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Statistically Convergent Difference Sequence Spaces of Fuzzy Real Numbers Defined by Orlicz Function[1]

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Abstract : The classes of statistically convergent difference sequence spaces of fuzzy real numbers defined by Orlicz functions are introduced. Some properties of these sequence spaces like solidness, symmetricity, convergence-free etc. are studied. We obtain some inclusion relations involving these sequence spaces.

Keywords : Orlicz function; symmetric space; solid space; convergence-free; metric space; completeness.
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1 Introduction

In order to extend the notion of convergence of sequences, statistical convergence of sequences was introduced by Fast [1] and Schoenberg [2] independently.

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It is also found in Zygmund [3]. Later on it was studied from sequence space point of view and linked with summability theory by Tripathy ([4, 5]), Tripathy and Sen [6], Rath and Tripathy [7], Fridy [8], Kwon [9], Nuray and Savas [10], Šalàt [11], Altin [12] and many others.

The notion depends on the idea of asymptotic density of subsets of the set N of natural numbers. A subset A of N is said to have *natural density* $\delta(A)$ if

$$\delta(A) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_A(k) \text{ exists},$$

where χ_A is the characteristic function of A. Clearly all-finite subsets of N have zero natural density and $\delta(A^c) = \delta(N - A) = 1 - \delta(A)$.

A sequence (x_k) is said to be *statistically convergent* to L, if for any $\varepsilon > 0$, we have $\delta\{(k \in N : |x_k - L| \geq \varepsilon)\} = 0$. We write $x_k \xrightarrow{stat} L$ or $stat - \lim_{k\to\infty} x_k = L$. For two sequences (x_k) and (y_k), we say that $x_k \neq y_k$ for almost all k (in short a.a.k.) if $\delta(\{k \in N : x_k = y_k\}) = 0$.

Let $x \in w$ and let p be a positive real number. The sequence x is said to be strongly p-Cesàro summable if there is a complex number L such that,

$$\lim_{n}\frac{1}{n}\sum_{k=1}^{n}|x_k - L|^p = 0.$$

We say that x is strongly p-Cesàro summable to L.

The concept of fuzzy set theory was introduced by Zadeh in the year 1965. Gradually the potential of the notion of fuzzy set was realized by the scientific group and many researchers were motivated for further investigation and its application. It has been applied for the studies in almost all the branches of science, where mathematics plays a role. Workers on sequence spaces have also applied the notion and introduced sequences of fuzzy real numbers and studied their different properties. The concept of fuzzyness has been applied in various fields like Cybernatics, Artificial Intelligence, Expert System and Fuzzy control, Pattern recognition, Operation research, Decision making, Image analysis, Projectiles, Probability theory, Agriculture, Weather forecasting etc. The fuzzyness of all the subjects of Mathematical Sciences have been investigated. Fuzzy probability theory is known as Possibility theory. The notion of statistical convergence of sequences has relationship with possibility theory. The distribution that is used in case of statistical convergence is uniform distribution. The notion of statistical convergence is same as the notion of almost sure convergence of probability theory. The results on almost sure convergence are of single sequence type.

The notion of fuzzy set theory has been applied for investigating different classes of sequences. We have listed some of the papers mostly recent in the list of references. We have observed that only a few papers have been published on sequence spaces of fuzzy numbers till date. Therefore we were motivated by the recent publications as well as the application of the notion of fuzzy set.

Sequences of fuzzy numbers have been discussed by Nuray and Savas [10], Kwon [9], Tripathy and Dutta ([13, 14]), Tripathy and Sarma [15], Altin [12], Altin et al. [16], Altin et al. [17] and many others. Nuray and Savas [10] introduced the concept of statistically convergent sequences of fuzzy real numbers.

Kizmaz [4] studied the notion of difference sequence spaces at the initial stage. Kizmaz [18] introduced and investigated the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ for crisp sets. The notion is defined as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\},$$

for $Z = \ell_\infty, c$ and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$.

The above spaces are Banach spaces, normed by,

$$\| x \|_\Delta = |x_1| + \sup_k |\Delta x_k|.$$

The idea of Kizmaz [18] was applied to introduce different type of difference sequence spaces and study their different properties by Tripathy [5], Tripathy et al. [19], Tripathy and Mahanta [20], Tripathy and Sen [6] and many others.

Tripathy and Esi [21] introduced the new type of difference sequence spaces, for fixed $m \in N$ by,

$$Z(\Delta_m) = \{x = (x_k) : (\Delta_m X_k) \in Z\},$$

for $Z = \ell_\infty, c$ and c_0 where $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$.

The above spaces are Banach spaces, normed by,

$$\| x \|_{\Delta_m} = \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k|.$$

Tripathy et al. [22] further generalized this notion and introduced the following. For $m \geq 1$ and $n \geq 1$,

$$Z(\Delta_m^n) = \{x = (x_k) : (\Delta_m^n x_k) \in Z\},$$

for $Z = \ell_\infty, c$ and c_0. Where $(\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ for all $k \in N$. This generalized difference notion has the following binomial representation,

$$\Delta_m^n x_k = \sum_{r=0}^{n} (-1)^r \binom{n}{r} x_{k+rm}. \tag{1.1}$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. An Orlicz function M is said to satisfy Δ_2-*condition* for all values of x, if there exists a constant $K > 0$, such that $M(Lx) \leq KLM(x)$, for all $x > 0$ and for $L > 1$. If the convexity of the Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called as modulus function. In the recent past different classes of sequences have been introduced and investigated using modulus function by Tripathy and Chandra [23], Esi and Tripathy [24], Lindenstrauss and Tzafriri [25], Et et al. [26] and many others.

Remark 1.1. *An Orlicz function satisfies the inequality* $M(\lambda x) \leq \lambda M(x)$ *for all* λ *with* $0 < \lambda < 1$.

Throughout the article $w^F, \ell^F, \ell_\infty^F$ represent the classes of *all*, *absolutely summable* and *bounded* sequences of fuzzy real numbers respectively.

2 Preliminaries

A fuzzy real number X is a fuzzy set on R, i.e. a mapping $X : R \to I(= [0, 1])$ associating each real number t with its grade of membership $X(t)$. A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal.* A fuzzy real number X is said to be *upper semi-continuous* if for each $\varepsilon > 0, X^{-1}([0, a + \varepsilon))$, for all $a \in I$, is open in the usual topology of R.

The class of all *upper semi-continuous, normal, convex* fuzzy real numbers is denoted by $R(I)$. For $X \in R(I)$, the α-level set X^α for $0 < \alpha \leq 1$ is defined by, $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. The 0-level of X *i.e.* X^0 is the closure of strong 0-cut, *i.e.* $cl\{t \in R : X(t) > 0\}$.

The absolute value of $X \in R(I)$ *i.e.* $|X|$ is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\} & \text{for } t \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

For $r \in R, \overline{r} \in R(I)$ is defined by

$$\overline{r}(t) = \begin{cases} 1 & \text{for } t = r; \\ 0 & \text{otherwise.} \end{cases}$$

The additive identity and multiplicative identity of $R(I)$ are denoted by $\overline{0}$ and $\overline{1}$ respectively. The zero sequence of fuzzy real numbers is denoted by $\overline{\theta}$.

Let D be the set of all closed bounded intervals $X = [X^L, X^R]$. Define $d : D \times D \to R$ by $d(X,Y) = \max\{|X^L - Y^L|, |X^R - Y^R|\}$. Then clearly (D, d) is a complete metric space. Define $\overline{d} : R(I) \times R(I) \to R$ by $\overline{d}(X, Y) = \sup_{0<\alpha\leq 1} d(X^\alpha, Y^\alpha)$, for $X, Y \in R(I)$. Then it is well known that $(R(I), \overline{d})$ is a complete metric space.

A sequence $X = (X_n)$ of fuzzy real numbers is said to converge to the fuzzy number X_0, if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\overline{d}(X_n, X_0) < \varepsilon$, for all $n \geq n_0$. A sequence space E is said to be *solid* if $(Y_n) \in E$, whenever $(X_n) \in E$ and $|Y_n| \leq |X_n|$, for all $n \in N$. A sequence space E is said to be *monotone* if E contains the canonical pre-images of all its step spaces.

Let $X = (X_n)$ be a sequence, then $S(X)$ denotes the set of all permutations of the elements of (X_n) i.e. $S(X) = \{(X_{\pi(n)}) : \pi \text{ is a permutation of } N\}$. A sequence space E is said to be *symmetric* if $S(X) \subset E$ for all $X \in E$. A sequence space E is said to be *convergence-free* if $(Y_n) \in E$ whenever $(X_n) \in E$ and $X_n = \overline{0}$ implies $Y_n = \overline{0}$. Sequences of fuzzy numbers have been investigated by Kwon [9], Nuray [10], Altin [12], Tripathy and Dutta [13, 14], Tripathy and Sarma [15], Et et al. [27] and others.

Remark 2.1. *A sequence space E is solid implies that E is monotone.*

Lindenstrauss and Tzafriri [25] used the notion of Orlicz function and introduced the sequence space:

$$\ell_M = \left\{(x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}.$$

The space ℓ_M with the norm,

$$\| (x_k) \| = \inf \left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space, which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p , which is an Orlicz sequence space with $M(x) = x^p$, for $1 \leq p < \infty$.

In the later stage different classes of Orlicz sequence spaces have been introduced and investigated by Tripathy et al. [19], Tripathy and Mahanta [20], Tripathy and Sarma [15] and many others.

In this article we introduce the following sequence spaces.

$$\overline{c}(M, \Delta_m^n)^F = \left\{(X_k) \in w^F : \text{ stat-lim } M\left(\frac{\overline{d}(\Delta_m^n X_k, L)}{\rho}\right) = 0, \text{ for some } \rho > 0, L \in R(I)\right\},$$

$$\overline{c}_0(M, \Delta_m^n)^F = \left\{(X_k) \in w^F : \text{ stat-lim } M\left(\frac{\overline{d}(\Delta_m^n X_k, \overline{0})}{\rho}\right) = 0, \text{ for some } \rho > 0\right\},$$

$$W(M, \Delta_m^n, p)^F = \left\{(X_k) \in w^F : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(M\left(\frac{\overline{d}(\Delta_m^n X_k, L)}{\rho}\right)\right)^p = 0, \right.$$
$$\left. \text{for some } \rho > 0, L \in R(I)\right\}.$$

Also, we define,

$$m^F(M) = \overline{c}(M, \Delta_m^n)^F \cap \ell_\infty(M, \Delta_m^n)^F,$$
$$m_0^F(M) = \overline{c}_0(M, \Delta_m^n)^F \cap \ell_\infty(M, \Delta_m^n)^F.$$

3 Main Results

Theorem 3.1. *The spaces $m^F(M)$ and $m_0^F(M)$ are complete metric spaces by the metric,*

$$\eta(X, Y) = \sum_{r=1}^{mn} \overline{d}(X_r, Y_r) + inf\left\{\rho > 0 : \sup_k M\left(\frac{\overline{d}(\Delta_m^n X_k, \Delta_m^n Y_k)}{\rho}\right) \leq 1\right\},$$

for $X, Y \in m^F(M)$ and $m_0^F(M)$.

Proof. Consider the class of sequences $m^F(M)$. Let $(X^{(i)})$ be a Cauchy sequence in $m^F(M)$ such that $X^{(i)} = (X_k^{(i)})_{k=1}^{\infty}$.

Let $\varepsilon > 0$ be given. For a fixed $x_0 > 0$, choose $r > 0$ such that $M\left(\frac{rx_0}{2}\right) \geq 1$. Then there exists a positive integer $n_0 = n_0(\varepsilon)$ such that

$$\eta(X^{(i)}, X^{(j)}) < \frac{\varepsilon}{rx_0},$$

for all $i, j \geq n_0$.

By the definition of η we get,

$$\sum_{r=1}^{mn} \overline{d}(X_r^{(i)}, X_r^{(j)}) + \inf\left\{\rho > 0 : \sup_k M\left(\frac{\overline{d}(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k^{(j)})}{\rho}\right) \leq 1\right\} < \varepsilon, \quad (3.1)$$

for all $i, j \geq n_0$. Which implies,

$$\sum_{r=1}^{mn} \overline{d}(X_r^{(i)}, Y_r^{(j)}) < \varepsilon, \text{ for all } i, j \geq n_0$$
$$\Rightarrow \overline{d}(X_r^{(i)}, Y_r^{(j)}) < \varepsilon, \text{ for all } i, j \geq n_0, r = 1, 2, 3, ..., mn.$$

Hence $(X_r^{(i)})$ is a Cauchy sequence in $R(I)$, so it is convergent in $R(I)$, by the completeness property of $R(I)$, for $r = 1, 2, 3, ..., mn$.

Let

$$\lim_{i\to\infty} X_r^{(i)} = X_r, \text{ for } r = 1, 2, 3, ..., mn. \quad (3.2)$$

Also,

$$\sup_k M\left(\frac{\overline{d}(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k^{(j)})}{\rho}\right) \leq 1, \text{ for all } i, j \geq n_0 \quad (3.3)$$
$$\Rightarrow M\left(\frac{\overline{d}(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k^{(j)})}{\eta(X^{(i)}, X^{(j)})}\right) \leq 1 \leq M\left(\frac{rx_0}{2}\right), \text{ for all } i, j \geq n_0.$$

Since M is continuous, we get

$$\overline{d}(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k^{(j)}) \leq \frac{rx_0}{2}.\eta(X^{(i)}, X^{(j)}), \text{ for all } i, j \geq n_0$$
$$\Rightarrow \overline{d}(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k^{(j)}) < \frac{rx_0}{2}.\frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2}, \text{ for all } i, j \geq n_0$$
$$\Rightarrow \overline{d}(\Delta_m^n X_k^{(i)}, \Delta_m^n X_k^{(j)}) < \frac{\varepsilon}{2}, \text{ for all } i, j \geq n_0.$$

Which implies $(\Delta_m^n X_k^{(i)})$ is a Cauchy sequence in $R(I)$ and so it is convergent in $R(I)$ by the completeness property of $R(I)$.

Let, $\lim_i \Delta_m^n X_k^{(i)} = Y_k$ (say), in $R(I)$, for each $k \in N$. We have to prove that,

$$\lim_i X^{(i)} = X \text{ and } X \in m^F(M).$$

For $k = 1$, we get, from (1.1) and (3.2),

$$\lim_{i\to\infty} X^{(i)}_{mn+1} = X_{mn+1}.$$

Proceeding in this way by using principle of induction, we get

$$\lim_{i\to\infty} X^{(i)}_k = X_k, \text{ for each } k \in N.$$

Also, $\lim_{i\to\infty} \Delta^n_m X^{(i)}_k = \Delta^n_m X_k$, for each $k \in N$.

Now, taking $j \to \infty$ and fixing i and using the continuity of M, it follows from (3.3),

$$\sup_k M\left(\frac{\overline{d}(\Delta^n_m X^{(i)}_k, \Delta^n_m X_k)}{\rho}\right) \le 1, \text{ for some } \rho > 0.$$

Now on taking the infimum of such ρ's, we get

$$\inf\left\{\rho > 0 : \sup_k M\left(\frac{\overline{d}(\Delta^n_m X^{(i)}_k, \Delta^n_m X_k)}{\rho}\right) \le 1\right\} < \varepsilon, \text{ for all } i \ge n_0 \text{ (by (1.1))}.$$

Hence, we get

$$\sum_{r=1}^{mn} \overline{d}(X^{(i)}_r, X_r) + \inf\left\{\rho > 0 : \sup_k M\left(\frac{\overline{d}(\Delta^n_m X^{(i)}_k, \Delta^n_m X_k)}{\rho}\right) \le 1\right\} < \varepsilon + \varepsilon = 2\varepsilon,$$

for all $i \ge n_0$. Which implies, $\eta(X^{(i)}, X) < 2\varepsilon$, for all $i \ge n_0$, i.e. $\lim_i X^{(i)} = X$. Now, it is to show that $X \in m^F(M)$.

Let $X^{(i)} \in m^F(M)$. Then, for each i there exists L_i such that

$$\text{stat-}\lim_{k\to\infty} M\left(\frac{\overline{d}(\Delta^n_m X^{(i)}_k, L_i)}{\rho}\right) = 0, \text{ for some } \rho > 0, L_i \in R(I), \quad \text{for each } i \in N.$$

We have to show that,

1. (L_i) converges to L, for $i \to \infty$.

2. stat-$\lim_{k\to\infty} M\left(\frac{\overline{d}(\Delta^n_m X_k, L)}{\rho}\right) = 0$, for some $\rho > 0, L \in R(I)$.

Since $(X^{(i)})$ is a convergent sequence of elements from $m^F(M)$, so for a given $\varepsilon > 0$, there exists $n_0 \in N$, such that,

$$\eta(X^{(i)}, X^{(j)}) < \frac{\varepsilon}{3}.$$

Again, for given $\varepsilon > 0$, we have

$$\delta(A_i) = \delta\left(\left\{k \in N : \eta(X^{(i)}, L_i) < \frac{\varepsilon}{3}\right\}\right) = 1.$$

and

$$\delta(A_j) = \delta\left(\left\{k \in N : \eta(X^{(j)}, L_i) < \frac{\varepsilon}{3}\right\}\right) = 1.$$

Let $A = A_i \cap A_j$, then $\delta(A) = 1$. We choose $k \in A$. Then for each $i, j \geq n_0$, we have

$$\begin{aligned}\eta(L_i, L_j) &\leq \eta(L_i, X^{(i)}) + \eta(X^{(i)}, X^{(j)}) + \eta(X^{(j)}, L_j)\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\\ &= \varepsilon.\end{aligned}$$

Since the sequence (L_i) fulfills the Cauchy condition for convergence, it must be convergent to a fuzzy real number L (say). Hence, $\lim_{i\to\infty} L_i = L$. Let $\xi > 0$. We show that

$$\delta(F) = \delta(\{k \in N : \eta(X_k, L) < \xi\}) = 1.$$

Since $X^{(n)} \to X$, there exists $q \in N$, such that

$$\eta(X^{(q)}, X) < \frac{\xi}{3}. \tag{3.4}$$

The number q can be chosen in such a way that together with (3.4) we have

$$\eta(L_q, L) < \frac{\xi}{3}.$$

Since stat-$\lim_{k\to\infty} M\left(\frac{\overline{d}(\Delta_m^n X_k^{(i)}, L_i)}{\rho}\right) = 0$, we have a subset B of N such that $\delta(B) = 1$, where, $B = \{k \in N : \eta(X^{(q)}, L_q) < \frac{\xi}{3}\}$. Therefore, for each $k \in B$, we have

$$\begin{aligned}\eta(X, L) &\leq \eta(X, X^{(q)}) + \eta(X^{(q)}, L_q) + \eta(L_q, L)\\ &< \frac{\xi}{3} + \frac{\xi}{3} + \frac{\xi}{3} = \xi.\end{aligned}$$

This completes the proof of the theorem.

Proof is similar for $m_0^F(M)$. □

Theorem 3.2. *The sequence spaces $\overline{c}(M, \Delta_m^n)^F, \overline{c}_0(M, \Delta_m^n)^F, m^F(M)$ and $m_0^F(M)$ are neither solid nor monotone in general.*

Proof. The result follows from the following example:

Example 3.3. *Consider the sequence space $\overline{c}_0(M, \Delta_m^n)^F$. Let $X_k = \overline{k}$, for all $k \in K \subset N$, with $\delta(K) = 1$. Let $m = 3$ and $n = 2$. Let $M(x) = |x|$, for all $x \in [0, \infty)$. Then, we have $\overline{d}(\Delta_3^2 X_k, \overline{0}) = 0$, for all $k \in K \subset N$. Hence, we get*

$$\text{stat-lim} M\left(\frac{\overline{d}(\Delta_3^2 X_k, \overline{0})}{\rho}\right) = 0, \text{ for some } \rho > 0.$$

Which implies that, $(X_k) \in \overline{c}(M, \Delta_3^2)^F$.

Consider the sequence (α_k) *of scalars defined by*

$$\alpha_k = \begin{cases} 1 & k = 2n, n \in N; \\ 0 & \text{otherwise.} \end{cases}$$

Now,

$$\alpha_k X_k = \begin{cases} \overline{k} & \text{for } k = 2n-1, n \in N; \\ 0 & \text{otherwise.} \end{cases}$$

So,

$$\overline{d}(\Delta \alpha_k X_k, \overline{0}) = \begin{cases} k & \text{for } k \text{ odd;} \\ k+1 & \text{for } k \text{ even.} \end{cases}$$

Which shows that, $(\alpha_k X_k) \notin \overline{c}(M, \Delta)^F$. *Hence,* $\overline{c}(M, \Delta_m^n)^F$ *is not solid, in general.*

Proofs are similar for the other spaces also. □

Theorem 3.4. *The sequence spaces* $\overline{c}(M, \Delta_m^n)^F, \overline{c}_0(M, \Delta_m^n)^F, m^F(M)$ *and* $m_0^F(M)$ *are neither symmetric nor convergence free.*

Proof. The result follows from the following example:

Example 3.5. *Consider the sequence space* $\overline{c}_0(M, \Delta_m^n)^F$. *Let* $m = 1, n = 1$. *Let* $M(x) = x^3$ *for all* $x \in [0, \infty)$. *Taking the focus at* a *and following the standard formula of parabolic type fuzzy number,*

$$\alpha = -(x-a)^2 + 1.$$

Consider the sequence (X_k) *defined as follows:*

For $k = i^2, i \in N, X_k(t) = -(t-k)^2 + 1$, *for* $t \in [-(1+k), (1+k)]$ *and* $X_k = \overline{0}$, *otherwise. Then, for* $k = i^2$ *and* $i^2 - 1, i \in N$,

$$\Delta X_k(t) = -(t-k)^2 + 1, \text{ for } t \in \left[\frac{-(2k^2+4k+1)}{k^2+k}, \frac{(2k^2+4k+1)}{k^2+k} \right]$$

and $\Delta X_k = \overline{0}$, *otherwise. So,* $(X_k) \in \overline{c}_0(M, \Delta_m^n)^F$. *Let* (Y_k) *be a re-arrangement of* (X_k), *such that, for* $k = 2i, i \in N, Y_k(t) = -(t-k)^2 + 1$, *for* $t \in [-(1+k), (1+k)]$ *and* $Y_k = \overline{0}$, *otherwise. So that, for* k *odd,*

$$\Delta Y_k(t) = -(t-k)^2 + 1, \text{ for } t \in \left[\frac{-(2k^2+4k+1)}{k^2+k}, \frac{(2k^2+4k+1)}{k^2+k} \right]$$

and $\Delta Y_k = \overline{0}$, *otherwise. For* k *even,*

$$\Delta Y_k(t) = -(t-k)^2 + 1, \text{ for } t \in \left[\frac{-(2k^2+4k+1)}{k^2+k}, \frac{(2k^2+4k+1)}{k^2+k} \right]$$

and $\Delta Y_k = \overline{0}$, *otherwise. Which implies,* $(Y_k) \notin \overline{c}_0(M, \Delta_m^n)^F$. *Hence* $\overline{c}_0(M, \Delta_m^n)^F$ *is not symmetric in general.*

Proof is similar for convergence- free also. □

Theorem 3.6. *Let M, M_1 and M_2 be Orlicz functions satisfying Δ_2-condition. Then,*

(i) $m^F(M_1) \subseteq m^F(M \circ M_1)$;

(ii) $m^F(M_1) \cap m^F(M_2) \subseteq m^F(M_1 + M_2)$.

Proof. (i) Let $(X_k) \in m^F(M_1)$. For $\varepsilon > 0$, there exists $\eta > 0$ such that $\varepsilon = M(\eta)$. Then there exists a set $K \subseteq N$, with $\delta(K) = 1$ such that

$$M_1\left(\frac{\overline{d}(\Delta_m^n X_k, L)}{\rho}\right) < \eta, \text{ for all } k \in K \text{ and for some } \rho > 0.$$

Let

$$Y_k = M_1\left(\frac{\overline{d}(\Delta_m^n X_k, L)}{\rho}\right), \text{ for some } \rho > 0.$$

Since M is continuous and non-decreasing, we get

$$M(Y_k) = M\left(M_1\left(\frac{\overline{d}(\Delta_m^n X_k, L)}{\rho}\right)\right) < M(\eta) = \varepsilon, \text{ for some } \rho > 0.$$

Which implies, $(X_k) \in m^F(M \circ M_1)$. This completes the proof.

(ii) Let $(X_k) \in m^F(M_1) \cap m^F(M_2)$. Then there exists a set $K \subseteq N$, with $\delta(K) = 1$ such that

$$M_1\left(\frac{\overline{d}(\Delta_m^n X_k, L)}{\rho}\right) < \varepsilon, \text{ for all } k \in K \text{ and for some } \rho > 0$$

and

$$M_2\left(\frac{\overline{d}(\Delta_m^n X_k, L)}{\rho}\right) < \varepsilon, \text{ for all } k \in K \text{ and for some } \rho > 0.$$

The rest of the proof follows from the equality,

$$\begin{aligned}(M_1 + M_2)\left(\frac{\overline{d}(\Delta_m^n X_k, L)}{\rho}\right) &= M_1\left(\frac{\overline{d}(\Delta_m^n X_k, L)}{\rho}\right) + M_2\left(\frac{\overline{d}(\Delta_m^n X_k, L)}{\rho}\right) \\ &< \varepsilon + \varepsilon = 2\varepsilon\end{aligned}$$

for some $\rho > 0$. Which implies that $(X_k) \in m^F(M + M_1)$. This completes the proof. □

Theorem 3.7.

(i) $W(M, \Delta_m^n, p)^F \subseteq \overline{c}(M, \Delta_m^n)^F$.

(ii) For $X = (X_k) \in \overline{c}(M, \Delta_m^n)^F$, (X_k) *is strongly p-Cesàro summable to* X_0, *if it is bounded.*

Proof. (*i*) Let $(X_k) \in W(M, \Delta_m^n, p)^F$. For any $\varepsilon > 0$ and $p \in R, 0 < p < \infty$ and using the continuity of M, we get

$$\sum_{k=1}^{n}\left[M\left(\frac{\overline{d}(\Delta_m^n X_k, X_0)}{\rho}\right)\right]^p \geq \varepsilon^p.\left|\left\{k \leq n : \left[M\left(\frac{\overline{d}(\Delta_m^n X_k, X_0)}{\rho}\right)\right] \geq \varepsilon\right\}\right|$$

for some $\rho > 0$. It follows that (X_k) is statistically convergent to X_0. Hence, $(X_k) \in \overline{c}(M, \Delta_m^n)^F \Rightarrow W(M, \Delta_m^n, p)^F \subseteq \overline{c}(M, \Delta_m^n)^F$. It completes the proof.

(*ii*) Let $\varepsilon > 0$ be given and let $K = \overline{d}(X_k, \overline{0}) + \overline{d}(X_0, \overline{0})$. Since $X = (X_k) \in m^F(M)$ is bounded and statistically convergent to X_0, there is a positive number $N(\varepsilon)$ such that

$$\frac{1}{n}\left|\left\{k \leq n : \left[M\left(\frac{\overline{d}(\Delta_m^n X_k, X_0)}{\rho}\right)\right]^p \geq \frac{\varepsilon}{2}\right\}\right| < \frac{\varepsilon}{K^p}, \text{ for all } n \geq N(\varepsilon).$$

Now, set

$$L_n = \left\{k \leq n : \left[M\left(\frac{\overline{d}(\Delta_m^n X_k, X_0)}{\rho}\right)\right]^p \geq \frac{\varepsilon}{2}\right\}.$$

Then for all $n \geq N(\varepsilon)$, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{\overline{d}(\Delta_m^n X_k, X_0)}{\rho}\right)\right]^p$$
$$= \frac{1}{n}\left(\sum_{k \in L_n}\left[M\left(\frac{\overline{d}(\Delta_m^n X_k, X_0)}{\rho}\right)\right]^p + \sum_{k \notin L_n, k \leq n}\left[M\left(\frac{\overline{d}(\Delta_m^n X_k, X_0)}{\rho}\right)\right]^p\right) < \varepsilon.$$

for some fixed $\rho > 0$. This completes the proof. □

The proof of the following result is easy, so omitted.

Corollary 3.8. *Let $p, q \in R, 0 \leq p < q < \infty$, then*

1. $W(M, \Delta_m^n, p)^F \supseteq W(M, \Delta_m^n, q)^F$;
2. $W(M, \Delta_m^n, p)^F \cap \ell_\infty^F(M, \Delta_m^n) = W(M, \Delta_m^n, q)^F \cap \ell_\infty^F(M, \Delta_m^n)$.

Proposition 3.9. *$Z(M, \Delta_m^i) \subset Z(M, \Delta_m^n)$, for $0 \leq i < n$, for $Z = \overline{c}$ and $\overline{c}_0$.*

Proof. Let $(X_k) \in \overline{c}^F(M, \Delta_m^{n-1})$. Then, we have

$$\text{stat-lim} M\left(\frac{\overline{d}(\Delta_m^{n-1} X_k, L)}{\rho}\right) = 0, \text{ for some } \rho > 0 \text{ and } L \in R(I).$$

Now, we have

$$\begin{aligned}
&\text{stat-lim} M\left(\frac{\overline{d}(\Delta_m^n X_k,\overline{0})}{2\rho}\right)\\
&\qquad = \text{stat-lim} M\left(\frac{\overline{d}(\Delta_m^{n-1} X_k - \Delta_m^{n-1} X_{k+1},\overline{0})}{2\rho}\right)\\
&\qquad \le \text{stat-lim}\frac{1}{2} M\left(\frac{\overline{d}(\Delta_m^{n-1} X_k,\overline{0})}{\rho}\right) + \text{stat-lim}\frac{1}{2} M\left(\frac{\overline{d}(\Delta_m^{n-1} X_{k+1},\overline{0})}{\rho}\right)\\
&\qquad = 0.
\end{aligned}$$

Proceeding in this way by induction, we have $Z(M,\Delta_m^i) \subset Z(M,\Delta_m^n)$, for $0 \le i < n$, for $Z = \ell_\infty^F, c^F$ and c_0^F. This completes the proof. □

4 Conclusion

In this paper, we have introduced the notions of different types of statistically convergent sequences of real numbers. We have investigated their different algebraic and topological properties like completeness, solidness, symmetricity, convergence free etc. The notion of statistical convergence is similar to the notion of almost sure convergence of the probability theory. These results can be applied for investigating the almost sure convergence of difference sequences of random variables. It is expected that it will attract workers from sequence spaces as well as from distribution theory for further investigation.

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