Thai Journal of Mathematics
Volume 11 (2013) Number 2 : 371–382
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Fuzzy Interior Ideals with Thresholds $(s, t]$ in Ordered Semigroups[1]

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Abstract : We investigate some interesting properties of fuzzy interior ideals with thresholds $(s, t]$ in ordered semigroups and characterize an ordered semigroup by means of fuzzy interior ideals with thresholds $(s, t]$. We also prove that in a regular (resp. intra-regular and semisimple) ordered semigroup the concept of fuzzy ideals with thresholds $(s, t]$ and fuzzy interior ideals with thresholds $(s, t]$ coincide.

Keywords : fuzzy interior ideals; regular ordered semigroup.
2010 Mathematics Subject Classification : 06F05.

1 Introduction

For any a set X, a fuzzy subset of X is a mapping $H : X \to [0, 1]$ where $[0, 1]$ is an unit closed interval of the real number. Since the concept of a fuzzy subset was introduced by Zadeh [1], many papers about fuzzy subsets were developed quickly and its application is used to real analysis, topology, logic, set theory, group theory, groupoid, semigroup, etc. Rosenfeld [2] introduced the notion of a fuzzy subgroup by using the idea of a fuzzy subset. Kuroki [3] investigated the ideal of a fuzzy semigroup and gave some properties of fuzzy ideals and fuzzy bi-ideals of semigroups. Hong, Jun and Meng [4] investigated some properties and considered the characterization of a fuzzy interior ideal of a semigroup. Jun and Song [5]

[1]This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.
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introduced the concept of an $(\in, \in\vee q)$-fuzzy interior ideal in semigroups by using the idea of "belongingness" and "quasi-coincidence" of a fuzzy point and a fuzzy subset. The concept of an $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy interior ideal and a fuzzy interior ideal with thresholds $(s, t]$ in semigroups have been introduced by Zhan and Jun in [6]. In particular, a fuzzy interior ideal with thresholds $(0, 1]$ of a semigroup is a fuzzy interior ideal of a semigroup and an $(\in, \in \vee q)$-fuzzy interior ideal of a semigroup is a fuzzy interior ideal with thresholds $(0, 0.5]$ of a semigroup. They described the relationships among fuzzy interior ideals, $(\in, \in\vee q)$-fuzzy interior ideals and $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy interior ideals of semigroups and gave some characterization of $[H]_p := \{x \in S | x_t \in \vee qH\}$ in term of an $(\in, \in\vee q)$-fuzzy interior ideal H of a semigroup S.

An ordered semigroup S is a semigroup together with a partial order $\leq$ that is compatible with the semigroup operation, i.e., $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$ for all $x, y, z \in S$. Kehayopulu and Tsingelis [7] first studied the fuzzy subsets in ordered groupoids and extended the notion of a fuzzy interior ideal in semigroups into ordered semigroups in [8]. Furthermore, they proved that in regular and intra-regular ordered semigroups, the concept of fuzzy ideals and the fuzzy interior ideals coincide. The notion of the $(\in, \in\vee q)$-fuzzy interior ideal in an ordered semigroup was introduced by Khan and Shabir [9] and the related properties of ordered semigroups in terms of $(\in, \in\vee q)$-fuzzy interior ideals were investigated.

A further aim of this paper is to introduce a new type of a fuzzy interior ideal of an ordered semigroup, called a fuzzy interior ideal with thresholds $(s, t]$ and extend the concept of fuzzy interior ideals of ordered semigroups to fuzzy interior ideals with thresholds $(s, t]$ of ordered semigroups. One of aims of this paper show that a fuzzy subset H in an ordered semigroup S is a fuzzy interior ideal with thresholds $(s, t]$ of S if and only if a level subset $U(H; p)(\neq \emptyset)$ of H is an interior ideal of S for all $p \in (s, t]$ and some interesting properties are investigated. Finally, we show that in a regular [resp. intra-regular, semisimple] ordered semigroup, the concept of a fuzzy interior ideal with thresholds $(s, t]$ and a fuzzy ideal with thresholds $(s, t]$ coincide.

2 Preliminaries

In this section we give some definitions and theorems, most of them are well known, which will be used in the next sections. An *ordered semigroup* is a poset S, at the same time a semigroup, such that $x \leq y$ then $xz \leq yz$ and $zx \leq zy$ for all $x, y, z \in S$. A nonempty subset A of an ordered semigroup S is called a *left ideal* [resp. *right ideal*] [8] of S if (1) $SA \subseteq A$ [resp. $AS \subseteq A$] and (2) if $x \in A$, $S \ni y \leq x$ then $y \in A$. By an *ideal*[8] of an ordered semigroup S, we mean a nonempty subset of S which is both left ideal and right ideal of S. A nonempty subset A of an ordered semigroup S is called an *interior ideal* [8] of S if (1) $A^2 \subseteq A$, (2) $SAS \subseteq A$ and (3) for $x, y \in S$ if $x \in A$ and $S \ni y \leq x$ then $y \in A$. By the definition of an interior ideal, we see that every ideal of an ordered semigroup S is an interior ideal of S.

For $A \subseteq S$, we denote $(A] := \{t \in S | t \leq a$ for some $a \in A\}$. It is often convenient to write $(a]$ instead of $(\{a\}]$ if $A = \{a\}$. An ordered semigroup S is called *regular* [10] if for each $x \in S$ there is $a \in S$ such that $x \leq xax$. Equivalent Definition: (1) $A \subseteq ASA$ for each $A \subseteq S$. (2) $x \in (xSx]$ for all $x \in S$. An ordered semigroup S is called *intra-regular* [10] if for each $x \in S$ there are $a, b \in S$ such that $x \leq axxb$. Equivalent Definition: (1) $A \subseteq SA^2S$ for each $A \subseteq S$. (2) $x \in (Sx^2S]$ for all $x \in S$. An ordered semigroup S is called *semisimple* [10] if for each $x \in S$ there exist $a, b, c \in S$ such that $x \leq axbxc$. Equivalent Definition: (1) $A \subseteq SASAS$ for each $A \subseteq S$. (2) $x \in (SxSxS]$ for each $x \in S$.

We see that every interior ideal of a regular [resp. intra-regular, semisimple] ordered semigroup S is an ideal of S, which is proved by Kuroki [3].

Let $(S, \cdot, \leq)$ be an ordered semigroup. By a *fuzzy subset* f of S, we mean a mapping $f : S \to [0,1]$ where $[0,1]$ is the unit closed interval of the real number.

A fuzzy subset H of an ordered semigroup S is said to be a *fuzzy left* [resp. *right*] *ideal* [8] of S if

(1) $H(xy) \geq H(y)$ [resp. $H(xy) \geq H(x)$ for all $x, y \in S$ and

(2) for $x, y \in S$ if $x \leq y$ then $H(x) \geq H(y)$.

Both a fuzzy left ideal and fuzzy right ideal is called a *fuzzy ideal* [8] of S.

A fuzzy subset H of an ordered semigroup S is said to be a *fuzzy interior ideal* [8] of S if

(1) $H(xy) \geq \min\{H(x), H(y)\}$ for all $x, y \in S$,

(2) $H(xhy) \geq H(h)$ for all $h, x, y \in S$ and

(3) for $x, y \in S$ if $x \leq y$ then $H(x) \geq H(y)$.

By above definition, it is clear that every fuzzy ideal of an ordered semigroup S is a fuzzy interior ideal of S.

A fuzzy subset H of an ordered semigroup S is said to be an $(\in, \in \vee q)$-*fuzzy left* [resp. *right*] *ideal* [10] of S if

(1) $H(xy) \geq \min\{H(y), 0.5\}$ [resp. $H(xy) \geq \min\{H(x), 0.5\}$ for all $x, y \in S$,

(2) for $x, y \in S$ if $x \leq y$ then $H(x) \geq \min\{H(y), 0.5\}$.

Both an $(\in, \in \vee q)$-fuzzy left ideal and fuzzy right ideal is called an $(\in, \in \vee q)$-*fuzzy ideal* [10] of S. It is easy to check that every fuzzy left (resp. right) ideal of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S.

A fuzzy subset H of an ordered semigroup S is said to be an $(\in, \in \vee q)$-*fuzzy interior ideal* [10] of S if

(1) $H(xy) \geq \min\{H(x), H(y), 0.5\}$ for all $x, y \in S$,

(2) $H(xhy) \geq \min\{H(h), 0.5\}$ for all $h, x, y \in S$ and

(3) for $x, y \in S$ if $x \leq y$ then $H(x) \geq \min\{H(y), 0.5\}$.

It is clear from this definition that every fuzzy interior ideal of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy interior ideal of S and every $(\in, \in \vee q)$-fuzzy ideal of S is an $(\in, \in \vee q)$-fuzzy interior ideal of S.

For any subset A of S, the *characteristic function* χ_A [8] is the fuzzy subset of S defined as follows :

$$\chi_A : S \to [0,1] | x \to \chi_A(x) := \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

As observe above, we have the characteristic function χ_A is a fuzzy subset of S.

For fuzzy subset H of an ordered semigroup S and $p \in (0, 1]$, we define

$$U(H; p) := \{x \in S | H(x) \geq p\}$$

which is called a *level subset* [11] of H.

3 Main Results

In this section, we introduce a fuzzy interior ideal with thresholds $(s, t]$ and give some interesting properties of a fuzzy interior with thresholds $(s, t]$. The following definition of a fuzzy interior ideal with thresholds $(s, t]$ is defined as follows.

Let $s, t \in [0, 1]$ and $s < t$. A fuzzy subset H of an ordered semigroup S is said to be a *fuzzy interior ideal with thresholds* $(s, t]$ of S if

(1) $\max\{H(xy), s\} \geq \min\{H(x), H(y), t\}$ for all $x, y \in S$,

(2) $\max\{H(xhy), s\} \geq \min\{H(h), t\}$ for all $h, x, y \in S$ and

(3) for $x, y \in S$,$x \leq y$ implies $\max\{H(x), s\} \geq \min\{H(y), t\}$.

By the definition we see that a fuzzy interior ideal with thresholds $(s, t]$ is a generalization of a fuzzy interior ideal and an $(\in, \in \vee q)$-fuzzy interior ideal.

The proof of the next lemma is straightforward and is omitted.

Lemma 3.1. *For any $s, t \in [0, 1]$ and $s < t$, every fuzzy interior ideal of an ordered semigroup S is a fuzzy interior ideal with thresholds $(s, t]$ of S.*

In the following theorem we characterize fuzzy interior ideals with thresholds $(s, t]$ in terms of its level subsets.

Theorem 3.2. *For any $s, t \in [0, 1]$ and $s < t$, a fuzzy subset H of an ordered semigroup S is a fuzzy interior ideal with thresholds $(s, t]$ of S if and only if $U(H; p)(\neq \emptyset)$ is an interior ideal of S for all $p \in (s, t]$.*

Proof. Suppose that H is a fuzzy interior ideal with thresholds $(s, t]$ of S. Let $h, x, y \in S$ and $p \in (s, t]$. If $x, y \in U(H; p)$ then $H(x) \geq p$ and $H(y) \geq p$. We then have min $\{H(x), H(y)\} \geq p$ and so

$$s < p = \min\{p, t\} \leq \min\{H(x), H(y), t\} \leq \max\{H(xy), s\} = H(xy).$$

This implies that $xy \in U(H;p)$. Next, if $h \in U(H;p)$ then $H(h) \geq p$. Thus

$$s < p = \min\{p,t\} \leq \min\{H(h),t\} \leq \max\{H(xhy),s\} = H(xhy).$$

It follows directly from assumption that $xhy \in U(H;p)$. Next, let $x \leq y$ and $y \in U(H;p)$. Thus $H(y) \geq p$ and so

$$s < p = \min\{p,t\} \leq \min\{H(y),t\} \leq \max\{H(x),s\} = H(x).$$

We obtain that $x \in U(H;p)$. Hence $U(H;p)$ is an interior ideal of S for all $p \in (s,t]$.

Conversely, let $U(H;p)(\neq \emptyset)$ is a fuzzy interior ideal with thresholds $(s,t]$ of S for all $p \in (s,t]$. Assume that there are $x,y \in S$ such that

$$\max\{H(xy),s\} < \min\{H(x),H(y),t\}.$$

We have $p \in (s,t]$, $H(xy) < p$ and $x,y \in U(H;p)$ where $p := \min\{H(x),H(y),t\}$. By hypothesis, we have $xy \in U(H;p)$ and so $H(xy) \geq p$, which contradicts with $H(xy) < p$. Therefore $\max\{H(xy),s\} \geq \min\{H(x),H(y),t\}$. Assume that there exist $h,x,y \in S$ such that

$$\max\{H(xhy),s\} < \min\{H(h),t\}.$$

Choose $q := \min\{H(h),t\}$, we get $q \in (s,t]$, $H(xy) < q$ and $x,y \in U(H;q)$. We have $xhy \in U(H;q)$ since hypothesis, and so $H(xhy) \geq q$, which is a contradiction. Hence $\max\{H(xhy),s\} \geq \min\{H(h),t\}$. Finally, assume that there are $x,y \in S$ such that

$$x \leq y \text{ and } \max\ \{H(x),s\} < \min\{H(y),t\}.$$

Thus $r \in (s,t]$, $H(x) < r$ and $y \in U(H;r)$ where $r := \min\{H(y),t\}$. This give $x \in U(H;r)$ by hypothesis, and so $H(x) \geq r$, which contradicts with $H(x) < r$. Hence $\max\{H(x),s\} \geq \min\ \{H(y),t\}$. By definition of a fuzzy interior ideal with thresholds $(s,t]$, we have H is a fuzzy interior ideal with thresholds $(s,t]$ of S. □

The following results may be deduced from above theorem.

Corollary 3.3. *A fuzzy subset H of an ordered semigroup S is a fuzzy interior ideal of S if and only if the set $U(H;p)(\neq \emptyset)$ is an interior ideal of S for all $p \in (0,1]$.*

Corollary 3.4. *A fuzzy subset H of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if the set $U(H;p)(\neq \emptyset)$ is an interior ideal of S for all $p \in (0,0.5]$.*

The proof of the following is straightforward.

Theorem 3.5. *Let H be a fuzzy subset of an ordered semigroup S, $s,t \in [0,1]$ and $s < t$. If $s \leq H(x) \leq t$ for all $x \in S$ then a fuzzy interior ideal with thresholds $(s,t]$ of S is a fuzzy interior ideal of S.*

Corollary 3.6. *Let H be a fuzzy subset of an ordered semigroup S. If $H(x) \leq 0.5$ for all $x \in S$ then an $(\in, \in \vee q)$-fuzzy interior ideal of S is a fuzzy interior ideal of S.*

In the next theorem we shall now derive some sufficient condition of a fuzzy subset H in order that H is a fuzzy interior ideal with thresholds $(s, t]$.

Theorem 3.7. *Let H be a fuzzy subset of an ordered semigroup S, $s, t \in [0, 1]$ and $s < t$. If A is an interior ideal of S such that*

(i) $H(x) \geq t$ for all $x \in A$ and

(ii) $H(x) \leq s$ for all $x \in S \backslash A$,

then H is a fuzzy interior ideal with thresholds $(s, t]$ of S.

Proof. Let A be an interior ideal with conditions (i) and (ii) hold. Assume that there are $x, y \in S$ such that $\max\{H(xy), s\} < \min\{H(x), H(y), t\}$. If $\min\{H(x), H(y)\} < t$ then $\min\{H(x), H(y)\} \leq s$. By assumption, we get

$$\max\{H(xy), s\} < \min\{H(x), H(y), t\} = \min\{H(x), H(y)\} \leq s,$$

which is a contradiction. If $\min\{H(x), H(y)\} \geq t$ then $x, y \in A$. We have $xy \in A$ since A is an interior ideal of S. We get

$$\max\{H(xy), s\} < \min\{H(x), H(y), t\} = t,$$

it follows that $\max\{H(xy), s\} \leq s$, which contradicts with $xy \in A$. We obtain that $\max\{H(xy), s\} \geq \min\{H(x), H(y), t\}$ for all $x, y \in S$. Next, assume that there exist $h, x, y \in S$ such that $\max\{H(xhy), s\} < \min\{H(h), t\}$. If $H(h) < t$ then $H(h) \leq s$. By assumption, we have

$$\max\{H(xhy), s\} < \min\{H(h), t\} = H(h) \leq s,$$

which is a contradiction. If $H(h) \geq t$ then $h \in A$. We get $xhy \in A$ because A is an interior ideal of S. We obtain that

$$\max\{H(xhy), s\} < \min\{H(h), t\} = t,$$

it follows that $\max\{H(xhy), s\} \leq s$, which contradicts with $xhy \in A$. We have $\max\{H(xhy), s\} \geq \min\{H(h), t\}$ for all $h, x, y \in S$. Finally, assume that there exist $x, y \in S$ such that $x \leq y$ and $\max\{H(x), s\} < \min\{H(y), t\}$. If $H(y) < t$ then $H(y) \leq s$. By assumption, we get

$$s \leq \max\{H(x), s\} < \min\{H(y), t\} = H(y) \leq s,$$

which is a contradiction. If $H(y) \geq t$ then $y \in A$. We have $x \in A$ since A is an interior ideal of S. We have

$$\max\{H(x), s\} < \min\{H(y), t\} = t,$$

it follows that $\max\{H(x), s\} \leq s$, which contradicts with $x \in A$. We obtain that $\max\{H(x), s\} \geq \min\{H(y), t\}$ for all $x, y \in S$. Therefore H is a fuzzy interior ideal with thresholds $(s, t]$ of S. □

It follows from above theorem that

Corollary 3.8. *Let H be a fuzzy subset of an ordered semigroup S. If A is an interior ideal of S such that*

(i) $H(x) \geq 0.5$ *for all* $x \in A$ *and*

(ii) $H(x) = 0$ *for all* $x \in S\backslash A$,

then H is an $(\in, \in \vee q)$-fuzzy interior ideal of S.

The following lemma is used in Theorem 3.10.

Lemma 3.9 ([9])**.** *Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is an interior ideal of S if and only if χ_A is a fuzzy interior ideal of S.*

The next theorem we discuss an interior ideal of an ordered semigroup S in term of a characteristic function of the interior ideal of S.

Theorem 3.10. *Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$, $s, t \in [0,1]$ and $s < t$. Then A is an interior ideal of S if and only if χ_A is a fuzzy interior ideal with thresholds $(s,t]$ of S.*

Proof. It is readily seen that $s < 1$ and $t > 0$. The if part follows from Lemma 3.1 and Lemma 3.9.

Conversely, suppose that χ_A is a fuzzy interior ideal with thresholds $(s,t]$ of S. Let $x, y \in A$ thus $\chi_A(x) = 1$ and $\chi_A(y) = 1$. By assumption, we have $\max\{\chi_A(xy), s\} \geq \min\{\chi_A(x), \chi_A(y), t\} = t$. Since $t > 0$ thus $\max\{\chi_A(xy), s\} > 0$, which implies that $\max\{\chi_A(xy), s\} = 1$. We deduce that $\chi_A(xy) = 1$ since $s < 1$, and implies that $xy \in A$. Next, let $h \in A$ thus $\chi_A(h) = 1$. We obtain that $\max\{\chi_A(xhy), s\} \geq \min\{\chi_A(h), t\} = t$ for all $x, y \in S$ by assumption. Then by $t > 0$ we have $\max\{\chi_A(xhy), s\} > 0$, which implies that $\max\{\chi_A(xhy), s\} = 1$. We get $\chi_A(xhy) = 1$ because $s < 1$. Hence $xhy \in A$. Finally, let $x \leq y \in A$. We have $\chi_A(y) = 1$. By assumption, we obtain that $\max\{\chi_A(x), s\} \geq \min\{\chi_A(y), t\} = t$. We get $\max\{\chi_A(x), s\} > 0$ since $t > 0$, which implies that $\max\{\chi_A(x), s\} = 1$, and so $\chi_A(x) = 1$ since $s < 1$. Hence $x \in A$. Therefore A is an interior ideal of S. □

Next, we will investigate relations between a fuzzy interior ideal with thresholds $(s,t]$ of an ordered semigroup S and a fuzzy ideal with thresholds $(s,t]$ of S. A fuzzy subset H of an ordered semigroup S is said to be a *fuzzy left [resp. right] ideal with thresholds (s,t]* [12] of S if

(1) $\max\{H(xy), s\} \geq \min\{H(y), t\}$ [resp. $\min\{H(x), t\}$] for all $x, y \in S$ and

(2) for $x, y \in S$,$x \leq y$ implies $\max\{H(x), s\} \geq \min\{H(y), t\}$.

A fuzzy subset S of an ordered semigroup S is called a *fuzzy ideal with thresholds* $(s,t]$ of S if it is both a fuzzy left ideal with thresholds $(s,t]$ and a fuzzy right ideal with thresholds $(s,t]$ of S.

Proposition 3.11. *Every fuzzy ideal with thresholds $(s,t]$ of an ordered semigroup S is a fuzzy interior ideal with thresholds $(s,t]$ of S.*

Proof. Let $h, x, y \in S$ and $s, t \in [0,1]$ such that $s < t$. By definition of a fuzzy ideal with thresholds $(s,t]$ of S, we obtain that

$$\max\{H(xy), s\} \geq \min\{H(x), t\} \geq \min\{H(x), H(y), t\},$$
$$x \leq y \text{ implies } \max\{H(x), s\} \geq \min\{H(y), t\} \text{ and}$$
$$\max\{H(xhy), s\} \geq \min\{H(hy), t\}.$$

If $H(hy) \geq s$ then

$$\begin{aligned}\max\{H(xhy), s\} &\geq \min\{H(hy), t\}\\ &= \min\{\max\{H(hy), s\}, t\}\\ &\geq \min\{\min\{H(h), t\}, t\}\\ &= \min\{H(h), t\}.\end{aligned}$$

If $H(hy) < s$ then

$$\begin{aligned}\max\{H(xhy), s\} &\geq s\\ &= \max\{H(hy), s\}\\ &\geq \min\{H(y), t\}.\\ &= \min\{H(h), t\}.\end{aligned}$$

Hence H is a fuzzy interior ideal with thresholds $(s,t]$ of S. □

Next, we will prove that the concepts of a fuzzy ideal with thresholds $(s,t]$ and a fuzzy interior ideal with thresholds $(s,t]$ in a regular (resp. intra-regular, semisimple) ordered semigroup coincide.

Theorem 3.12. *Let S be a regular ordered semigroup and $s, t \in [0,1]$ such that $s < t$. Then every fuzzy interior ideal with thresholds $(s,t]$ of S is a fuzzy ideal with thresholds $(s,t]$ of S.*

Proof. Let H be a fuzzy interior ideal with thresholds $(s,t]$ of a regular ordered semigroup S and $x, y \in S$. There exists $a \in S$ such that $x \leq xax$. Since $xy \leq xaxy$, it follows that $\max\{H(xy), s\} \geq \min\{H(xaxy), t\}$.
Case $H(xaxy) < s$. We obtain that $\max\{H(xaxy), s\} = s$. Thus

$$\max\{H(xy), s\} \geq s = \max\{H(xaxy), s\} \geq \min\{H(x), t\}.$$

Case $H(xaxy) \geq s$. We have $\max\{H(xaxy), s\} = H(xaxy)$. If $H(xaxy) < t$ then

$$\begin{aligned}\max\{H(xy), s\} &\geq \min\{H(xaxy), t\}\\ &= H(xaxy)\\ &= \max\{H(xaxy), s\}\\ &\geq \min\{H(x), t\}.\end{aligned}$$

If $H(xaxy) \geq t$ then

$$\max\{H(xy), s\} \geq \min\{H(xaxy), t\} = t \geq \min\{H(x), t\}.$$

Thus H is a fuzzy right ideal with thresholds $(s, t]$ of S. Similarly, we obtain that H is a fuzzy left ideal with thresholds $(s, t]$ of S. Hence H is a fuzzy ideal with thresholds $(s, t]$ of S. □

Combining Proposition 3.11 and Theorem 3.12, we obtain the following:

Theorem 3.13. *In a regular ordered semigroup S, the concept of a fuzzy ideal with thresholds $(s, t]$ of S and a fuzzy interior ideal with thresholds $(s, t]$ of S coincide where $s, t \in [0, 1]$ and $s < t$.*

By above theorem, we obtain that

Corollary 3.14. *In a regular ordered semigroup S, the concept of a fuzzy ideal of S and a fuzzy interior ideal of S coincide.*

Corollary 3.15. *In a regular ordered semigroup S, the concept of an $(\in, \in \vee q)$-fuzzy ideal of S and an $(\in, \in \vee q)$-fuzzy interior of S coincide.*

Next, we consider fuzzy interior ideals with thresholds $(s, t]$ of an intra-regular ordered semigroup. Then we have

Theorem 3.16. *Let S be an intra-regular ordered semigroup and $s, t \in [0, 1]$ such that $s < t$. Then every fuzzy interior ideal with thresholds $(s, t]$ of S is a fuzzy ideal with thresholds $(s, t]$ of S.*

Proof. Let H be a fuzzy interior ideal with thresholds $(s, t]$ of S and $x, y \in S$. There exist $a, b \in S$ such that $x \leq axxb$. Since now $xy \leq axxby$ we obtain that $\max\{H(xy), s\} \geq \min\ \{H(axxby), t\}$. Case $H(axxby) < s$. Then we have max $\{H(axxby), s\} = s$. Thus

$$\max\{H(xy), s\} \geq s = \max\{H(axxby), s\} \geq \min\{H(x), t\}.$$

Case $H(axxby) \geq s$. Then we get max $\{H(axxby), s\} = H(axxby)$.
If $H(axxby) < t$ then

$$\begin{aligned}\max\{H(xy), s\} &\geq \min\{H(axxby), t\} \\ &= H(axxby) \\ &= \max\{H(axxby), s\} \\ &\geq \min\{H(x), t\}.\end{aligned}$$

If $H(axxby) \geq t$ then

$$\max\{H(xy), s\} \geq \min\{H(axxby), t\} = t \geq \min\{H(x), t\}.$$

Thus H is a fuzzy right ideal with thresholds $(s, t]$ of S. Similarly, we can show that H is a fuzzy left ideal with thresholds $(s, t]$ of S. Hence H is a fuzzy ideal with thresholds $(s, t]$ of S. □

The following results may be deduced from Proposition 3.11 and Theorem 3.16.

Theorem 3.17. *In intra-regular ordered semigroup, the concept of a fuzzy ideal with thresholds $(s, t]$ of S and a fuzzy interior ideal with thresholds $(s, t]$ of S coincide where $s, t \in [0, 1]$ and $s < t$.*

By Theorem 3.17 we have the following corollary.

Corollary 3.18. *In an intra-regular ordered semigroup, the concept of fuzzy ideals and fuzzy interior ideals coincide.*

Corollary 3.19. *In an intra-regular ordered semigroup, the concept of $(\in, \in \vee q)$-fuzzy ideals and $(\in, \in \vee q)$-fuzzy interior ideals coincide.*

Theorem 3.20. *Let S be a semisimple ordered semigroup and $s, t \in [0, 1]$ such that $s < t$. Then every fuzzy interior ideal with thresholds $(s, t]$ of S is a fuzzy ideal with thresholds $(s, t]$ of S.*

Proof. Let H be a fuzzy interior ideal with thresholds $(s, t]$ of a semisimple ordered semigroup S and $x, y \in S$. There are $a, b, c \in S$ such that $x \leq axbxc$. Since $xy \leq axbxcy$, it follows that

$$\max\{H(xy), s\} \geq \min\{H(axbxcy), t\}.$$

Case $H(axbxcy) < s$. Then we have max $\{H(axbxcy), s\} = s$. Thus

$$\max\{H(xy), s\} \geq s = \max\{H(axbxcy), s\} \geq \min\{H(x), t\}.$$

Case $H(axbxcy) \geq s$. Then we obtain that $\max\{H(axbxcy), s\} = H(axbxcy)$. If $H(axbxcy) < t$ then

$$\begin{aligned} \max\{H(xy), s\} &\geq \min\{H(axbxcy), t\} \\ &= H(axbxcy) \\ &= \max\{H(axbxcy), s\} \\ &\geq \min\{H(x), t\}. \end{aligned}$$

If $H(axbxcy) \geq t$ then

$$\begin{aligned} \max\{H(xy), s\} &\geq \min\{H(axbxcy), t\} \\ &= t \\ &\geq \min\{H(x), t\}. \end{aligned}$$

Thus H is a fuzzy right ideal with thresholds $(s, t]$ of S. Likewise we have that H is a fuzzy left ideal with thresholds $(s, t]$ of S. Hence H is a fuzzy ideal with thresholds $(s, t]$ of S. □

Combining Proposition 3.11 and Theorem 3.20, we have

Theorem 3.21. *In semisimple ordered semigroup S, the concept of a fuzzy ideal with thresholds $(s, t]$ of S and a fuzzy interior ideal with thresholds $(s, t]$ of S coincide where $s, t \in [0, 1]$ and $s < t$.*

Next corollary follows directly from above theorem.

Corollary 3.22. *In semisimple ordered semigroup S, the concept of a fuzzy ideal of S and a fuzzy interior ideal of S coincide.*

Corollary 3.23. *In semisimple ordered semigroup S, the concept of an $(\in, \in \vee q)$-fuzzy ideal of S and an $(\in, \in \vee q)$-fuzzy interior ideal of S coincide.*

Acknowledgement : This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

References

[1] L.A. Zadeh, Fuzzy sets, Inf. Control. 8 (1965) 338–353.

[2] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.

[3] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets Syst. 5 (1981) 203–215.

[4] S.M. Hong, Y.B. Jun, J. Meng, Fuzzy interior ideals in semigroups, Indian J. pure appl. Math. 26 (1995) 859–863.

[5] Y.B. Jun, S.Z. Song, Generalized fuzzy interior ideals in semigroups, Inf. Sci. 176 (2006) 3079–3093.

[6] J. Zhan, Y.B. Jun, Generalized fuzzy interior ideals of semigroups, Neural Comput. Applic. 19 (2010) 515–519.

[7] N. Kehayopulu, M. Tsingelis, Fuzzy sets in ordered groupoids, Semigroup Forum 65 (2002) 128–132.

[8] N. Kehayopulu, M. Tsingelis, Fuzzy interior ideals in ordered semigroups, Lobacheskii J. Math. 21 (2006) 65–71.

[9] A. Khan, M. Shabir, (α, β)-fuzzy interior ideals in ordered semigroups, Lobachevskii J. Math. 30 (2009) 30–39.

[10] A. Khan, Y.B. Jun, M.Z. Abbas, Characterizations of ordered semigroups in terms of $(\in, \in\vee q)$-fuzzy interior ideals, Neural comput. Applic. 21 (3) (2012) 433–440.

[11] X.Y. Xie, J. Tang, Fuzzy radicals and prime fuzzy ideals of ordered semigroups, Inf. Sci. 178 (2008) 4357–4374.

[12] Y. Yin, J. Zhan, D. Xu, Generalized intuitionistic fuzzy ideals of ordered semigroups, Bull. Malays. Math. Sci. Soc. 3 (2011) 649–663.

(Received 26 January 2012)
(Accepted 22 April 2013)