A Class of a System of Multi-Valued Extended General Quasi-Variational Inequality Problems

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Abstract : In this paper, we consider a system of multi-valued extended general quasi-variational inequality problems (SMEGQVIP) in real Hilbert spaces. Using the projection operator technique, it is observed that the SMEGQVIP is equivalent to the system of projection equations. This alternative equivalence formulation is used to suggest an iterative algorithm for the SMEGQVIP. Further, we prove the existence of a solution of SMEGQVIP and discuss the convergence analysis of iterative sequences generated by given algorithm. The approach used in this paper may be treated as an extension and unification of approaches for studying existence results for various important classes of system of variational inequality problems given by many authors in this direction.

Keywords : system of multi-valued extended general quasi-variational inequality; relaxed (d, e)-cocoercive mappings; strongly monotone mappings; mixed Lipschitz continuous mappings; μ-$\mathcal{H}$-Lipschitz continuous mapping; projection operator technique; iterative algorithm; existence and convergence analysis.

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1   Introduction

Variational inequality theory introduced by Stampacchia [1] and Fichera [2], has become a rich source of inspiration and motivation for the study of a large number of problems arising in mechanics, elasticity and optimization and control problems, boundary value problems etc., see [1-17]. In the last four decades, considerable interest has been shown in developing various classes of variational inequalities and system of variational inequality problems. One of the most important and interesting problem in the theory of variational inequalities is the development of numerical methods which provide an efficient and implementable algorithm for solving variational inequalities and its generalization. This theory provides a simple, natural and unified framework for a general treatment of unrelated problems, which have motivated a large number of mathematicians to generalize and extend the variational inequalities and related optimization problems in several directions using novel techniques, see [1-17].

By using the projection technique, Noor [3], Noor et al. [4], Verma [5] studied the existence of solution for some classes of variational and quasi-variational inequalities involving single and multi-valued mappings in Banach spaces. Recently, by using the projection technique, Noor [6, 7] studied the existence of solution for some classes of extended general variational inequalities in the setting of Hilbert and Banach spaces.

Very recently, by using the projection technique, Chang et al. [8, 9], Cho et al. [10], Feng et al. [11], Huang et al. [12], Noor et al. [13], Verma [14] and Zou et al. [15] studied the existence theory for various classes of system of general variational inequalities and system of variational inclusions in the setting of Hilbert and Banach spaces.

Inspired by recent research works in this area, in this paper, we consider a system of multi-valued extended general quasi-variational inequality problems (SMEGQVIP, for short) in real Hilbert spaces. Using the projection operator technique, it is observed that the SMEGQVIP is equivalent to the system of projection equations. This alternative equivalence formulation is used to suggest an iterative algorithm for the SMEGQVIP. Further, we prove the existence of a solution of SMEGQVIP and discuss the convergence analysis of iterative sequences generated by given algorithm. The technique and results presented in this paper generalize and improve the corresponding technique and results given in [3–15].

2   Preliminaries

Let H be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively; let 2^H be the family of all nonempty subsets of H and let $CB(H)$ be the family of all nonempty, closed and bounded subsets of H. The Hausdorff metric $\mathcal{H}(\cdot,\cdot)$ on $CB(H)$ is defined by

$$\mathcal{H}(C,D) = \max\left\{\sup_{x\in C}\inf_{y\in D} d(x,y),\ \sup_{y\in D}\inf_{x\in C} d(x,y)\right\},\ \ C, D \in CB(H).$$

From now onwards, unless or otherwise stated, let $I = \{1, 2\}$ be an index set and for each $i \in I$, let H_i be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle_i$ and $\|.\|_i$, respectively. Let $A, C : H_1 \to CB(H_1)$, $B, D : H_2 \to CB(H_2)$ be multi-valued mappings and $N_i : H_1 \times H_2 \to H_i$, $g_i, h_i : H_i \to H_i$ be nonlinear mappings. Let $K_1 : H_1 \to 2^{H_1}$ and $K_2 : H_2 \to 2^{H_2}$ be such that for each fixed $x \in H_1, y \in H_2$, $K_1(x)$ and $K_2(y)$ are nonempty closed convex sets in H_1 and H_2, respectively, such that $(g_1(x), g_2(y)) \in \text{domain}(K_1(x), K_2(y)), \forall (x, y) \in H_1 \times H_2$. We consider the following system of multi-valued extended general quasi-variational inequality problems (SMEGQVIP):

Find $(x, y) \in H_1 \times H_2$, $u \in A(x)$, $v \in B(y)$, $w \in C(x)$, $z \in D(y)$: $h_1(x) \in K_1(x)$, $h_2(y) \in K_2(y)$ such that

$$\langle N_1(u, v) + h_1(x) - g_1(x), \, g_1(v_1) - h_1(x) \rangle_1 \geq 0, \quad \forall v_1 \in H_1 : g_1(v_1) \in K_1(x), \quad (2.1)$$

$$\langle N_2(w, z) + h_2(y) - g_2(y), \, g_2(v_2) - h_2(y) \rangle_2 \geq 0, \quad \forall v_2 \in H_2 : g_2(v_2) \in K_2(y). \quad (2.2)$$

The corresponding quasi-variational inequality problem has been studied in many practical problems, $K(x)$ has the following form $K(x) \equiv m(x) + K, \, \forall x \in H$, where $m : H \to H$ is a single-valued mapping and K is a nonempty, closed and convex set of H.

Some Special Cases of SMEGQVIP (2.1)-(2.2):

(1) If $h_1 \equiv g_1$; $h_2 \equiv g_2$, then SMEGQVIP (2.1)-(2.2) reduces to the problem of finding $(x, y) \in H_1 \times H_2$, $u \in A(x)$, $v \in B(y)$, $w \in C(x)$, $z \in D(y)$ such that

$$\langle N_1(u, v), \, g_1(v_1) - g_1(x) \rangle_1 \geq 0, \quad \forall v_1 \in H_1 : g_1(v_1) \in K_1(x), \quad (2.3)$$

$$\langle N_2(w, z), \, g_2(v_2) - g_2(y) \rangle_2 \geq 0, \quad \forall v_2 \in H_2 : g_2(v_2) \in K_2(y), \quad (2.4)$$

which is known as system of multi-valued quasi-variational inequality problems, similar type problem has been studied by many authors, see [9–14].

(2) If $H \equiv H_1 \equiv H_2$; $T(x, x) \equiv N_1(\cdot, \cdot) \equiv N_2(\cdot, \cdot)$; $h \equiv h_1 \equiv h_2$; $g \equiv g_1 \equiv g_2$; $K(x) \equiv K$ and $x = y$, then SMEGQVIP (2.1)-(2.2) reduces to the problem of finding $x \in H$, $h(x) \in K$ such that

$$\langle T(x, x) + h(x) - g(x), \, g(v) - h(x) \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \quad (2.5)$$

which is known as extended variational inequality, similar type problem has been studied by Noor [6, 7].

Further, it is remarked that for a suitable choice of the mappings A, B, C, D, g_1, g_2, h_1, h_2, K_1, K_2, N_1, N_2 and the spaces H_1, H_2, one can obtain many other known systems of variational inequalities, variational inequalities from SMEGQVIP (2.1)-(2.2), see for example [3–15] and the references therein.

Now, we give the following known concepts and results which are needed in the sequel:

Lemma 2.1. *Let K be a closed and convex set in H. Then for a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,$$

if and only if $u = P_K(z)$, where P_K is the projection of H onto the closed convex set K in H.

It is well known that the projection operator P_K is nonexpansive operator, i.e.,

$$\|P_K(x) - P_K(y)\| \leq \|x - y\|, \ \forall x, y \in H.$$

Definition 2.2 ([8])**.** A multi-valued mapping $T : H \rightarrow CB(H)$ is said to be *ξ-$\mathcal{H}$-Lipschitz continuous* if there exists a constant $\xi > 0$ such that

$$\mathcal{H}(T(x), T(y)) \leq \xi \|x - y\|, \ \forall\, x, y \in H,$$

where $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric space on $CB(H)$.

Lemma 2.3 ([16])**.**

(a) *Let $A : H \rightarrow CB(H)$ be a multi-valued mapping. Then for any given $\xi > 0$ and for any given $x, y \in H$ and $u \in A(x)$, there exists $v \in A(y)$ such that*

$$d(u, v) \leq (1 + \xi)\, \mathcal{H}(A(x), A(y));$$

(b) *If $T : H \rightarrow C(H)$, then above inequality holds for $\xi = 0$.*

Definition 2.4. A mapping $g : H \rightarrow H$ is said to be

(i) *σ-strongly monotone* if there exists a constant $\sigma > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \sigma \|x - y\|^2 \ \forall x, y \in H;$$

(ii) *δ-Lipschitz continuous* if there exists a constant $\delta > 0$ such that

$$\|g(x) - g(y)\| \leq \delta \|x - y\| \ \forall x, y \in H.$$

Definition 2.5. Let $A, C : H_1 \rightarrow CB(H_1)$, $B, D : H_2 \rightarrow CB(H_2)$. A mapping $N_1 : H_1 \times H_2 \rightarrow H_1$ is said to be

(i) *α_1-strongly monotone with respect to A in the first argument* if there exists a constant $\alpha_1 > 0$ such that

$$\langle N_1(u_1, v) - N_1(u_2, v), x_1 - x_2 \rangle_1 \geq \alpha_1 \|x_1 - x_2\|_1^2,$$

$$\forall x_1, x_2 \in H_1, y \in H_2, u_1 \in A(x_1), u_2 \in A(x_2), v \in B(y);$$

(ii) (β_1, γ_1)-*mixed Lipschitz continuous* if there exist constants $\beta_1, \gamma_1 > 0$ such that

$$\|N_1(u_1, v_1) - N_1(u_2, v_2)\|_1 \leq \beta_1 \|u_1 - u_2\|_1 + \gamma_1 \|v_1 - v_2\|_2,$$

$$x_1, x_2 \in H_1, y_1, y_2 \in H_2, u_1 \in A(x_1), u_2 \in A(x_2), v_1 \in B(y_1), v_2 \in B(y_2).$$

Definition 2.6. A mapping $h : H \to H$ is said to be *relaxed (d, e)-cocoercive* if there exist constants $d, e > 0$ such that

$$\langle h(x_1) - h(x_2), x_1 - x_2 \rangle \geq -d\|h(x_1) - h(x_2)\|^2 + e\|x_1 - x_2\|^2, \ \forall x_1, x_2 \in H.$$

Remark 2.7. *The class of relaxed (d, e)-cocoercive mappings is more general than the class of strongly monotone mappings, see [5, 9–14].*

Lemma 2.8. *Let H be a Hilbert space. Then for any $x, y \in H$,*

$$\|x + y\|^2 \leq \|x\|^2 + \langle y, x + y \rangle.$$

Lemma 2.9 ([15]). *Let $\{c_n\}$ and $\{k_n\}$ be two real sequences of nonnegative numbers that satisfy the following conditions:*

(i) $0 \leq k_n < 1$ *for $n = 0, 1, 2, ...,$ and $\limsup_n k_n < 1$,*

(ii) $c_{n+1} \leq k_n c_n$ *for $n = 0, 1, 2, ...$*

Then $\{c_n\}$ converges to 0 as $n \to \infty$.

Assumption 2.10. *The operator $P_{K_1(x)}$ satisfies the condition:*

$$\|P_{K_1(x_1)}(z) - P_{K_1(x_2)}(z)\| \leq \nu_1 \|x_1 - x_2\|, \forall x_1, x_2, z \in H_1, \ \nu_1 > 0 \ \text{ is a constant.}$$

3   Main Results

First we establish an equivalence between SMEGQVIP (2.1)-(2.2) and system of projection equations and then using this equivalence to prove the existence of a solution of SMEGQVIP (2.1)-(2.2).

Lemma 3.1. *For any given $(x, y) \in H_1 \times H_2$, $u \in A(x), v \in B(y), w \in C(x), z \in D(y) : h_1(x) \in K_1(x), h_2(y) \in K_2(y)$, (x, y, u, v, w, z) is a solution of SMEGQVIP (2.1)-(2.2) if and only if (x, y, u, v, w, z) satisfies the system of projection equations*

$$h_1(x) = P_{K_1(x)}[g_1(x) - \rho_1 N_1(u, v)],$$

$$h_2(y) = P_{K_2(y)}[g_2(y) - \rho_2 N_2(w, z)],$$

where $\rho_1, \rho_2 > 0$ are constants.

Using Lemma 2.3 and Lemma 3.1, we suggest and analyze the following iterative algorithm for finding the approximate solution of SMEGQVIP (2.1)-(2.2) in Hilbert spaces.

Iterative Algorithm 3.2. *For given* $(x_0, y_0) \in H_1 \times H_2$, $u_0 \in A(x_0), v_0 \in B(y_0), w_0 \in C(x_0), z_0 \in D(y_0)$, *compute approximate solution* $(x_n, y_n, u_n, v_n, w_n, z_n)$ *given by iterative schemes:*

$$h_1(x_{n+1}) = P_{K_1(x_n)}[g_1(x_n) - \rho_1 N_1(u_n, v_n)], \tag{3.1}$$

$$h_2(y_{n+1}) = P_{K_2(y_n)}[g_2(y_n) - \rho_2 N_2(w_n, z_n)], \tag{3.2}$$

$$u_n \in A(x_n): \|u_{n+1} - u_n\| \leq \left(1 + (1+n)^{-1}\right) \mathcal{H}_1(A(x_{n+1}), A(x_n)), \tag{3.3}$$

$$v_n \in B(y_n): \|v_{n+1} - v_n\| \leq \left(1 + (1+n)^{-1}\right) \mathcal{H}_2(B(y_{n+1}), B(y_n)), \tag{3.4}$$

$$w_n \in C(x_n): \|w_{n+1} - w_n\| \leq \left(1 + (1+n)^{-1}\right) \mathcal{H}_1(C(x_{n+1}), C(x_n)), \tag{3.5}$$

$$z_n \in D(y_n): \|z_{n+1} - z_n\| \leq \left(1 + (1+n)^{-1}\right) \mathcal{H}_2(D(y_{n+1}), D(y_n)), \tag{3.6}$$

where $n = 0, 1, 2, ...;$ $\rho_1, \rho_2 > 0$ *are constants.*

Now, we prove the existence of a solution of SMEGQVIP (2.1)-(2.2) for some relaxed (d, e)-cocoercive mappings which are not Lipschitz continuous and discuss the convergence analysis for Iterative Algorithm 3.2.

Theorem 3.3. *Let* $A, C : H_1 \to CB(H_1)$ *be* μ_1-$\mathcal{H}_1$-*Lipschitz, μ_2-$\mathcal{H}_1$-Lipschitz continuous and* $B, D : H_2 \to CB(H_2)$ *be* η_1-$\mathcal{H}_2$-*Lipschitz, η_2-$\mathcal{H}_2$-Lipschitz continuous, respectively. Let the mapping N_1 is α_1-strongly monotone in the first argument with respect to A and (β_1, γ_1)-mixed Lipschitz continuous; N_2 be α_2-strongly monotone in the second argument with respect to D and (β_2, γ_2)-mixed Lipschitz continuous. For each $i = 1, 2$, let h_i be relaxed (d_i, e_i)-cocoercive mappings; g_i be σ_i-strongly monotone and δ_i-Lipschitz continuous mappings. Let $K_1 : H_1 \to 2^{H_1}$ and $K_2 : H_2 \to 2^{H_2}$ be such that for each fixed $x \in H_1, y \in H_2$, $K_1(x)$ and $K_2(y)$ are nonempty closed convex sets in H_1 and H_2, respectively. Suppose that there are constants $\nu_1, \nu_2 > 0$ such that*

$$\|P_{K_1(x_1)}(x) - P_{K_1(x_2)}(x)\| \leq \nu_1 \|x_1 - x_2\|, \ \forall x, x_1, x_2 \in H_1, \tag{3.7}$$

$$\|P_{K_2(y_1)}(y) - P_{K_1(y_2)}(y)\| \leq \nu_2 \|y_1 - y_2\|, \ \forall y, y_1, y_2 \in H_2, \tag{3.8}$$

and $\rho_1, \rho_2 > 0$ *satisfy the following condition:*

$$\begin{cases} \sqrt{\frac{1+2d_1}{2e_1+3}} \left(\sqrt{1 - 2\sigma_1 + \delta_1^2} + \sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2\mu_1^2} + \nu_1 + \rho_2\beta_2\mu_2 \right) < 1; \\ \\ \sqrt{\frac{1+2d_2}{2e_2+3}} \left(\sqrt{1 - 2\sigma_2 + \delta_2^2} + \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\gamma_2^2\eta_2^2} + \nu_2 + \rho_1\gamma_1\eta_1 \right) < 1. \end{cases} \tag{3.9}$$

Then iterative sequence $\{(x_n, y_n, u_n, v_n, w_n, z_n)\}$ *generated by Iterative Algorithm 3.2 converges strongly to* (x, y, u, v, w, z), *a solution of SMEGQVIP (2.1)-(2.2).*

Proof. Since for each $i = 1, 2$, h_i is relaxed (d_i, e_i)-cocoercive and by using Lemma 2.8, we have the following estimate:

$$\|x_{n+1} - x_n\|_1^2$$

$$= \|h_1(x_{n+1}) - h_1(x_n) + x_{n+1} - x_n - (h_1(x_{n+1}) - h_1(x_n))\|_1^2$$

$$\leq \|h_1(x_{n+1}) - h_1(x_n)\|_1^2 - 2\langle h_1(x_{n+1}) - h_1(x_n) - x_{n+1} + x_n, x_{n+1} - x_n\rangle_1$$

$$\leq (1 + 2d_1)\|h_1(x_{n+1}) - h_1(x_n)\|_1^2 - (2 + 2e_1)\|x_{n+1} - x_n\|_1^2$$

which implies that

$$\|x_{n+1} - x_n\|_1 \leq \sqrt{\frac{1 + 2d_1}{2e_1 + 3}}\, \|h_1(x_{n+1}) - h_1(x_n)\|_1. \tag{3.10}$$

Similarly, we have

$$\|y_{n+1} - y_n\|_2 \leq \sqrt{\frac{1 + 2d_2}{2e_2 + 3}}\, \|h_2(y_{n+1}) - h_2(y_n)\|_2. \tag{3.11}$$

Now, we have

$$\|h_1(x_{n+1}) - h_1(x_n)\|_1$$

$$= \|P_{K_1(x_n)}(g_1(x_n) - \rho_1 N_1(u_n, v_n)) - P_{K_1(x_{n-1})}(g_1(x_{n-1}) - \rho_1 N_1(u_{n-1}, v_{n-1}))\|_1$$

$$\leq \|P_{K_1(x_n)}(g_1(x_n) - \rho_1 N_1(u_n, v_n)) - P_{K_1(x_n)}(g_1(x_{n-1}) - \rho_1 N_1(u_{n-1}, v_{n-1}))\|_1$$

$$\quad + \|P_{K_1(x_n)}(g_1(x_{n-1}) - \rho_1 N_1(u_{n-1}, v_{n-1}))$$

$$\quad\quad - P_{K_1(x_{n-1})}(g_1(x_{n-1}) - \rho_1 N_1(u_{n-1}, v_{n-1}))\|_1$$

$$\leq \|g_1(x_n) - g_1(x_{n-1}) - (x_{n-1} - x_n)\|_1$$

$$\quad + \|x_{n-1} - x_n - \rho_1(N_1(u_n, v_n) - N_1(u_{n-1}, v_n))\|_1$$

$$\quad + \rho_1\|N_1(u_{n-1}, v_n) - N_1(u_{n-1}, v_{n-1})\|_1 + \nu_1\|x_n - x_{n-1}\|_1. \tag{3.12}$$

Next, using α_1-strongly monotonicity with respect to A in the first argument and (β_1, γ_1)-mixed Lipschitz continuity of $N_1(\cdot, \cdot)$; μ_1-$\mathcal{H}_1$-Lipschitz continuity of A; η_1-$\mathcal{H}_2$-Lipschitz continuity of B, it follows that

$$\|x_n - x_{n-1} - \rho_1(N_1(u_n, v_n) - N_1(u_{n-1}, v_n))\|_1^2$$

$$\leq \|x_n - x_{n-1}\|_1^2 - 2\rho_1\langle N_1(u_n, v_n) - N_1(u_{n-1}, v_n), x_n - x_{n-1}\rangle_1$$

$$\quad + \rho_1^2\|N_1(u_n, v_n) - N_1(u_{n-1}, v_n)\|_1^2$$

$$\leq \|x_n - x_{n-1}\|_1^2 - 2\rho_1\alpha_1\|x_n - x_{n-1}\|_1^2 + \rho_1^2\beta_1^2\mu_1^2(1 + (1+n)^{-1})^2\|x_n - x_{n-1}\|_1^2$$

$$\leq (1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2\mu_1^2(1 + (1+n)^{-1})^2)\|x_n - x_{n-1}\|_1^2, \tag{3.13}$$

and

$$\|N_1(u_{n-1}, v_n) - N_1(u_{n-1}, v_{n-1})\|_1 \leq \gamma_1\eta_1(1 + (1+n)^{-1})\|y_n - y_{n-1}\|_2. \tag{3.14}$$

Similarly, we estimate

$$\|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\|_1^2 \leq (1 - 2\sigma_1 + \delta_1^2)\|x_n - x_{n-1}\|_1^2, \tag{3.15}$$

where g_1 is σ_1-strongly monotone and δ_1-mixed Lipschitz continuous.

From (3.10) and (3.12)-(3.15), we have

$$\|x_{n+1} - x_n\|_1$$
$$\leq \sqrt{\frac{1 + 2d_1}{2e_1 + 3}} \left[\left(\sqrt{1 - 2\sigma_1 + \delta_1^2} + \sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2\mu_1^2(1 + (1 + n)^{-1})^2} + \nu_1 \right) \right.$$
$$\left. \times \|x_n - x_{n-1}\|_1 + \rho_1\gamma_1\eta_1 \left(1 + \frac{1}{1 + n} \right) \|y_n - y_{n-1}\|_2^2 \right]. \qquad (3.16)$$

Also, we have

$$\|h_2(y_{n+1}) - h_2(y_n)\|_2$$
$$= \|P_{K_2(y_n)}(g_2(y_n) - \rho_2 N_2(w_n, z_n)) - P_{K_2(y_{n-1})}(g_2(y_{n-1}) - \rho_2 N_2(w_{n-1}, z_{n-1}))\|_2$$
$$\leq \|P_{K_2(y_n)}(g_2(y_n) - \rho_2 N_2(w_n, z_n)) - P_{K_2(y_n)}(g_2(y_{n-1}) - \rho_2 N_2(w_{n-1}, z_{n-1}))\|_2$$
$$\quad + \|P_{K_2(y_n)}(g_2(y_{n-1}) - \rho_2 N_2(w_{n-1}, z_{n-1})) - P_{K_2(y_{n-1})}(g_2(y_{n-1})$$
$$\qquad - \rho_2 N_2(w_{n-1}, z_{n-1}))\|_2$$
$$\leq \|g_2(y_n) - g_2(y_{n-1}) - (y_{n-1} - y_n)\|_2$$
$$\quad + \|y_{n-1} - y_n - \rho_2(N_2(w_n, z_n) - N_2(w_n, z_{n-1}))\|_2$$
$$\quad + \rho_2\|N_2(w_n, z_{n-1}) - N_2(w_{n-1}, z_{n-1})\|_2 + \nu_2\|y_n - y_{n-1}\|_2. \qquad (3.17)$$

Next, using α_2-strongly monotonicity with respect to D in the second argument and (β_2, γ_2)-mixed Lipschitz continuity of $N_2(\cdot, \cdot)$; μ_2-$\mathcal{H}_1$-Lipschitz continuity of C; η_2-$\mathcal{H}_2$-Lipschitz continuity of D, it follows that

$$\|y_n - y_{n-1} - \rho_2(N_2(w_n, z_n) - N_2(w_n, z_{n-1})\|_2^2$$
$$\quad \leq \|y_n - y_{n-1}\|_2^2 - 2\rho_2\langle N_2(w_n, z_n) - N_2(w_n, z_{n-1}), y_n - y_{n-1}\rangle_2$$
$$\qquad + \rho_2^2\|N_2(w_n, z_n) - N_2(w_n, z_{n-1})\|_2^2$$
$$\quad \leq \|y_n - y_{n-1}\|_2^2 - 2\rho_2\alpha_2\|y_n - y_{n-1}\|_2^2 + \rho_2^2\gamma_2^2\eta_2^2(1 + (1 + n)^{-1})^2\|y_n - y_{n-1}\|_2^2$$
$$\quad \leq (1 - 2\rho_2\alpha_2 + \rho_2^2\gamma_2^2\eta_2^2(1 + (1 + n)^{-1})^2)\|y_n - y_{n-1}\|_2^2, \qquad (3.18)$$

and

$$\|N_2(w_n, z_{n-1}) - N_2(w_{n-1}, z_{n-1})\|_2 \leq \beta_2\mu_2(1 + (1 + n)^{-1})\|x_n - x_{n-1}\|_1. \qquad (3.19)$$

Similarly, we estimate

$$\|y_n - y_{n-1} - (g_2(y_n) - g_2(y_{n-1}))\|_2^2 \leq (1 - 2\sigma_2 + \delta_2^2)\|y_n - y_{n-1}\|_2^2, \qquad (3.20)$$

where g_2 is σ_2-strongly monotone and δ_2-mixed Lipschitz continuous.

From (3.11) and (3.17)-(3.20), we have

$$\|y_{n+1} - y_n\|_2$$
$$\leq \sqrt{\frac{1 + 2d_2}{2e_2 + 3}} \left[\left(\sqrt{1 - 2\sigma_2 + \delta_2^2} + \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\gamma_2^2\eta_2^2(1 + (1 + n)^{-1})^2} + \nu_2 \right) \right.$$
$$\left. \times \|y_n - y_{n-1}\|_2 + \rho_2\beta_2\mu_2 \left(1 + \frac{1}{1 + n} \right) \|x_n - x_{n-1}\|_1^2 \right]. \qquad (3.21)$$

From (3.16) and (3.21), we have

$$\|x_{n+1} - x_n\|_1 + \|y_{n+1} - y_n\|_2 = k_1^n\|x_n - x_{n-1}\|_1 + k_2^n\|y_n - y_{n-1}\|_2$$
$$\leq \theta^n(\|x_n - x_{n-1}\|_1 + \|y_n - y_{n-1}\|_2), \qquad (3.22)$$

where $\theta^n = \max\{k_1^n, k_2^n\}$,

$$\begin{cases} k_1^n := \sqrt{\frac{1+2d_1}{2e_1+3}} \left(\sqrt{1 - 2\sigma_1 + \delta_1^2} + \sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2\mu_1^2(L^n)^2} + \nu_1 + \rho_2\beta_2\mu_2 L^n \right), \\[2mm] k_2^n := \sqrt{\frac{1+2d_2}{2e_2+3}} \left(\sqrt{1 - 2\sigma_2 + \delta_2^2} + \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\gamma_2^2\eta_2^2(L^n)^2} + \nu_2 + \rho_1\gamma_1\eta_1 L^n \right), \\[2mm] L^n := (1 + (1 + n)^{-1}). \end{cases}$$
$$(3.23)$$

Letting $\theta^n \to \theta$ as $n \to \infty$ ($k_1^n \to k_1, k_2^n \to k_2$ as $n \to \infty$), where $\theta = \max\{k_1, k_2\}$;

$$\begin{cases} k_1 := \sqrt{\frac{1+2d_1}{2e_1+3}} \left(\sqrt{1 - 2\sigma_1 + \delta_1^2} + \sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2\mu_1^2} + \nu_1 + \rho_2\beta_2\mu_2 \right), \\[2mm] k_2 := \sqrt{\frac{1+2d_2}{2e_2+3}} \left(\sqrt{1 - 2\sigma_2 + \delta_2^2} + \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\gamma_2^2\eta_2^2} + \nu_2 + \rho_1\gamma_1\eta_1 \right). \end{cases}$$
$$(3.24)$$

Now, define the norm $\|\cdot\|_*$ on $H_1 \times H_2$ by

$$\|(x, y)\|_* = \|x\|_1 + \|y\|_2, \quad \forall(x, y) \in H_1 \times H_2. \qquad (3.25)$$

It is observe that $(H_1 \times H_2, \|\cdot\|_*)$ is a Banach space. Hence (3.22) implies that

$$\|(x_{n+1}, y_{n+1}) - (x_n, y_n)\|_* \leq \theta\|(x_n, y_n) - (x_{n-1}, y_{n-1})\|_*. \qquad (3.26)$$

By condition (3.24), it follows that $\theta < 1$. Hence $\theta_n < 1$ for sufficiently large n. Therefore, (3.26) implies that $\{(x_n, y_n)\}$ is a Cauchy sequence in $H_1 \times H_2$. Let $(x_n, y_n) \to (x, y) \in H_1 \times H_2$ as $n \to \infty$. By μ_1-$\mathcal{H}$-Lipschitz continuity of A and Iterative Algorithm 3.2, we have

$$\|u_n - u_{n-1}\|_1 \leq (1 + (1 + n)^{-1}) \, \mathcal{H}_1(A(x_n), A(x_{n-1}))$$
$$\leq (1 + (1 + n)^{-1}) \, \mu_1\|x_n - x_{n-1}\|_1. \qquad (3.27)$$

Since $\{x_n\}$ is a Cauchy sequence in H_1. Hence there exists $u \in H_1$ such that $\{u_n\} \to u$ as $n \to \infty$. Similarly, we can show that $\{v_n\} \in H_2, \{w_n\} \in H_1$ and $\{z_n\} \in H_2$ are Cauchy sequences and hence there exist $v \in H_2, w \in H_1$ and $z \in H_2$ such that $\{v_n\} \to v, \{w_n\} \to w$ and $\{z_n\} \to z$ as $n \to \infty$.

Next, we claim that $u \in A(x)$. Since $u_{n-1} \in A(x_{n-1})$, we have

$$d(u, A(x)) \leq \|u - u_{n-1}\|_1 + d(u_{n-1}, A(x))$$
$$\leq \|u - u_{n-1}\|_1 + \mathcal{H}_1(A(x_{n-1}), A(x))$$
$$\leq \|u - u_{n-1}\|_1 + \mu_1 \|x_{n-1} - x\|_1 \to 0 \text{ as } n \to \infty. \qquad (3.28)$$

Since $A(x)$ is closed, we have $u \in A(x)$. Similarly, we can show that $v \in B(y), w \in C(x)$ and $z \in D(y)$. Furthermore, continuity of the mappings $A, B, C, D, g_1, g_2, h_1, h_2, K_1, K_2, N_1, N_2, P_{K_1(x)}, P_{K_2(y)}$ and Iterative Algorithm 3.2 gives that

$$h_1(x) = P_{K_1(x)}[g_1(x) - \rho_1 N_1(u, v)], \tag{3.29}$$
$$h_2(x) = P_{K_2(x)}[g_2(y) - \rho_2 N_2(w, z)]. \tag{3.30}$$

Finally, we define

$$w_1 = P_{K_1(x)}[g_1(x) - \rho_1 N_1(u, v)], \tag{3.31}$$
$$w_2 = P_{K_2(x)}[g_2(y) - \rho_2 N_2(w, z)]. \tag{3.32}$$

Now, we estimate:

$$\|h_1(x_{n+1}) - w_1\|_1 \leq \left(\sqrt{1 - 2\sigma_1 + \delta_1^2} + \sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2\mu_1^2(L^n)^2} + \nu_1 \right) \|x_n - x\|_1$$
$$+ \rho_1\gamma_1\eta_1 L^n \|y_n - y\|_2^2, \tag{3.33}$$

and

$$\|h_2(x_{n+1}) - w_2\|_2 \leq \left(\sqrt{1 - 2\sigma_2 + \delta_2^2} + \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\gamma_2^2\eta_2^2(L^n)^2} + \nu_2 \right) \|y_n - y\|_2$$
$$+ \rho_2\beta_2\mu_2 L^n \|x_n - x\|_1^2. \tag{3.34}$$

Now, it follows from (3.25), (3.33) and (3.34) that

$$\|(h_1(x_{n+1}), h_2(y_{n+1})) - (w_1, w_2)\|_* = \|h_1(x_{n+1}) - w_1\|_1 + \|h_2(y_{n+1}) - w_2\|_2$$
$$\leq \theta^n (\|x_n - x\|_1 + \|y_n - y\|_2)$$
$$\to 0, \text{ as } n \to \infty. \tag{3.35}$$

Thus,

$$h_1(x) = w_1 = P_{K_1(x)}[g_1(x) - \rho_1 N_1(u, v)], \tag{3.36}$$
$$h_2(y) = w_2 = P_{K_2(y)}[g_2(y) - \rho_2 N_2(w, z)]. \tag{3.37}$$

By Lemma 3.1, it follows that (x, y, u, v, w, z) is a solution of SMEGQVIP (2.1)-(2.2). This completes the proof. $\square$

Remark 3.4.

(i) *For $i = 1, 2$, it is clear that $\sigma_i \leq \delta_i$. Further, $\theta < 1$ and condition (3.9) holds for some suitable set values of constants, for example,*

- $\alpha_1 = 0.4, \beta_1 = 0.4, \gamma_1 = 0.1, \sigma_1 = 0.1, \delta_1 = 0.2, \mu_1 = 1, \eta_1 = 1, \nu_1 = 0.1, \rho_1 = 0.1.$

- $\alpha_2 = 0.2, \beta_2 = 0.3, \gamma_2 = 0.2, \sigma_2 = 0.2, \delta_2 = 0.3, \mu_2 = 1, \eta_2 = 1, \nu_2 = 0.2, \rho_2 = 0.2.$

(ii) *The proof of theorem presented in this paper for SMEGQVIP (2.1)-(2.2) under the assumption of relaxed (d, e)-cocoercivity on mappings h_1 and h_2, need further research effort.*

(iii) *Using the method presented in this paper, one can extend the existence result for the system of n-generalized quasi-variational inequality problems.*

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