Thai Journal of Mathematics
Volume 11 (2013) Number 2 : 337–345
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On Characterization of Inextensible Flows of Tangent Developable Surfaces of Timelike Biharmonic General Helices in the Lorentzian $\mathbb{E}(1,1)$

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Abstract : In this paper, we study inextensible flows of tangent developable surfaces of timelike biharmonic general helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. We obtain the corresponding equations for the inextensible flow of tangent developable surfaces of timelike biharmonic general helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

Keywords : biharmonic curve; bienergy; general helix; geometric flows; harmonic map.
2010 Mathematics Subject Classification : 53C41; 53A10.

1 Introduction

Geometric flows have been extensively used in mathematics. In particular, surface flows based on functional minimization (i.e. evolving a surface so as to progressively decrease an energy functional) is a common methodology in geometry processing with applications spanning surface diffusion, shape optimization and surface design, minimal surfaces, (geodesic) shortest paths, and animation, [1–4].

Physically, inextensible curve and surface flows are characterized by the

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absence of any strain energy induced from themotion. The earliest (and most mathematically tractable), is as an equilibrium of moments, drawing on a fundamental principle of statics. Another approach, ultimately yielding the same equation for the curve, is as a minimum of bending energy in the elastic curve. A forcebased approach finds that normal, compression, and shear forces are also in equilibrium; this approach is useful when considering specific constraints on the endpoints, which are often intuitively expressed in terms of these forces, [5–7]. On the other hand, harmonic maps $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \mathrm{trace}\nabla df. \tag{1.2}$$

Bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [8], showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace}R^N(df, \tau(f))\,df \\ &= 0,\end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic [9–13]. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

This study is organised as follows: Firstly, we study tangent developable surfaces of timelike biharmonic general helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Finally, we obtain parametric equation of tangent developable surfaces of timelike biharmonic general helices in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

2 Preliminaries

Let $\mathbb{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $\mathbb{E}(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$\mathbb{E}(1,1) \longrightarrow \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x, y, z),$$

It's Lie algebra has a basis consisting of

$$\mathbf{X}_1 = \frac{\partial}{\partial x},\ \mathbf{X}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z},\ \mathbf{X}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3,\ [\mathbf{X}_2, \mathbf{X}_3] = 0,\ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2.$$

Put

$$x^1 = x,\ x^2 = \frac{1}{2}(y+z),\ x^3 = \frac{1}{2}(y-z).$$

Then, we get

$$\mathbf{X}_1 = \frac{\partial}{\partial x^1},\ \mathbf{X}_2 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} + e^{-x^1}\frac{\partial}{\partial x^3}\right),\ \mathbf{X}_3 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} - e^{-x^1}\frac{\partial}{\partial x^3}\right). \tag{2.1}$$

The bracket relations are

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3,\ [\mathbf{X}_2, \mathbf{X}_3] = 0,\ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2. \tag{2.2}$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. We consider left-invariant Lorentzian metric [14], given by

$$g = -\left(dx^1\right)^2 + \left(e^{-x^1}dx^2 + e^{x^1}dx^3\right)^2 + \left(e^{-x^1}dx^2 - e^{x^1}dx^3\right)^2, \tag{2.3}$$

where

$$g(\mathbf{X}_1, \mathbf{X}_1) = -1,\ g(\mathbf{X}_2, \mathbf{X}_2) = g(\mathbf{X}_3, \mathbf{X}_3) = 1. \tag{2.4}$$

Let coframe of our frame be defined by

$$\theta^1 = dx^1,\ \theta^2 = e^{-x^1}dx^2 + e^{x^1}dx^3,\ \theta^3 = e^{-x^1}dx^2 - e^{x^1}dx^3.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ -\mathbf{X}_3 & 0 & -\mathbf{X}_1 \\ -\mathbf{X}_2 & -\mathbf{X}_1 & 0 \end{pmatrix}, \tag{2.5}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{X}_k, k = 1,2,3\} = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}.$$

3 Timelike Biharmonic General Helices in the Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a non geodesics timelike curve in the group of rigid motions $\mathbb{E}(1,1)$ parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the group of rigid motions $\mathbb{E}(1,1)$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ) and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\
\nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\
\nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N},
\end{aligned} \tag{3.1}$$

where κ is the curvature of γ, τ is its torsion and

$$\begin{aligned}
&g(\mathbf{T},\mathbf{T}) = -1,\ g(\mathbf{N},\mathbf{N}) = 1,\ g(\mathbf{B},\mathbf{B}) = 1, \\
&g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0.
\end{aligned} \tag{3.2}$$

With respect to the orthonormal basis $\{\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3\}$ we can write

$$\begin{aligned}
\mathbf{T} &= T_1\mathbf{X}_1 + T_2\mathbf{X}_2 + T_3\mathbf{X}_3, \\
\mathbf{N} &= N_1\mathbf{X}_1 + N_2\mathbf{X}_2 + N_3\mathbf{X}_3, \\
\mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{X}_1 + B_2\mathbf{X}_2 + B_3\mathbf{X}_3.
\end{aligned} \tag{3.3}$$

Theorem 3.1 ([11])**.** *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic timelike biharmonic general helix in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the parametric equations of γ are*

$$\begin{aligned}
x^1(s) &= \cosh\eth\kappa s + \wp_3, \\
x^2(s) &= \frac{\sinh\eth e^{\cosh\eth\kappa s+\wp_3}}{2\left(\wp_1^2+\cosh^2\eth\right)}\{(\cosh\eth-\wp_1)\cos(\wp_1\kappa s+\wp_2) \\
&\quad + (\cosh\eth+\wp_1)\sin(\wp_1\kappa s+\wp_2)\} + \wp_4, \\
x^3(s) &= \frac{\sinh\eth e^{-\cosh\eth\kappa s-\wp_3}}{2\left(\wp_1^2+\sinh^2\eth\right)}\{-(\cosh\eth-\wp_1)\cos(\wp_1\kappa s+\wp_2) \\
&\quad + (\cosh\eth+\wp_1)\sin(\wp_1\kappa s+\wp_2)\} + \wp_5,
\end{aligned} \tag{3.4}$$

where $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$, $\wp_5$ are constants of integration.

4 Inextensible Flows of Tangent Developable Surfaces of Timelike Biharmonic General Helices in the Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

Developable surfaces are defined as the surfaces on which the Gaussian curvature is 0 everywhere. The developable surfaces are useful since they can be made out of sheet metal or paper by rolling a flat sheet of material without stretching it. Most large-scale objects such as airplanes or ships are constructed using un-stretched sheet metals, since sheet metals are easy to model and they have good stability and vibration properties. Moreover, sheet metals provide good fluid dynamic properties. In ship or airplane design, the problems usually stem from engineering concerns and in engineering design there has been a strong interest in developable surfaces.

The tangent developable of γ is a ruled surface

$$\Pi(s,u)=\gamma(s)+u\gamma'(s). \tag{4.1}$$

Let ϖ be the standard unit normal vector field on a surface Π defined by

$$\varpi=\frac{\Pi_s\wedge\Pi_u}{\left|g\left(\Pi_s\wedge\Pi_u,\Pi_s\wedge\Pi_u\right)\right|^{\frac{1}{2}}}.$$

Then, the first fundamental form $\mathbf{I}$ is defined by

$$\mathbf{I}=\mathbf{E}ds^2+2\mathbf{F}dsdu+\mathbf{G}dt^2,$$

where

$$\mathbf{E}=g\left(\Pi_s,\Pi_s\right),\quad \mathbf{F}=g\left(\Pi_s,\Pi_u\right),\quad \mathbf{G}=g\left(\Pi_u,\Pi_u\right).$$

Theorem 4.1. *Let $\gamma:I\longrightarrow\mathbb{E}(1,1)$ is a non geodesic timelike biharmonic general helix in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the parametric equations of tangent developable of γ are*

$$\begin{aligned}
x_{\Pi}^{1}(s,u)&=\cosh\eth\kappa s+u\cosh\eth+\wp_3,\\
x_{\Pi}^{2}(s,u)&=\frac{\sinh\eth e^{\cosh\eth\kappa s+\wp_3}}{2\left(\wp_1^2+\cosh^2\eth\right)}\{(\cosh\eth-\wp_1)\cos(\wp_1\kappa s+\wp_2)\\
&\quad+(\cosh\eth+\wp_1)\sin(\wp_1\kappa s+\wp_2)\}\\
&\quad+\frac{u\sinh\eth e^{\cosh\eth\kappa s+\wp_3}}{2}[\cos(\wp_1\kappa s+\wp_2)+\sin(\wp_1\kappa s+\wp_2)]+\wp_4,\\
x_{\Pi}^{3}(s,u)&=\frac{\sinh\eth e^{-\cosh\eth\kappa s-\wp_3}}{2\left(\wp_1^2+\cosh^2\eth\right)}\{-(\cosh\eth-\wp_1)\cos(\wp_1\kappa s+\wp_2)\\
&\quad+(\cosh\eth+\wp_1)\sin(\wp_1\kappa s+\wp_2)\}\\
&\quad+\frac{u\sinh\eth e^{-\cosh\eth\kappa s-\wp_3}}{2}[\cos(\wp_1\kappa s+\wp_2)-\sin(\wp_1\kappa s+\wp_2)]+\wp_5,
\end{aligned}\tag{4.2}$$

where $\wp_1, \wp_2, \wp_3, \wp_4, \wp_5$ *are constants of integration.*

Proof. By Theorem (3.1), we immediately arrive at

$$\mathbf{T} = \Big(\cosh \eth, \frac{1}{2}e^{\cosh \eth \kappa s + \wp_3} \sinh \eth [\cos(\wp_1 \kappa s + \wp_2) + \sin(\wp_1 \kappa s + \wp_2)], \\ \frac{1}{2}e^{-\cosh \eth \kappa s - \wp_3} \sinh \eth [\cos(\wp_1 \kappa s + \wp_2) - \sin(\wp_1 \kappa s + \wp_2)]\Big), \tag{4.3}$$

where $\wp_3$ is constant of integration. From (4.1) and (4.3), by direct calculation we have (4.2), which proves the theorem. □

Definition 4.2 ([6])**.** A surface evolution $\Pi(s,u,t)$ and its flow $\frac{\partial \Pi}{\partial t}$ are said to be *inextensible* if its first fundamental form $\{\mathbf{E},\mathbf{F},\mathbf{G}\}$ satisfies

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{F}}{\partial t} = \frac{\partial \mathbf{G}}{\partial t} = 0. \tag{4.4}$$

This definition states that the surface $\Pi(s,u,t)$ is, for all time t, the isometric image of the original surface $\Pi(s,u,t_0)$ defined at some initial time t_0. For a developable surface, $\Pi(s,u,t)$ can be physically pictured as the parametrization of a waving flag. For a given surface that is rigid, there exists no nontrivial inextensible evolution.

Definition 4.3. We can define the following one-parameter family of developable ruled surface

$$\Pi(s,u,t) = \gamma(s,t) + u\gamma'(s,t). \tag{4.5}$$

Hence, we have the following theorem.

Theorem 4.4. *Let* Π *is the one-parameter family of tangent developable surface associated with non geodesic timelike biharmonic general helix in the Lorentzian group of rigid motions* $\mathbb{E}(1,1)$, *then* $\frac{\partial \Pi}{\partial t}$ *is inextensible if and only if*

$$\frac{\partial}{\partial t}[\cosh \eth - 2u(\sinh^2 \eth \cos(\wp_1 \kappa s + \wp_2)\sin(\wp_1 \kappa s + \wp_2))]^2 \tag{4.6}$$
$$= \frac{\partial}{\partial t}\sinh^2 \eth \left[\cos(\wp_1 \kappa s + \wp_2) - u \sin(\wp_1 \kappa s + \wp_2)\left(\frac{1}{\wp_1 \kappa} + \cosh \eth\right)\right]^2$$
$$+ \frac{\partial}{\partial t}\sinh^2 \eth \left[\sin(\wp_1 \kappa s + \wp_2) + u \cos(\wp_1 \kappa s + \wp_2)\left(\frac{1}{\wp_1 \kappa} - \cosh \eth\right)\right]^2,$$

where $\kappa, \eth, \wp_1, \wp_2$ *are function of time* t.

Proof. Assume that $\Pi(s,u,t)$ be a one-parameter family of ruled surface. We show that $\frac{\partial \Pi}{\partial t}$ is inextensible.

On the other hand, from Theorem 3.1, we have

$$\mathbf{T} = \cosh \eth \mathbf{e}_1 + \sinh \eth \cos(\wp_1 \kappa s + \wp_2)\mathbf{e}_2 + \sinh \eth \sin(\wp_1 \kappa s + \wp_2)\mathbf{e}_3. \tag{4.7}$$

Using first equation of (3.3) and basis, we have

$$\nabla_{\mathbf{T}}\mathbf{T}=\left(T_1'-2T_2T_3\right)\mathbf{X}_1+\left(T_2'-T_1T_3\right)\mathbf{X}_2+\left(T_3'-T_1T_2\right)\mathbf{X}_3. \tag{4.8}$$

Further, substituting components of (4.7) in above equation we get

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T}=&\left(-2\sinh^2\eth\cos\left(\wp_1\kappa s+\wp_2\right)\sin\left(\wp_1\kappa s+\wp_2\right)\right)\mathbf{X}_1\\&-\sinh\eth\sin\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}+\cosh\eth\right)\mathbf{X}_2\\&+\sinh\eth\cos\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}-\cosh\eth\right)\mathbf{X}_3.\end{aligned}$$

From the Frenet formula (3.1), we have

$$\mathbf{N}=\frac{1}{\kappa}\nabla_{\mathbf{T}}\mathbf{T}.$$

Since

$$\begin{aligned}\mathbf{N}=&-\frac{2}{\kappa}\left(\sinh^2\eth\cos\left(\wp_1\kappa s+\wp_2\right)\sin\left(\wp_1\kappa s+\wp_2\right)\right)\mathbf{X}_1\\&-\frac{1}{\kappa}\sinh\eth\sin\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}+\cosh\eth\right)\mathbf{X}_2\\&+\frac{1}{\kappa}\sinh\eth\cos\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}-\cosh\eth\right)\mathbf{X}_3.\end{aligned}$$

Furthermore, we have the natural frame $\{\Pi_s,\Pi_u\}$ given by

$$\begin{aligned}\Pi_s=&\left[\cosh\eth-2u\left(\sinh^2\eth\cos\left(\wp_1\kappa s+\wp_2\right)\sin\left(\wp_1\kappa s+\wp_2\right)\right)\right]\mathbf{X}_1\\&+\sinh\eth\left[\cos\left(\wp_1\kappa s+\wp_2\right)-u\sin\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}+\cosh\eth\right)\right]\mathbf{X}_2\\&+\sinh\eth\left[\sin\left(\wp_1\kappa s+\wp_2\right)+u\cos\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}-\cosh\eth\right)\right]\mathbf{X}_3\end{aligned}$$

and

$$\Pi_u=\cosh\eth\mathbf{X}_1+\sinh\eth\cos\left(\wp_1\kappa s+\wp_2\right)\mathbf{X}_2+\sinh\eth\sin\left(\wp_1\kappa s+\wp_2\right)\mathbf{X}_3.$$

The components of the first fundamental form are

$$\begin{aligned}\mathbf{E}=g(\Pi_s,\Pi_s)=&-\left[\cosh\eth-2u\left(\sinh^2\eth\cos\left(\wp_1\kappa s+\wp_2\right)\sin\left(\wp_1\kappa s+\wp_2\right)\right)\right]^2\\&+\sinh^2\eth\left[\cos\left(\wp_1\kappa s+\wp_2\right)-u\sin\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}+\cosh\eth\right)\right]^2\\&+\sinh^2\eth\left[\sin\left(\wp_1\kappa s+\wp_2\right)+u\cos\left(\wp_1\kappa s+\wp_2\right)\left(\frac{1}{\wp_1\kappa}-\cosh\eth\right)\right]^2,\end{aligned}$$

$$\mathbf{F} = g(\Pi_s, \Pi_u) = -1,$$
$$\mathbf{G} = g(\Pi_u, \Pi_u) = -1.$$

Using second and third equation of above system, we have

$$\frac{\partial \mathbf{F}}{\partial t} = 0,$$
$$\frac{\partial \mathbf{G}}{\partial t} = 0.$$

Hence, $\frac{\partial \Pi}{\partial t}$ is inextensible if and only if (4.6) is satisfied. This concludes the proof of theorem. □

We can use Mathematica in Theorem 4.1, yields

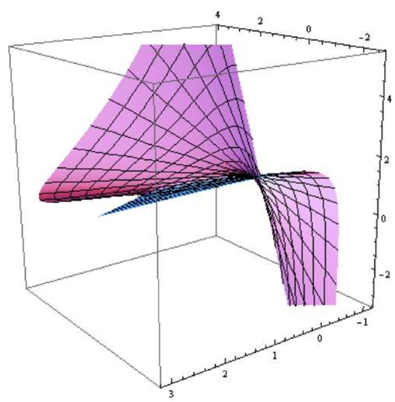

Figure 1.

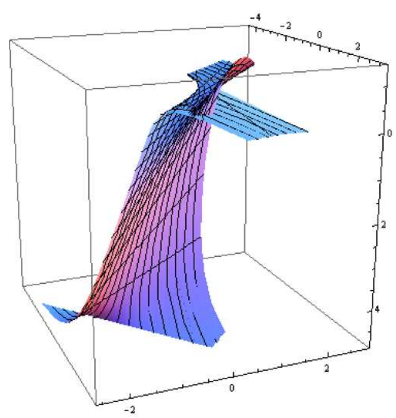

Figure 2.

Acknowledgement : The authors thank to the referee for useful suggestions and remarks for the revised version.

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(Received 17 October 2011)
(Accepted 3 April 2012)