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On the Spaces of λ-Convergent Sequences and Almost Convergence

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Abstract : The sequence space l_∞^λ has been defined and the various classes of infinite matrices have been characterized by Mursaleen and Noman (see [1]). In this paper we characterize the classes $(l_\infty^\lambda : f_\infty)$, $(l_\infty^\lambda : f)$ and $(l_\infty^\lambda : f_0)$, where f_∞, f and f_0 denote respectively almost bounded sequences, almost convergent sequences and almost convergent null sequences.

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1 Preliminaries, Background and Notations

A sequence space is defined to be a linear space of real or complex sequences. Throughout the paper N, R and C denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let ω denote the space of all sequences (real or complex); l_∞ and c respectively, denotes the space of all bounded sequences and the space of convergent sequences. Also, by cs we denote the space of all convergent series.

Let X, Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in N$. Then, the matrix A defines the $A-$transformation from X *into* Y, if for every sequence $x = (x_k) \in X$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk}x_k$.

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For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(X : Y)$, we denote the class of all such matrices. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{x = (x_k) : x = (x_k) \in \omega\}. \tag{1.1}$$

Let $S : l_\infty \to l_\infty$ be the shift operator defined by $(Sx)_n = x_{n+1}$ for all $n \in N$. A Banach limit L is defined on l_∞ as a non negative linear functional such that $L(Sx) = L(x)$ and $L(e) = 1, e = (1, 1, 1, ...)$ (see [2]). A sequence space is said to be almost convergent to the generalized limit α if all Banach limits of x are α (see [3]). We denote the set of almost convergent sequences by f, *i.e.*

$$f = \left\{x \in l_\infty : \lim_m t_{mn}(x) = \alpha, \text{ uniformly in } n\right\}$$

where

$$t_{mn}(x) = \frac{1}{m+1}\sum_{j=0}^{m} x_{j+n},\ t_{-1,n} = 0 \text{ and } \alpha = f - \lim x.$$

Nanda (see [4]) has defined a new set of sequences f_∞. We call it as the set of all f-bounded sequences, that is

$$f_\infty = \left\{x \in l_\infty : \sup_m |t_{mn}(x)| < \infty\right\}.$$

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors viz., (see [1, 5–7]).

Let $\lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive reals tending to infinity, that is $0 < \lambda_0 < \lambda_1 < \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$. The sequence spaces l_∞^λ have been defined as the sets of all sequences whose $\wedge$-transform is in l_∞, *i.e.*

$$l_\infty^\lambda = \left\{x \in \omega : \sup_n |\wedge_n (x)| < \infty\right\},$$

where

$$\wedge_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})x_k,\ (k \in N).$$

With the notation of (1.1) that $l_\infty^\lambda = (l_\infty)_\lambda$.

2 Main Results

We shall assume throughout the text that the sequences $x = (x_k)$ and $y = (y_k)$ are connected by the relation, that is y is the $\wedge$-transform of x, where

$$x_k = \sum_{j=k-1}^{k}(-1)^{k-j}\frac{\lambda_j}{\lambda_k - \lambda_{k-1}}y_j,\ (k \in N). \tag{2.1}$$

For brevity in notation, we shall write

$$t_{mn}(Ax)=\frac{1}{m+1}\sum_{j=0}^{m}A_{n+j}(x)=\sum_{k}a(n,k,m)x_k$$

where

$$a(n,k,m)=\frac{1}{m+1}\sum_{j=0}^{m}a_{n+j,k};\ (n,k,m\in N).$$

Also,

$$\widehat{a}(n,k,m)=\widehat{\triangle}\left[\frac{a(n,k,m)}{\lambda_k-\lambda_{k-1}}\right]\lambda_k$$

where

$$\widehat{\triangle}\left[\frac{a(n,k,m)}{\lambda_k-\lambda_{k-1}}\right]\lambda_k=\left[\frac{a(n,k,m)}{\lambda_k-\lambda_{k-1}}-\frac{a(n,k+1,m)}{\lambda_{k+1}-\lambda_k}\right]\lambda_k,\ (k\in N).$$

We denote by X^β the $\beta-$ dual of the sequence space X and mean the set of all sequences $x=(x_k)$ such that $xy=(x_ky_k)\in cs$ for all $y=(y_k)\in X$. Now, we give the following lemmas which will be needed in proving the main results.

Lemma 2.1 ([1])**.** *Define the sets a_1^λ and a_2^λ as follows,*

$$a_1^\lambda=\left\{a=(a_k)\in\omega:\sum_k\left|\widehat{\triangle}\left(\frac{a_k}{\lambda_k-\lambda_{k-1}}\right)\lambda_k\right|<\infty\right\}$$

and

$$a_2^\lambda=\left\{a=(a_k)\in\omega:\lim_k\left(\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}a_k\right)=0\right\}.$$

Then, $\left[l_\infty^\lambda\right]^\beta=a_1^\lambda\cap a_2^\lambda$.

Lemma 2.2 ([8])**.** $f\subset f_\infty$.

Theorem 2.3. $A\in\left(l_\infty^\lambda:f_\infty\right)$ *if and only if*

$$\sup_{n,m\in N}\sum_k|\widehat{a}(n,k,m)|<\infty \tag{2.2}$$

and

$$\left(\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}a_{nk}\right)\in c_0, \tag{2.3}$$

for all $n\in N$.

Proof. Sufficiency: Suppose the conditions (2.2) and (2.3) holds and $x \in l_\infty^\lambda$. Then $\{a_{n,k}\}_{n,k\in N} \in \left[l_\infty^\lambda\right]^\beta$ for every $n \in N$, the A- transform of x exists. Since $x \in (l_\infty^\lambda)$, by hypothesis, and $l_\infty^\lambda \cong l_\infty$ (see [1]), we have $y \in l_\infty$. Thus, we can find $K > 0$ such that $\sup_k |y_k| \leq K$,

$$|t_{mn}(Ax)| = \left|\sum_k a(n,k,m)x_k\right| = \left|\sum_k \widehat{a}(n,k,m)y_k\right| \\ \leq \sum_k |\widehat{a}(n,k,m)|\,|y_k| \leq K\sum_k \ |\widehat{a}(n,k,m)|\,.$$

Taking $\sup_{n,m}$ on both sides, we get $Ax \in f_\infty$ for every $x \in l_\infty^\lambda$.

Necessity: Suppose $A \in (l_\infty^\lambda : f_\infty)$. Then Ax exists for every $x \in l_\infty^\lambda$ and this implies that $\{a_{n,k}\}_{k\in N} \in \left[l_\infty^\lambda\right]^\beta$ for every $n \in N$, the necessity of (2.3) is immediate. Now, $\sum_k a(n,k,m)x_k$ exists for each m, n and $x \in l_\infty^\lambda$, the sequence $\{a(n,k,m)\}_{n,m\in N}$ define the continuous linear functionals ϕ_{mn} on l_∞^λ, by

$$\phi_{mn}(x) = \sum_k a(n,k,m)x_k,\ (n,k,m \in N).$$

Since $l_\infty^\lambda \cong l_\infty$ (see [1]), it should follow with (2.1) that $\|\phi_{mn}(x)\| = \|\widehat{a}(n,k,m)\|$ holds for every $k \in N$. This implies that the functionals defined by ϕ_{mn} on l_∞^λ are pointwise bounded, so by uniform boundedness principle, there exists $M > 0$ such that $\|\phi_{mn}(x)\| \leq M$ for every $m, n \in N$. Thus, we conclude that

$$\sup_{m,n}|\phi_{mn}(x)| = \sup_{m,n}\left|\sum_k a(n,k,m)x_k\right| = \sup_{m,n}\left|\sum_k \widehat{a}(n,k,m)y_k\right| \leq M.$$

This gives that $\sup_{m,n}\sum_k|\widehat{a}(n,k,m)| < \infty$, which shows the necessity of the condition (2.2) and the result follows. □

Theorem 2.4. $A \in \left(l_\infty^\lambda : f\right)$ *if and only if (2.2), (2.3) and*

$$\lim_m \widehat{a}(n,k,m) = \widehat{\beta}_k,\ \textit{uniformly in } n \textit{ and for each } k \in\ N, \tag{2.4}$$

$$\lim_m \sum_k \left|\widehat{a}(n,k,m) - \widehat{\beta}_k\right| = 0,\ \textit{uniformly in } n. \tag{2.5}$$

Proof. Sufficiency: Suppose the conditions (2.2), (2.3), (2.4) and (2.5) holds and $x \in l_\infty^\lambda$. Then Ax exists and at this stage we observe with the help of (2.4) and (2.5) that

$$\sum_{j=0}^{k}\left|\widehat{\beta}_j\right| = \sup_{m,n}\sum_j|\widehat{a}(n,j,m)| < \infty,$$

holds for every $k \in N$. Now for $x \in l_\infty$ there exists $K > 0$ such that $\sup_k |y_k| < K$. Now, for $\epsilon > 0$, choose a fixed $k_0 \in N$, there is some $m_0 \in N$ such that

$$\left|\sum_{k=0}^{k_0}\left(\widehat{a}(n,k,m) - \widehat{\beta}_k\right) y_k\right| < \frac{\epsilon}{2}$$

holds for every $m \geq m_0$ and $k_0 \in N$. Also, by (2.5), there is some $m_1 \in N$ such that

$$\sum_{k=k_{0+1}}^{\infty}\left|\widehat{a}(n,k,m) - \widehat{\beta}_k\right| < \frac{\epsilon}{2K}$$

holds for every $m \geq m_1$ and uniformly in n. Thus, we have

$$\begin{aligned}
&\left|\frac{1}{m+1}\sum_{j=0}^{m}(Ax)_{n+j} - \sum_k \widehat{\beta}_k y_k\right| \\
&\qquad = \left|\sum_k\left(\widehat{a}(n,k,m) - \widehat{\beta}_k\right) y_k\right| \\
&\qquad \leq \left|\sum_{k=0}^{k_0}\left(\widehat{a}(n,k,m) - \widehat{\beta}_k\right) y_k\right| + \sum_{k=k_0+1}^{\infty}\left|\left(\widehat{a}(n,k,m) - \widehat{\beta}_k\right) y_k\right| \\
&\qquad < \frac{\epsilon}{2} + \sum_{k=k_0+1}^{\infty}\left|\left(\widehat{a}(n,k,m) - \widehat{\beta}_k\right)\right| |y_k| \\
&\qquad < \frac{\epsilon}{2} + K\frac{\epsilon}{2K} = \epsilon,
\end{aligned}$$

for sufficiently large m and uniformly in n. Hence, $Ax \in f$, which proves the sufficiency.

Necessity: Suppose that $A \in \left(l_\infty^\lambda : f\right)$. Then, since $f \subset f_\infty$ (by Lemma 2.2), the necessities of (2.2) and (2.3) are immediately obtained from Theorem 2.3. To prove the necessity of (2.4), consider the sequence $B = (b_{nk})$ for every $n, k \in N$, where

$$b_{nk} = \begin{cases} (-1)^{n-k}\frac{\lambda_k}{\lambda_n - \lambda_{n-1}}, & \text{for } \ n-1 \leq k \leq n, \\ 0, & \text{for } \ k < n-1 \ \textit{or} \ k > n. \end{cases}$$

Since Ax exists and is in f for each $x \in l_\infty^\lambda$, one can easily see that

$$Ab = \left\{\widehat{\triangle}\left(\frac{a_{nk}}{\lambda_k - \lambda_{k-1}}\right)\lambda_k\right\}_{n\in N} \in f$$

for all $k \in N$, which proves the necessity of (2.5). Similarly, by taking $x = e \in l_\infty^\lambda$, we shall get

$$Ax = \left\{\sum_k \widehat{\triangle}\left(\frac{a_{nk}}{\lambda_k - \lambda_{k-1}}\right)\lambda_k\right\}_{n\in N} \in f$$

which proves the necessity of (2.4). This concludes the proof. □

Note that if we replace f by f_0, then Theorem 2.3 is reduced to the following corollary:

Corollary 2.5. *$A \in \left(l_\infty^\lambda : f\right)$ if and only if (2.2), (2.3), (2.4) and (2.5) holds with $\widehat{\beta}_k = 0$ for each $k \in N$.*

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