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Common Fixed Point Theorems Using Property (E.A) in Complex-Valued Metric Spaces

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Abstract : In this paper, we introduce the concept of property (E.A) in a complex-valued metric space to prove some common fixed point results for two pairs of weakly compatible mappings, satisfying a contractive condition of 'max' type. Further, we prove a common fixed point theorem for two pairs of self-mappings satisfying the common limit property in the range of a mapping *called (CLR)-property* by Sintunavarat and Kumam [Common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric space, J. Appl. Math., Vol. 2011 (2011), Article ID 637958, 14 pages]. The related result generalizes various theorems of ordinary metric spaces.

Keywords : Banach contraction principal; common fixed point; complex-valued metric space; contractive condition; metric space; property (E.A); partial order; weakly compatible mappings.

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1 Introduction and Preliminaries

Banach fixed point theorem [1] in a complete metric space has been generalized in many spaces. In 2011, Azam et al. [2] introduced the notion of complex-valued metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. The idea of complex-valued metric spaces can be exploited to define complex-valued normed spaces and complex-valued Hilbert spaces; additionally it offers numerous research activities in mathematical analysis. The theorems proved by Azam et al. [2] and Bhatt et al. [3] uses the rational inequality in a complex-valued metric space as contractive condition. In this paper, we introduce the concept of property (E.A) in a complex-valued metric space, to prove some common fixed point results for a quadruple of self-mappings satisfying a contractive condition of 'max' type. Our results generalizes various theorems of ordinary metric spaces.

An ordinary metric d is a real-valued function from a set $X \times X$ into $\mathbb{R}$, where X is a nonempty set. That is, d : $X \times X \to \mathbb{R}$. A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first co-ordinate is called $Re(z)$ and second coordinate is called $Im(z)$. Thus a complex-valued metric d is a function from a set $X \times X$ into $\mathbb{C}$, where X is a nonempty set and $\mathbb{C}$ is the set of complex number. That is, $d : X \times X \to \mathbb{C}$. Let $z_1, z_2 \in \mathbb{C}$, define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \le Re(z_2),\ \ Im(z_1) \le Im(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2),\ Im(z_1 = Im(z_2)$,

(iii) $Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$.

In (i), (ii) and (iii), we have $|z_1| < |z_2|$. In (iv), we have $|z_1| = |z_2|$. So $|z_1| \le |z_2|$. In particular, $z_1 \precnsim z_2$ if $z_1 \ne z_2$ and one of (i), (ii), (iii) is satisfy. In this case $|z_1| < |z_2|$. We will write $z_1 \prec z_2$ if only (iii) satisfy. Further,

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2 \ \text{ and } \ z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

Azam et al. [2] defined the complex-valued metric space (X, d) in the following way:

Definition 1.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(C1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(C3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a *complex-valued metric on* X, and (X, d) is called a complex-valued metric space.

A point $x \in X$ is called an *interior point* of $A \subseteq X$ if there exists $r \in \mathbb{C}$, where $0 \prec r$, such that

$$B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A.$$

A point $x \in X$ is called a *limit point* of $A \subseteq X$, if for every $0 \prec r \in \mathbb{C}$,

$$B(x,r) \cap (A \ - X) \neq \phi.$$

The set A is called *open* whenever each element of A is an interior point of A. A subset B is called *closed* whenever each limit point of B belongs to B.

The family $\mathcal{F} := \{B(x,r) : x \in X, 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is called *convergent*. Also, $\{x_n\}$ *converges* to x (written as, $x_n \to x$ or $\lim_{n\to\infty} x_n = x$); and x is the *limit point* of $\{x_n\}$. The sequence $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} |d(x_n, x)| = 0$.

If for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called *Cauchy sequence* in (X, d). If every Cauchy sequence converges in X, then X is called a *complete complex-valued metric space.* The sequence $\{x_n\}$ is called Cauchy if and only if $\lim_{n\to\infty} |d(x_n, x_{n+m})| = 0$.

Definition 1.2 ([3])**.** A pair of self-mappings $A, S : X \to X$ is called *weakly-compatible* if they commute at their coincidence points. That is, if there be a point $u \in X$ such that $Au = Su$, then $ASu = SAu$, for each $u \in X$.

Example 1.3. *Let $X = \mathbb{C}$. Define complex-metric $d : X \times X \to \mathbb{C}$ by: $d(z_1, z_2) := e^{ia}|z_1 - z_2|$, where a is any real constant. Then (X, d) is a complex-valued metric space. Suppose $A, S : X \to X$ be defined as: $Az = 2e^{i\pi/4}$ if Re(z)$\neq 0$, $Az = 3e^{i\pi/3}$ if Re(z) $= 0$, and $Sz = 2e^{i\pi/4}$ if Re(z)$\neq 0$, $Sz = 4e^{i\pi/6}$ if Re(z) $= 0$.*

Then observe that: A and S are coincident when Re(z)$\neq$0 and $Az = Sz = 2e^{i\pi/4}$. At this point $ASz = SAZ = 2e^{i\pi/4}$. Hence pair (A, S) commutes at their coincidence point, so it is weakly compatible at all $z \in \mathbb{C}$ with Re(z)$\neq$0.

Definition 1.4. We define the 'max' function for the partial order relation $\precsim$ by:

(1) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$.

(2) $z_1 \precsim \max\{z_2, z_3\} \Rightarrow z_1 \precsim z_2,$ or $z_1 \precsim z_3$.

(3) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ or $|z_1| \leq |z_2|$.

Using Definition 1.4 we have the following lemma:

Lemma 1.5. *Let $z_1, z_2, z_3, ... \in \mathbb{C}$ and the partial order relation $\precsim$ is defined on $\mathbb{C}$. Then following statements are easy to prove:*

(i) If $z_1 \precsim \max\{z_2, z_3\}$ then $z_1 \precsim z_2$ if $z_3 \precsim z_2$;

(ii) If $z_1 \precsim \max\{z_2, z_3, z_4\}$ then $z_1 \precsim z_2$ if $\max\{z_3, z_4\} \precsim z_2$;

(iii) If $z_1 \precsim \max\{z_2, z_3, z_4, z_5\}$ then $z_1 \precsim z_2$ if $\max\{z_3, z_4, z_5\} \precsim z_2$, and so on.

Now, we give the following definition of property (E.A), like [4] in complex-valued metric space:

Definition 1.6. Let $A, S : X \to X$ be two self-maps of a complex-valued metric space (X, d). The pair (A, S) is said to satisfy *property (E.A)*, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

Pathak et al. has shown in [5] that weakly compatibility and property (E.A) are independent to each other (see Ex.2.5, Ex.2.6, Ex.2.7 of [5]).

Example 1.7. *Let $X = \mathbb{C}$ and d be any complex-valued metric on X. Define $f, g : X \to X$ by: $fz = \frac{1}{2}z^2$ and $gz = -bz$, for all $z \in X$, where b is a fixed complex number, $b \neq 0$. Consider a sequence $\{z_n\} = \{\frac{1}{n}\}_{n\geq 1}$ in X, then $\lim_{n\to\infty} fz_n = 0$ and $\lim_{n\to\infty} gz_n = \lim_{n\to\infty}(-\frac{b}{n}) = 0$, as $b \neq 0$.*

Similarly, for another sequence $\{w_n\} = \{-2b + \frac{1}{n}\}_{n\geq 1}$ in X, we have $\lim_{n\to\infty} fw_n = \frac{1}{2}(-2b + \frac{1}{n})^2 = 2b^2$ and $\lim_{n\to\infty} gw_n = \lim_{n\to\infty} -bw_n = \lim_{n\to\infty}(2b^2 - \frac{b}{n}) = 2b^2$. Hence, the pair (f, g) satisfies property (E.A) for the sequences $\{z_n\}$ and $\{w_n\}$ in X with $t = 0$, $2b^2 \in X$ respectively.

2 Main Results

2.1 Fixed Point Theorem Using (E.A)-Property

Theorem 2.1. *Let (X, d) be a complex-valued metric space and $A, B, S, T : X \to X$ be four self-mappings satisfying:*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,

(ii) $d(Ax, By) \precsim k \max\big\{d(Sx, Ty), d(By, Sx), d(By, Ty)\big\}$, $\forall x, y \in X$, $0 < k < 1$,

(iii) the pairs (A, S) and (B, T) are weakly compatible,

(iv) one of the pair (A, S) or (B, T) satisfy property (E.A).

If the range of one of the mappings $S(X)$ or $T(X)$ is a complete subspace of X, then mappings A, B, S and T have a unique common fixed point in X.

Proof. First suppose that the pair (B, T) satisfy property (E.A). Then, by definition 1.6, there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$. Further, since $B(X) \subseteq S(X)$, there exist a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence $\lim_{n\to\infty} Sy_n = t$. We claim that $\lim_{n\to\infty} Ay_n = t$. If not, then putting $x = y_n,\ y = x_n$ in condition (ii), we have

$$\begin{aligned} d(Ay_n, Bx_n) &\precsim k \max\big\{d(Sy_n, Tx_n), d(Bx_n, Sy_n), d(Bx_n, Tx_n)\big\} \\ &= k \max\big\{d(Bx_n, Tx_n), 0, d(Bx_n, Tx_n)\big\}. \end{aligned}$$

Thus $|d(Ay_n, Bx_n)| \leq k\,|\max\big\{d(Bx_n, Tx_n), 0, d(Bx_n, Tx_n)\big\}| = k\,|d(Bx_n, Tx_n)|$, which is a contradiction. Letting $n \to \infty$ we have

$$\lim_{n\to\infty} |d(Ay_n, Bx_n)| \leq k \cdot 0 = 0,$$

which is a contradiction. Thus $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = t$.

Now, suppose first that $S(X)$ is a complete subspace of X, then $t = Su$ for some $u \in X$. Subsequently, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = t = Su. \tag{2.1}$$

We claim that $Au = Su$. For, putting $x = u$ and $y = x_n$ in (ii) we have

$$d(Au, Bx_n) \precsim k \max\big\{d(Su, Tx_n), d(Bx_n, Su), d(Bx_n, Tx_n)\big\},$$

letting $n \to \infty$ and using eq.(2.1), we have

$$d(Au, t) \precsim k \max\big\{d(t, t), d(t, t), d(t, t)\big\} = k \cdot 0 = 0,$$

whence $Au = t = Su$. Hence u is a coincidence point of (A, S). Now, the weak compatibility of pair (A, S) implies that $ASu = SAu$, or $At = St$.

On the other hand, since $A(X) \subseteq T(X)$, there exist v in X such that $Au = Tv$. Thus $Au = Su = Tv = t$. Let us show that v is a coincidence point of (B, T), i.e., $Bv = Tv = t$. If not, then putting $x = u$, $y = v$ in (ii), we have

$$d(Au, Bv) \precsim k \max\big\{d(Su, Tv), d(Bv, Su), d(Bv, Tv)\big\},$$

or,

$$d(t, Bv) \precsim k \max\big\{d(t, t), d(Bv, t), d(Bv, t)\big\};$$

whence $|d(t, Bv)| \leq k\,|\max\big\{d(t, t), d(Bv, t), d(Bv, t)\big\}| \leq k\,|d(Bv, t)| < |d(Bv, t)|$, a contradiction. Thus $Bv = t$. Hence $Bv = Tv = t$, and v is a coincidence point of B and T. Further, the weak compatibility of pair (B, T) implies that $BTv = TBv$, or $Bt = Tt$. Therefore t is a common coincidence point of A, B, S and T.

In order to show that t is a common fixed point, let us put $x = u$ and $y = t$ in (ii) we have

$$\begin{aligned} d(t, Bt) = d(Au, Bt) &\precsim k \max\big\{d(Su, Tt), d(Bt, Su), d(Bt, Tt)\big\} \\ &= k \max\big\{d(t, Bt), d(Bt, t), 0\big\}, \end{aligned}$$

or

$$|d(t, Bt)| \leq k \,|\max\{d(t, Bt), d(Bt, t), 0\}| \leq k|d(t, Bt)| < |d(t, Bt)|,$$

a contradiction. Thus $Bt = t$. Hence $At = Bt = St = Tt = t$.

Similar argument arises if we assume that $T(X)$ is a complete subspace of X. Similarly, the property (E.A) of the pair (A, S) will give the similar result.

For uniqueness of common fixed point, let us assume that w be another common fixed point of A, B, S, T. Then, putting $x = w,\ y = t$ in (ii) we have

$$\begin{aligned} d(w, t) = d(Aw, Bt) &\precsim k \max\{d(Sw, Tt), d(Bt, Sw), d(Bt, Tt)\} \\ &= k \max\{d(w, t), d(t, w), 0\}, \end{aligned}$$

whence,

$$|d(t, w)| \leq k \,|\max\{d(w, t), d(t, w), 0\}| = k\,|d(t, w)| < |d(t, w)|,$$

a contradiction. Thus $w = t$. Hence $At = Bt = St = Tt = t$, and t is the unique common fixed point of A, B, S, T. This completes the proof. □

Remark 2.2. *Continuity of mappings A, B, S, T is relaxed in Theorem 2.1.*

Remark 2.3. *Completeness of space X is relaxed in Theorem 2.1.*

If $A = B$ and $S = T$ in Theorem 2.1, we have the following result:

Corollary 2.4. *Let (X, d) be any complex-valued metric space and $A, S : X \to X$ be two self-mappings satisfying:*

(i) $A(X) \subseteq S(X)$,

(ii) $d(Ax, Ay) \precsim k \max\{d(Sx, Sy), d(Ay, Sx), d(Ay, Sy)\}$, $\forall x, y \in X$, $0 < k < 1$,

(iii) the pairs (A, S) is weakly compatible,

(iv) the pair (A, S) satisfy property (E.A).

If $S(X)$ is complete, then A and S have unique common fixed point in X.

2.2 Fixed Point Theorem Using (CLR)-Property

The notion of (CLR)-property was defined by Sintunavarat and Kumam [6] in a metric space for a pair of self-mappings, which have the common limit in the range of one of the mappings.

Definition 2.5 (The (CLR)-property [6])**.** Suppose that (X, d) is a metric space and $f, g : X \to X$. Two mappings f and g are said to satisfy the *common limit in the range of g* property if $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$, for some $x \in X$.

In the complex-valued metric space, the definition will be same but the space X will be a complex-valued metric space.

Example 2.6. *Let $X = \mathbb{C}$ and d be any complex-valued metric on X. Define $f, g : X \to X$ by: $fz = z + 3i$ and $gz = 4z$, for all $z \in X$. Consider a sequence $\{z_n\} = \{i + \frac{1}{n}\}_{n\geq 1}$ in X, then*
$\lim_{n\to\infty} fz_n = \lim_{n\to\infty} z_n + 3i = \lim_{n\to\infty}(i + \frac{1}{n}) + 3i = 4i$, *and*
$\lim_{n\to\infty} gz_n = \lim_{n\to\infty} 4(i + \frac{1}{n}) = 4i = g(0 + i)$.
Hence, the pair (f, g) satisfies property (CLRg) in X with $x = 0 + i \in X$.

Some papers related to (CLR) property and the complex-valued metric spaces can be found in [7–9] of Sintunavarat and Kumam. Here is our main theorem using (CLR) property for two pairs of self-mappings in complex-valued metric space:

Theorem 2.7. *Let (X, d) be a complex-valued metric space and $A, B, S, T : X \to X$ be four self-mappings satisfying:*

(i) $A(X) \subseteq T(X)$,

(ii) $d(Ax, By) \precsim k \max\big\{d(Sx, Ty), d(By, Sx), d(By, Ty)\big\}$, $\forall x, y \in X$, $0 < k < 1$,

(iii) the pairs (A, S) and (B, T) are weakly compatible.

If the pair (A, S) satisfy (CLR_A) property, or the pair (B, T) satisfy (CLR_B) property, then mappings A, B, S and T have a unique common fixed point in X.

Proof. First suppose that the pair (B, T) satisfy the (CLR_B) property; then by Definition 2.5, there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = Bx. \tag{2.2}$$

for some $x \in X$. Further, since $BX \subseteq SX$, we have $Bx = Su$ for some $u \in X$. We claim that $Au = Su$ $(= t$ say). If not, then putting $x = u$ and $y = x_n$ in (ii) we have

$$d(Au, Bx_n) \precsim k \max\big\{d(Su, Tx_n), d(Bx_n, Su), d(Bx_n, Tx_n)\big\}$$

letting $n \to \infty$ and using (2.2) we have

$$d(Au, Bx) \precsim k \max\big\{d(Bx, Bx), d(Bx, Bx), d(Bx, Bx)\big\} = k.\ 0 = 0$$

whence $|d(Au, Bx)| \leq 0$, which is a contradiction. Thus $Au = Su$. Hence $Au = Su = Bx = t$. It shows that u is a coincidence point of (A, S). Also the weak compatibility of (A, S) implies that $ASu = SAu = At = St$. Further, since $AX \subseteq TX$, there exist some $v \in X$ such that $Au = Tv$. We claim that $Bv = t$. If not, then from (ii), we have

$$d(Au, Bv) \precsim k \max\big\{d(Su, Tv), d(Bv, Su), d(Bv, Tv)\big\}$$

i.e.,

$$d(t, Bv) \precsim k \max\big\{d(t, t), d(Bv, t), d(Bv, t)\big\}.$$

So,

$$|d(t, Bv)| \leq k| \max\{0, d(Bv, t), d(Bv, t)\}| \leq k|d(Bv, t)| < |d(Bv, t)|,$$

which is a contradiction. Thus $Bv = t$. Hence $Au = Su = t = Bv = Tv$. It shows that v is a coincidence point of pair (B, T). Since, the pair (B, T) is weakly compatible, we have $BTv = TBv$, or, $Bt = Tt$. Thus t is a common coincidence point of (A, S) and (B, T). We claim that t is a common fixed point of A, B, S, T. If not, then from (ii) we have

$$\begin{aligned} d(t, Bt) = d(Au, Bt) &\precsim k \max\{d(Su, Tt), d(Bt, Su), d(Bt, Tt)\} \\ &= k \max\{d(t, Bt), d(Bt, t), 0\}, \end{aligned}$$

whence $|d(t, Bt)| \leq k|d(Bt, t)| < |d(Bt, t)|$, which is a contradiction. Thus $Bt = t$. Hence t is a common fixed point of A, B, S and T. The uniqueness of common fixed point t follows easily. In the similar way, the argument that the pair (A, S) satisfy property (CLR_A) will also give the unique common fixed point of A, B, S and T. Hence in both cases we conclude the same result of existence and uniqueness of common fixed point of A, B, S and T. This completes the proof. □

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