxed-accretive*, if there exists a constant $\beta>0$ such that

$$\langle g(x)-g(y),J_q(x-y)\rangle\ge (-\beta)\|x-y\|^q,\ \forall\ x,y\in E;$$

(v) *Lipschitz continuous*, if there exists a constant $\lambda_g > 0$ such that

$$\|g(x) - g(y)\| \leq \lambda_g \|x - y\|, \ \forall \ x, y \in E;$$

(vi) $\eta-$*expansive*, if there exists a constant $\eta > 0$ such that

$$\|g(x) - g(y)\| \geq \eta \|x - y\|, \ \forall \ x, y \in E;$$

if $\eta = 1$, then it is expansive.

(vii) *cocoercive*, if there exists a constant $\mu > 0$ such that

$$\langle g(x) - g(y), J_q(x - y)\rangle \geq \mu \|g(x) - g(y)\|^q, \ \forall \ x, y \in E;$$

(viii) *relaxed-cocoercive*, if there exists a constant $\gamma > 0$ such that

$$\langle g(x) - g(y), J_q(x - y)\rangle \geq (-\gamma)\|g(x) - g(y)\|^q, \ \forall \ x, y \in E.$$

Definition 3.2. A multi-valued mapping $G : E \to CB(E)$ is said to be $\mathcal{D}-$*Lipschitz continuous*, if for any $x, y \in E$, there exists a constant $\lambda_{D_G} > 0$ such that

$$\mathcal{D}(G(x), G(y)) \leq \lambda_{D_G} \|x - y\|.$$

Definition 3.3 ([14])**.** Let $H : E \times E \to E$ and $A, B : E \to E$ be mappings. Then

(i) $H(A, \cdot)$ is said to be *cocoercive with respect to* A, if there exists a constant $\mu_1 > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), J_q(x - y)\rangle \geq \mu_1 \|Ax - Ay\|^q, \ \forall \ x, y, u \in E;$$

(ii) $H(\cdot, B)$ is said to be *relaxed-cocoercive with respect to* B, if there exists a constant $\gamma_1 > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x - y)\rangle \geq (-\gamma_1)\|Bx - By\|^q, \ \forall \ x, y, u \in E;$$

(iii) $H(A, \cdot)$ is said to be r_1-*Lipschitz continuous with respect to* A, if there exists a constant $r_1 > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), J_q(x - y)\rangle \leq r_1 \|x - y\|, \ \forall \ x, y, u \in E;$$

(iv) $H(\cdot, B)$ is said to be r_2-*Lipschitz continuous with respect to* B, if there exists a constant $r_2 > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x - y)\rangle \leq r_2\|x - y\|,\ \forall\ x, y, u \in E;$$

(v) $H(A, B)$ is said to be *symmetric cocoercive with respect to* A *and* B, if $H(A, \cdot)$ is cocoercive with respect to A and $H(\cdot, B)$ is relaxed-cocoercive with respect to B.

Example 3.4. *Let* $E = \mathbb{R}^2$*, with an inner product defined by*

$$\langle (x_1, x_2), (y_1, y_2)\rangle = x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2.$$

Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ *be mappings defined by*

$$A(x_1, x_2) = \left(\frac{2}{3}x_1 + \frac{1}{3}x_2, \frac{1}{3}x_1 + \frac{2}{3}x_2\right),\ \forall\ (x_1, x_2) \in \mathbb{R}^2,$$

$$B(y_1, y_2) = \left(-\frac{1}{2}y_1 - y_2, -y_1 - \frac{1}{2}y_2\right),\ \forall\ (y_1, y_2) \in \mathbb{R}^2.$$

Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ *be a mapping defined by*

$$H(Ax, Bx) = Ax + Bx,\ \forall\ x \in \mathbb{R}^2.$$

Then for any $u \in \mathbb{R}^2$*, it is easy to verify that*

$$\langle H(Ax, u) - H(Ay, u), x - y\rangle \geq 3\|Ax - Ay\|^2.$$

and

$$\langle H(u, Bx) - H(u, By), x - y\rangle \geq (-2)\|Bx - By\|^2.$$

Thus $H(A, B)$ *is symmetric cocoercive with respect to* A *and* B*.*

Definition 3.5 ([17])**.** Let $f, g : E \to E$ be the mappings and $M : E \times E \to 2^E$ be a multi-valued mapping. Then

(i) $M(f, \cdot)$ is said to be *strongly accretive with respect to* f, if there exists a constant $\alpha > 0$ such that
$\langle u - v, J_q(x - y)\rangle \geq \alpha\|x - y\|^q,\ \forall\, x, y, w \in E$ and $\forall u \in M(f(x), w), v \in M(f(y), w)$;

(ii) $M(\cdot, g)$ is said to be *relaxed-accretive with respect to* g, if there exists a constant $\beta > 0$ such that
$\langle u - v, J_q(x - y)\rangle \geq (-\beta)\|x - y\|^q,\ \forall\, x, y, w \in E$ and $\forall u \in M(w, g(x))$, $v \in M(w, g(y))$;

(iii) $M(f,g)$ is said to be *symmetric accretive with respect to f and g*, if $M(f,\cdot)$ is strongly accretive with respect to f and $M(\cdot,g)$ is relaxed-accretive with respect to g.

Now we define the following $H(\cdot,\cdot)$-co-accretive mapping.

Definition 3.6. Let $A,B,f,g:E\to E$ and $H:E\times E\to E$ be single-valued mappings. Let $M:E\times E\to 2^E$ be a multi-valued mapping. The mapping M is said to be *$H(\cdot,\cdot)$-co-accretive with respect to A,B,f and g*, if $H(A,B)$ is symmetric cocoercive with respect to A and B, $M(f,g)$ is symmetric accretive with respect to f and g and $(H(A,B)+\lambda M(f,g))(E)=E$, for all $\lambda>0$.

Definition 3.7. Let $A,B,f,g:E\to E$ and $H:E\times E\to E$ be single-valued mappings. Let $M:E\times E\to 2^E$ be an $H(\cdot,\cdot)$-co-accretive mapping with respect to A,B,f and g. The resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}:E\to E$ is defined by

$$R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)=[H(A,B)+\lambda M(f,g)]^{-1}(u),\ \forall u\in E \text{ and } \lambda>0.$$

In the rest of the paper, whenever we mention that M is $H(\cdot,\cdot)$-co-accretive mapping, we mean that $H(A,B)$ is symmetric cocoercive with respect to A and B with constants μ and γ, respectively and $M(f,g)$ is symmetric accretive with respect to f and g with constants α and β, respectively.

Next, we prove that the resolvent operator is single-valued and Lipschitz continuous.

Theorem 3.8. *Let $A,B,f,g:E\to E$ and $H:E\times E\to E$ be single-valued mappings. Let $M:E\times E\to 2^E$ be an $H(\cdot,\cdot)$-co-accretive mapping with respect to A,B,f and g. Let A be η-expansive and B be σ-Lipschitz continuous and let $\alpha>\beta,\mu>\gamma$ and $\eta>\sigma$. Then the resolvent operator*

$$R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)=[H(A,B)+\lambda M(f,g)]^{-1}(u),\ \forall u\in E,\ \lambda>0,$$

is single-valued.

Proof. For any given $u\in E$, let $x,y\in[H(A,B)+\lambda M(f,g)]^{-1}(u)$. It follows that

$$-H(A(x),B(x))+u\in\lambda M(f(x),g(x)),$$

and

$$-H(A(y),B(y))+u\in\lambda M(f(y),g(y)).$$

Since M is $H(\cdot,\cdot)$-co-accretive with respect to A, B, f and g, we have

$$\begin{aligned}(\alpha-\beta)\|x-y\|^q &\leq \langle -H(A(x),B(x))+u-(-H(A(y),B(y))+u), J_q(x-y)\rangle \\ &= \langle -H(A(x),B(x))-(-H(A(y),B(y))), J_q(x-y)\rangle \\ &= -\langle H(A(x),B(x))-H(A(y),B(x)), J_q(x-y)\rangle \\ &\quad -\langle H(A(y),B(x))-H(A(y),B(y)), J_q(x-y)\rangle \\ &\leq (-\mu)\|A(x)-A(y)\|^q+\gamma\|B(x)-B(y)\|^q.\end{aligned} \tag{3.1}$$

Since A is η-expansive and B is σ-Lipschitz continuous, thus (3.1) becomes

$$0 \leq (\alpha-\beta)\|x-y\|^q \leq -\mu\eta^q\|x-y\|^q+\gamma\sigma^q\|x-y\|^q,$$

which implies that

$$0 \leq [(\alpha-\beta)+(\mu\eta^q-\gamma\sigma^q)]\|x-y\|^q \leq 0.$$

Since $\alpha > \beta, \mu > \gamma$ and $\eta > \sigma$, it follows that $x = y$ and so the resolvent operator defined by $[H(A,B)+\lambda M(f,g)]^{-1}$ is single-valued. This completes the proof. □

Theorem 3.9. *Let $A, B, f, g : E \to E$ and $H : E \times E \to E$ be single-valued mappings. Suppose $M : E \times E \to 2^E$ is an $H(\cdot,\cdot)$-co-accretive mapping with respect to A, B, f and g. Let A be η-expansive and B be σ-Lipschitz continuous such that $\alpha > \beta, \mu > \gamma$ and $\eta > \sigma$. Then the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)} : E \to E$ is Lipschitz continuous with constant θ, that is,*

$$\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\| \leq \theta\|u-v\|, \quad \forall\ u,v \in E \text{ and } \lambda > 0,$$

where $\theta = \frac{1}{\lambda(\alpha-\beta)+(\mu\eta^q-\gamma\sigma^q)}$.

Proof. Let u, v be any given points in E. It follws that

$$R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u) = [H(A,B)+\lambda M(f,g)]^{-1}(u)$$

and

$$R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v) = [H(A,B)+\lambda M(f,g)]^{-1}(v)$$

and so

$$\tfrac{1}{\lambda}(u-H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)),B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)))) \in M(f(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)),g(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)))$$

and

$\frac{1}{\lambda}(v-H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)))) \in M(f(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), g(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)))$.

Since M is symmetric accretive with respect to f and g, we have

$$\begin{aligned}(\alpha-\beta)\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-&R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q \\ &\leq \langle \frac{1}{\lambda}(u-H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)))) \\ &\quad - \frac{1}{\lambda}(v-H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)))), \\ &\quad J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle \\ &\leq \frac{1}{\lambda}\langle u-v, J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle \\ &\quad - \frac{1}{\lambda}\langle H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u))) \\ &\quad - H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u))), \\ &\quad J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle \\ &\quad - \frac{1}{\lambda}\langle H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u))) \\ &\quad - H(A(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)), B(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))), \\ &\quad J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle.\end{aligned}$$

Since H is symmetric cocoercive with respect to A and B, A is η-expansive and B is σ-Lipschitz continuous, we have

$$\begin{aligned}(\alpha-\beta)\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-&R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q \\ &\leq \frac{1}{\lambda}\langle u-v, J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle \\ &\quad - \frac{1}{\lambda}(\mu\eta^q-\gamma\sigma^q)\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q,\end{aligned}$$

or

$$\begin{aligned}(\alpha-\beta)+\frac{1}{\lambda}(\mu\eta^q-\gamma\sigma^q)\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q \\ \leq \frac{1}{\lambda}\langle u-v, J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\langle u-v, J_q(R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-&R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v))\rangle \\ &\geq [\lambda(\alpha-\beta)+(\mu\eta^q-\gamma\sigma^q)]\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q,\end{aligned}$$

$$\|u-v\|\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^{q-1}$$
$$\geq [\lambda(\alpha-\beta)+(\mu\eta^q-\gamma\sigma^q)]\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\|^q.$$

So

$$\|R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(u)-R^{H(\cdot,\cdot)}_{\lambda,M(\cdot,\cdot)}(v)\| \leq \theta\|u-v\|,$$

where

$$\theta = \frac{1}{\lambda(\alpha-\beta)+(\mu\eta^q-\gamma\sigma^q)}.$$

This completes the proof. □

Remark 3.10. *Definitions 3.6, 3.7 and Theorems 3.8, 3.9 generalize the corresponding concepts and results in [13, 16, 18–20].*

Now we consider our main results.

4 System of Variational Inclusions

In this section, we apply $H(\cdot,\cdot)$-co-accretive mapping for solving a system of variational inclusions and we assume that for each $i=1,2, E_i$ are q_i-uniformly smooth Banach spaces with norm $\|\cdot\|_i$.

Let $A_1,B_1,f_1,g_1 : E_1 \to E_1; A_2,B_2,f_2,g_2 : E_2 \to E_2$ be nonlinear mappings. Let $S,H_1 : E_1\times E_2 \to E_1$ and $T,H_2 : E_1\times E_2 \to E_2$ be nonlinear mappings and let $M_1 : E_1\times E_1 \to 2^{E_1}$ be an $H_1(\cdot,\cdot)$-co-accretive mapping with respect to A_1,B_1,f_1 and g_1 and $M_2 : E_2\times E_2 \to 2^{E_2}$ be an $H_2(\cdot,\cdot)$-co-accretive mapping with respect to A_2,B_2,f_2 and g_2. Let $P : E_1 \to 2^{E_1}$ and $G : E_2 \to 2^{E_2}$ be multi-valued mappings. We consider the following system of variational inclusions.

Find $(x,y)\in E_1\times E_2, u\in P(x)$ and $v\in G(y)$ such that

$$\begin{aligned} &\theta_1 \in S(x,v)+M_1(f_1(x),g_1(x));\\ &\theta_2 \in T(u,y)+M_2(f_2(y),g_2(y)), \end{aligned} \tag{4.1}$$

where θ_1 and θ_2 are the zero vectors of E_1 and E_2, respectively.

Note that for suitable choices of mappings $A_1,B_1,A_2,B_2,f_1,g_1,f_2,g_2,S,$ H_1,T,H_2,M_1,M_2,P,G and the spaces E_1,E_2, the system of variational inclusions (4.1) reduces to various systems of variational inclusions (inequalities) existing in the literature.

Lemma 4.1. *For any given* $(x,y) \in E_1 \times E_2, u \in P(x), v \in G(y), (x,y,u,v)$ *is a solution of the system of variational inclusions (4.1) if and only if* (x,y,u,v) *satisfies*

$$x = R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}[H_1(A_1,B_1)(x) - \lambda_1 S(x,v)];$$

$$y = R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}[H_2(A_2,B_2)(y) - \lambda_2 T(u,y)],$$

where λ_1, λ_2 *are two constants,* $R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}(x) = [H_1(A_1,B_1)+\lambda_1 M_1(f_1,g_1)]^{-1}(x)$ *and* $R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}(y) = [H_2(A_2,B_2) + \lambda_2 M_2(f_2,g_2)]^{-1}(y),\ \forall\ x \in E_1, y \in E_2$.

Proof. Proof is an immediate consequence of definitions of $R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}$ and $R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}$. □

Based on Lemma 4.1, we define the following iterative Algorithm for solving the system of variational inclusions (4.1).

Algorithm 4.2. *For any given* $(x_0,y_0) \in E_1 \times E_2, u_0 \in P(x_0), v_0 \in G(y_0)$, *compute* $(x_n,y_n) \in E_1 \times E_2, u_n \in P(x_n)$ *and* $v_n \in G(y_n)$ *such that*

$$x_{n+1} = R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}[H_1(A_1,B_1)(x_n) - \lambda_1 S(x_n,v_n)];$$

$$y_{n+1} = R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}[H_2(A_2,B_2)(y_n) - \lambda_2 T(u_n,y_n)];$$

$$u_n \in P(x_n), \|u_n - u_{n+1}\| \leq \mathcal{D}(P(x_n), P(x_{n+1}));$$

$$v_n \in G(y_n), \|v_n - v_{n+1}\| \leq \mathcal{D}(G(y_n), G(y_{n+1}));$$

where $n = 0,1,2,\ldots$ *and* $\lambda_1, \lambda_2 > 0$ *are two constants.*

Theorem 4.3. *For each* $i = 1,2$, *let* E_i *be* q_i*-uniformly smooth Banach spaces,* A_i *be* η_i*-expansive mappings and* B_i *be* σ_i*-Lipschitz continuous mappings. Let* $S, H_1 : E_1 \times E_2 \to E_1$ *be mappings such that* H_1 *is* r_1*-Lipschitz continuous with respect to* A_1 *and* r_2*-Lipschitz continuous with respect to* $B_1; S$ *is* δ_S*-strongly accretive in the first argument and* $\lambda_{S_1}, \lambda_{S_2}$*-Lipschitz continuous in the first and second arguments, respectively. Let* $T, H_2 : E_1 \times E_2 \to E_2$ *be mappings such that* H_2 *is* r_3*-Lipschitz continuous with respect to* A_2 *and* r_4*-Lipschitz continuous with respect to* $B_2; T$ *is* δ_T*-strongly accretive in the second argument and* $\lambda_{T_1}, \lambda_{T_2}$*-Lipschitz continuous in the first and second arguments, respectively. Let* $P : E_1 \to CB(E_1)$ *is* $\mathcal{D}$*-Lipschitz continuous with constant* λ_{D_P} *and* $G : E_2 \to CB(E_2)$ *is* $\mathcal{D}$*-Lipschitz continuous with constant* λ_{D_G}. *Suppose that* $M_1 : E_1 \times E_1 \to 2^{E_1}$

be $H_1(\cdot,\cdot)$-co-accretive with respect to A_1, B_1, f_1 and g_1 with $\alpha_1 > \beta_1, \mu_1 > \gamma_1$ and $\eta_1 > \sigma_1$; $M_2 : E_2 \times E_2 \to 2^{E_2}$ be $H_2(\cdot,\cdot)$-co-accretive with respect to A_2, B_2, f_2 and g_2 with $\alpha_2 > \beta_2, \mu_2 > \gamma_2$ and $\eta_2 > \sigma_2$. Assume that there exist constants $\lambda_1, \lambda_2 > 0$ satisfying the following condition:

$$\begin{aligned} &\theta_1 \sqrt[q_1]{L_1} + \theta_2\lambda_2\lambda_{T_1}\lambda_{D_P} < 1; \\ &\theta_2 \sqrt[q_2]{L_2} + \theta_1\lambda_1\lambda_{S_2}\lambda_{D_G} < 1, \end{aligned} \tag{4.2}$$

where

$$\theta_1 = \frac{1}{\lambda_1(\alpha_1 - \beta_1) + (\mu_1\eta_1^{q_1} - \gamma_1\sigma_1^{q_1})}, \theta_2 = \frac{1}{\lambda_2(\alpha_2 - \beta_2) + (\mu_2\eta_2^{q_2} - \gamma_2\sigma_2^{q_2})};$$

$$L_1 = [(r_1 + r_2)^{q_1} - \lambda_1 q_1\delta_S + \lambda_1 q_1\lambda_{S_1}(r_1 + r_2)^{q_1-1} + \lambda_1 q_1\lambda_{S_1} + \lambda_1^{q_1} C q_1\lambda_{S_1}^{q_1}];$$

$$L_2 = [(r_3 + r_4)^{q_2} - \lambda_2 q_2\delta_T + \lambda_2 q_2\lambda_{T_2}(r_3 + r_4)^{q_2-1} + \lambda_2 q_2\lambda_{T_2} + \lambda_2^{q_2} C q_2\lambda_{T_2}^{q_2}].$$

Then $(x, y) \in E_1 \times E_2, u \in P(x), v \in G(y)$ is a solution of the system of variational inclusions (4.1) and the sequences $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ generated by the Algorithm 4.2 converge strongly to x, y, u and v, respectively.

Proof. From Algorithm 4.2 and Theorem 3.9, we have

$$\begin{aligned} &\|x_{n+1} - x_n\|_1 \\ &\quad = \|R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}[H_1(A_1(x_n), B_1(x_n)) - \lambda_1 S(x_n, v_n)] \\ &\qquad - R^{H_1(\cdot,\cdot)}_{\lambda_1, M_1(\cdot,\cdot)}[H_1(A_1(x_{n-1}), B_1(x_{n-1})) - \lambda_1 S(x_{n-1}, v_{n-1})]\|_1 \\ &\quad \le \theta_1\|H_1(A_1(x_n), B_1(x_n)) - H_1(A_1(x_{n-1}), B_1(x_{n-1})) \\ &\qquad - \lambda_1(S(x_n, v_n) - S(x_{n-1}, v_n) + S(x_{n-1}, v_n) - S(x_{n-1}, v_{n-1}))\|_1 \\ &\quad \le \theta_1\|H_1(A_1(x_n), B_1(x_n)) - H_1(A_1(x_{n-1}), B_1(x_{n-1})) - \lambda_1(S(x_n, v_n) \\ &\qquad - S(x_{n-1}, v_n)\|_1 + \theta_1\lambda_1\|S(x_{n-1}, v_n) - S(x_{n-1}, v_{n-1})\|_1, \end{aligned} \tag{4.3}$$

where

$$\theta_1 = \frac{1}{\lambda_1(\alpha_1 - \beta_1) + (\mu_1{\eta_1}^{q_1} - \gamma_1{\sigma_1}^{q_1})}.$$

Since H_1 is r_1-Lipschitz continuous with respect to A_1 and r_2-Lipschitz continuous with respect to B_1, S is δ_S-strongly accretive in the first argument and λ_{S_1}-Lipschitz continuous in the first argument and using Lemma

2.4, we have

$$\begin{aligned}
&\|H_1(A_1(x_n),B_1(x_n)) - H_1(A_1(x_{n-1}),B_1(x_{n-1})) - \lambda_1(S(x_n,v_n) - S(x_{n-1},v_n))\|_1^{q_1} \\
&\quad \le \|H_1(A_1(x_n),B_1(x_n)) - H_1(A_1(x_{n-1}),B_1(x_{n-1}))\|_1^{q_1} \\
&\qquad - \lambda_1 q_1 \langle S(x_n,v_n) - S(x_{n-1},v_n), J_{q_1}[H_1(A_1(x_n),B_1(x_n)) \\
&\qquad - H_1(A_1(x_{n-1}),B_1(x_{n-1}))]\rangle + \lambda_1^{q_1} C_{q_1}\|S(x_n,v_n) - S(x_{n-1},v_n)\|_1^{q_1} \\
&\quad \le \|H_1(A_1(x_n),B_1(x_n)) - H_1(A_1(x_{n-1}),B_1(x_{n-1}))\|_1^{q_1} \\
&\qquad - \lambda_1 q_1 \langle S(x_n,v_n) - S(x_{n-1},v_n), J_{q_1}(x_n - x_{n-1})\rangle \\
&\qquad - \lambda_1 q_1 \langle S(x_n,v_n) - S(x_{n-1},v_n), J_{q_1}[H_1(A_1(x_n),B_1(x_n)) \\
&\qquad - H_1(A_1(x_{n-1}),B_1(x_{n-1}))] - J_{q_1}(x_n - x_{n-1})\rangle \\
&\qquad + \lambda_1^{q_1} C_{q_1}\|S(x_n,v_n) - S(x_{n-1},v_n)\|_1^{q_1} \\
&\quad \le (r_1+r_2)^{q_1}\|x_n - x_{n-1}\|_1^{q_1} - \lambda_1 q_1 \delta_S \|x_n - x_{n-1}\|_1^{q_1} \\
&\qquad + \lambda_1 q_1 \|S(x_n,v_n) - S(x_{n-1},v_n)\|_1 \times [\|H_1(A_1(x_n),B_1(x_n)) \\
&\qquad - H_1(A_1(x_{n-1}),B_1(x_{n-1}))\|_1^{q_1-1} + \|x_n - x_{n-1}\|^{q_1-1}] \\
&\qquad + \lambda_1^{q_1} C q_1 \lambda_{S_1}^{q_1}\|x_n - x_{n-1}\|_1^{q_1-1} \\
&\quad \le (r_1+r_2)^{q_1}\|x_n - x_{n-1}\|_1^{q_1} - \lambda_1 q_1 \delta_S \|x_n - x_{n-1}\|_1^{q_1} \\
&\qquad + \lambda_1 q_1 \lambda_{S_1}\|x_n - x_{n-1}\|_1 \times [(r_1+r_2)^{q_1-1}\|x_n - x_{n-1}\|_1^{q_1-1} \\
&\qquad + \|x_n - x_{n-1}\|_1^{q_1-1}] + \lambda_1^{q_1} C q_1 \lambda_{S_1}^{q_1}\|x_n - x_{n-1}\|_1^{q_1} \\
&\quad = L_1\|x_n - x_{n-1}\|_1^{q_1},
\end{aligned} \tag{4.4}$$

where

$$L_1 = [(r_1+r_2)^{q_1} - \lambda_1 q_1 \delta_S + \lambda_1 q_1 \lambda_{S_1}(r_1+r_2)^{q_1-1} + \lambda_1 q_1 \lambda_{S_1} + \lambda_1^{q_1} C q_1 \lambda_{S_1}^{q_1}].$$

Since S is λ_{S_2}-Lipschitz continuous in the second argument and G is $\mathcal{D}$-Lipschitz continuous with constant λ_{D_G}, by using Algorithm 4.2, we have

$$\begin{aligned}
\|S(x_{n-1},v_n) - S(x_{n-1},v_{n-1})\|_1 &\le \lambda_{S_2}\|v_n - v_{n-1}\|_2 \\
&\le \lambda_{S_2}\mathcal{D}(G(y_n),G(y_{n-1})) \\
&\le \lambda_{S_2}\lambda_{D_G}\|y_n - y_{n-1}\|_2.
\end{aligned} \tag{4.5}$$

Due to (4.4) and (4.5), (4.3) becomes

$$\|x_{n+1} - x_n\|_1 \le \theta_1 \sqrt[q_1]{L_1}\|x_n - x_{n-1}\|_1 + \theta_1\lambda_1\lambda_{S_2}\lambda_{D_G}\|y_n - y_{n-1}\|_2. \tag{4.6}$$

Again from Algorithm 4.2 and Theorem 3.9, we have

$$\begin{aligned}
\|y_{n+1}-y_n\|_2 &= \|R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}[H_2(A_2(y_n),B_2(y_n))-\lambda_2 T(u_n,y_n)] \\
&\quad - R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}[H_2(A_2(y_{n-1}),B_2(y_{n-1}))-\lambda_2 T(u_{n-1},y_{n-1})]\|_2 \\
&\le \theta_2\|H_2(A_2(y_n),B_2(y_n))-H_2(A_2(y_{n-1}),B_2(y_{n-1})) \\
&\quad -\lambda_2(T(u_n,y_n)-T(u_n,y_{n-1})+T(u_n,y_{n-1})-T(u_{n-1},y_{n-1}))\|_2 \\
&\le \theta_2\|H_2(A_2(y_n),B_2(y_n))-H_2(A_2(y_{n-1}),B_2(y_{n-1})) \\
&\quad -\lambda_2(T(u_n,y_n)-T(u_n,y_{n-1})\|_2 \\
&\quad +\theta_2\lambda_2\|T(u_n,y_{n-1})-T(u_{n-1},y_{n-1})\|_2.
\end{aligned} \tag{4.7}$$

Since H_2 is r_3-Lipschitz continuous with respect to A_2 and r_4-Lipschitz continuous with respect to B_2, T is δ_T-strongly accretive in the second argument and λ_{T_2}-Lipschitz continuous in the second argument, by using Lemma 2.4, we have

$$\begin{aligned}
&\|H_2(A_2(y_n),B_2(y_n))-H_2(A_2(y_{n-1}),B_2(y_{n-1}))-\lambda_2(T(u_n,y_n)-T(u_n,y_{n-1})\|_2^{q_2} \\
&\le \|H_2(A_2(y_n),B_2(y_n))-H_2(A_2(y_{n-1}),B_2(y_{n-1}))\|_2^{q_2} \\
&\quad -\lambda_2 q_2\langle T(u_n,y_n)-T(u_n,y_{n-1}),J_{q_2}[H_2(A_2(y_n),B_2(y_n)) \\
&\quad -H_2(A_2(y_{n-1}),B_2(y_{n-1}))]\rangle+\lambda_2^{q_2}C_{q_2}\|T(u_n,y_n)-T(u_n,y_{n-1})\|_2^{q_2} \\
&\le \|H_2(A_2(y_n),B_2(y_n))-H_2(A_2(y_{n-1}),B_2(y_{n-1}))\|_2^{q_2} \\
&\quad -\lambda_2 q_2\langle T(u_n,y_n)-T(u_n,y_{n-1}),J_{q_2}(y_n-y_{n-1})\rangle \\
&\quad -\lambda_2 q_2\langle T(u_n,y_n)-T(u_n,y_{n-1}),J_{q_2}[H_2(A_2(y_n),B_2(y_n)) \\
&\quad -H_2(A_2(y_{n-1}),B_2(y_{n-1}))]-J_{q_2}(y_n-y_{n-1})\rangle \\
&\quad +\lambda_2^{q_2}C_{q_2}\|T(u_n,y_n)-T(u_n,y_{n-1})\|_2^{q_2} \\
&\le (r_3+r_4)^{q_2}\|y_n-y_{n-1}\|_2^{q_2}-\lambda_2 q_2\delta_T\|y_n-y_{n-1}\|_2^{q_2} \\
&\quad +\lambda_2 q_2\|T(u_n,y_n)-T(u_n,y_{n-1})\|_2\times[\|H_2(A_2(y_n),B_2(y_n)) \\
&\quad -H_2(A_2(y_{n-1}),B_2(y_{n-1}))\|_2^{q_2-1}+\|y_n-y_{n-1}\|_2^{q_2-1}] \\
&\quad +\lambda_2^{q_2}Cq_2\lambda_{T_2}^{q_2}\|y_n-y_{n-1}\|_2^{q_2} \\
&\le (r_3+r_4)^{q_2}\|y_n-y_{n-1}\|_2^{q_2}-\lambda_2 q_2\delta_T\|y_n-y_{n-1}\|_2^{q_2} \\
&\quad +\lambda_2 q_2\lambda_{T_2}\|y_n-y_{n-1}\|_2\times[(r_3+r_4)^{q_2-1}\|y_n-y_{n-1}\|_2^{q_2-1} \\
&\quad +\|y_n-y_{n-1}\|_2^{q_2-1}]+\lambda_2^{q_2}Cq_2\lambda_{T_2}^{q_2}\|y_n-y_{n-1}\|_2^{q_2} \\
&= L_2\|y_n-y_{n-1}\|_2^{q_2},
\end{aligned} \tag{4.8}$$

where

$$L_2=[(r_3+r_4)^{q_2}-\lambda_2 q_2\delta_T+\lambda_2 q_2\lambda_{T_2}(r_3+r_4)^{q_2-1}+\lambda_2 q_2\lambda_{T_2}+\lambda_2^{q_2}Cq_2\lambda_{T_2}^{q_2}].$$

Since T is λ_{T_1}-Lipschitz continuous in the first argument and P is $\mathcal{D}$-

Lipschitz continuous with constant λ_{D_P}, by using Algorithm 4.2, we have

$$\begin{aligned}\|T(u_n, y_{n-1}) - T(u_{n-1}, y_{n-1})\|_2 &\leq \lambda_{T_1}\|u_n - u_{n-1}\|_1 \\ &\leq \lambda_{T_1}\mathcal{D}(P(x_n), P(x_{n-1})) \\ &\leq \lambda_{T_1}\lambda_{D_P}\|x_n - x_{n-1}\|_1.\end{aligned} \tag{4.9}$$

Due to (4.8) and (4.9), (4.7) becomes

$$\|y_{n+1} - y_n\|_2 \leq \theta_2 \sqrt[q_2]{L_2}\|y_n - y_{n-1}\|_2 + \theta_2\lambda_2\lambda_{T_1}\lambda_{D_P}\|x_n - x_{n-1}\|_1. \tag{4.10}$$

Combining (4.6) and (4.10), we have

$$\begin{aligned}\|x_{n+1} - x_n\|_1 + \|y_{n+1} - y_n\|_2 &\leq [\theta_1 \sqrt[q_1]{L_1} + \theta_2\lambda_2\lambda_{T_1}\lambda_{D_P}]\|x_n - x_{n-1}\|_1 \\ &\quad + [\theta_2 \sqrt[q_2]{L_2} + \theta_1\lambda_1\lambda_{S_2}\lambda_{D_G}]\|y_n - y_{n-1}\|_2 \\ &\leq \phi(\theta)[\|x_n - x_{n-1}\|_1 + \|y_n - y_{n-1}\|_2],\end{aligned} \tag{4.11}$$

where

$$\phi(\theta) = \max[\theta_1 \sqrt[q_1]{L}_1 + \theta_2\lambda_2\lambda_{T_1}\lambda_{D_P}, \theta_2 \sqrt[q_2]{L}_2 + \theta_1\lambda_1\lambda_{S_2}\lambda_{D_G}].$$

From (4.2), it follows that $0 < \phi(\theta) < 1$ and so (4.11) implies that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Thus there exist $x, y \in E$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now we prove that $u_n \to u \in P(x)$ and $v_n \to v \in G(y)$. In fact, it follows from the $\mathcal{D}$-Lipschitz continuities of P, G and Algorithm 4.2 that

$$\|u_{n+1} - u_n\|_1 \leq \mathcal{D}(P(x_{n+1}), P(x_n)) \leq \lambda_{D_P}\|x_{n+1} - x_n\|_1 \tag{4.12}$$

and

$$\|v_{n+1} - v_n\|_2 \leq \mathcal{D}(G(y_{n+1}), G(y_n)) \leq \lambda_{D_G}\|y_{n+1} - y_n\|_2. \tag{4.13}$$

From (4.12) and (4.13), we know that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences in E. Thus there exist $u, v \in E$ such that $u_n \to u$ and $v_n \to v$ as $n \to \infty$.

Furthermore,

$$\begin{aligned}d(u, P(x)) &\leq \|u - u_n\|_1 + d(u_n, P(x)) \\ &\leq \|u - u_n\|_1 + \mathcal{D}(P(x_n), P(x)) \\ &\leq \|u - u_n\|_1 + \lambda_{D_P}\|x_n - x\|_1 \to 0, \text{ as } n \to \infty,\end{aligned}$$

which implies that $d(u, P(x)) = 0$. Since $P(x) \in CB(E)$, it follows that $u \in P(x)$. Similarly, we can show that $v \in G(y)$. By continuity of $H_1, H_2, A_1, A_2, B_1, B_2, M_1, M_2, S, T, P, G, R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}, R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}$ and Algorithm 4.2, we have

$$x = R^{H_1(\cdot,\cdot)}_{\lambda_1,M_1(\cdot,\cdot)}[H_1(A_1, B_1)(x) - \lambda_1 S(x, v)],$$

$$y = R^{H_2(\cdot,\cdot)}_{\lambda_2,M_2(\cdot,\cdot)}[H_2(A_2, B_2)(y) - \lambda_2 T(u, y)].$$

By Lemma 4.1, (x, y, u, v) is a solution of problem (4.1). This completes the proof. □

Corollary 4.4. *Let P, G be the single-valued, identity mappings, $M_1 : E_1 \times E_1 \to 2^{E_1}$ be generalized $H_1(\cdot,\cdot)$-accretive mapping, $M_2 : E_2 \times E_2 \to 2^{E_2}$ be generalized $H_2(\cdot,\cdot)$-accretive mapping and all other conditions are same as in Theorem 4.3, then one can obtain Theorem 6.1 of Kazmi et al. [17] without uniqueness.*

Acknowledgement : The authors would like to thank the referees for their comments and suggestions on the manuscript.

References

[1] R.P. Agarwal, Y.J. Cho, N.J. Huang, Sensitivity analysis for strongly nonlinear quasi- variational inclusions, Appl. Math. Lett. 13 (6) (2000) 19–24.

[2] R.P. Agarwal, N.J. Huang, Y.J. Cho, Generalized nonlinear mixed implicit quasi-variational inclusions with set-valued mappings, J. Inequal. Appl. 7 (6) (2002) 807–828.

[3] R. Ahmad, A.H. Siddiqi, Zubair Khan, Proximal point algorithm for generalized multi-valued nonlinear quasi-variational-like inclusions in Banach spaces, Appl. Math. Comput. 163 (2005) 295–308.

[4] R. Ahmad, Q.H. Ansari, An iterative algorithm for generalized nonlinear variational inclusions, Appl. Math. Lett. 13 (2000) 23–26.

[5] S.S. Chang, Y.J. Cho, H.Y. Zhou, *Iterative Methods for Nonlinear Operator Equations in Banach Spaces*, Nova Sci., New York, 2002.

[6] J.Y. Chen, N.C. Wong, J.C. Yao, Algorithm for generalized co-complementarity problems in Banach spaces, Comput. Math. Appl. 43 (1) (2002) 49–54.

[7] X.P. Ding, C.L. Lou, Perturbed proximal point algorithms for general quasi-variational like inclusions, J. Comput. Appl. Math. 210 (2000) 153–165.

[8] H.Y. Lan, Y.J. Cho, R.U. Verma, Nonlinear relaxed cocoercive variational inclusions involving (A,η)-accretive mappings in Banach spaces, Comput. Math. Appl. 51 (2006) 1529–1538.

[9] H.Y. Lan, J.H. Kim, Y.J. Cho, On a new system of nonlinear A-monotone multi-valued variational inclusions, J. Math. Anal. Appl. 327 (2007) 481–493.

[10] H.Y. Lan, J.I. Kang, Y.J. Cho, Nonlinear (A,η)-monotone operator inclusion systems involving non-monotone set-valued mappings, Taiwanese J. Math. 11 (2007) 683–701.

[11] B.S. Lee, M.K. Kang, S.J. Lee, K.H. Yang, Variational inequalities for L-pseudo- monotone maps, Nonlinear Anal. Forum 6 (2001) 417–426.

[12] M.A. Noor, Generalized set-valued variational inclusions and resolvent equations, J. Math. Anal. Appl. 228 (1998) 206–220.

[13] Y.Z. Zou, N.J. Huang, $H(\cdot,\cdot)$-accretive operator with an application for solving variational inclusions in Banach spaces, Appl. Math. Comput. 204 (2008) 809–816.

[14] R. Ahmad, M. Dilshad, M.M. Wong, J.C. Yao, $H(\cdot,\cdot)$-cocoercive operator and an application for solving generalized variational inclusions, Abs. Appl. Anal., Vol. 2011 (2011), Article ID 261534, 12 pages.

[15] H.K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal. 16 (12) (1991) 1127–1138.

[16] L.C. Zeng, S.M. Guu, J.C. Yao, Characterization of H-monotone operators with applications to variational inclusions in Banach spaces, Comput. Math. Appl. 50 (2005) 329–337.

[17] K.R. Kazmi, F.A. Khan, M. Shahzad, A system of generalized variational inclusions involving generalized $H(\cdot,\cdot)$-accretive mapping in real q-uniformly smooth Banach spaces, Appl. Math. Comput. 217 (2011) 9679–9688.

[18] Y.P. Fang, N.J. Huang, H-monotone operator and resolvent operator technique for variational inclusions, Appl. Math. Comput. 145 (2003) 795–803.

[19] Y.P. Fang, N.J. Huang, Iterative algorithm for a system of variational inclusions involving H-accretive operators in Banach spaces, Acta Math. Hungar. 108 (3) (2005) 183–195.

[20] X.P. Luo, N.J. Huang, A new class of variational inclusions with B-monotone operators in Banach spaces, J. Comput. Appl. Math. 233 (2010) 1888–1896.

(Accepted 4 July 2012)