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S_β-Open Sets and S_β-Continuity in Topological Spaces

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Abstract : In this paper we introduce a subclass of semi open sets called S_β-open sets in topological spaces. This class of sets used to define and study the concept of S_β-continuous functions.

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1 Introduction and Preliminaries

Throughout this paper, a space means a topological space on which no separation axioms are assumed unless explicitly stated. In 1963 [1] Levine was initiated semi open sets and their properties, Mathematicians gives in several papers interesting and different new types of sets. In [2], Abd-El-Moonsef in 1983 defined the class of β-open set. In 2010, Shareef [3] introduced a new class of semi-open sets called S_P-open sets. We recall the following definitions and characterizations. The closure (resp., interior) of a subset A of X is denoted by clA (resp., $intA$). A subset A of X is said to be semi-open [1] (resp., pre-open [4], α-open [5], β-open [3], regular open [6] and regular β-open [7]) set if $A \subseteq clintA$ (resp., $A \subseteq intclA$, $A \subseteq intclintA$, $A \subseteq clintclA$, $A = intclA$ and $A = \beta int\beta clA$). The complement

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of semi-open (resp., pre-open, α-open, β-open, regular open, regular β-open) set is said to be semi-closed (resp., pre- closed, α-closed, β-closed, regular closed, regular β-closed). The intersection of all semi-closed (resp., pre-closed, β-closed) sets of X containing a subset A is called the semi-closure (resp., pre-closure, β-closure) of A and denoted by $sclA$ (resp., $pclA$, βclA). The union of all semi-open (resp., pre-open, β-open) set of X contained in A is called the semi-interior (resp., pre-interior, β-interior) of A and denoted by $sintA$ (resp., $pintA$, $\beta intA$). The family of all semi-open (resp., pre-open, α-open, β-open, regular β-open, regular open, semi-closed, pre-closed, α-closed, β-closed, regular β-closed, and regular closed) subset of a topological space X is denoted by $SO(X)$ (resp., $PO(X)$, α $O(X)$, β $O(X)$,$R\beta$ $O(X)$ $RO(X), SC(X), PC(X)$, α $C(X)$, β $C(X)$, $R\beta$ $C(X)$ and $RC(X)$). A subset A of X is called δ-open [8] if for each $x \in A$, there exists an open set B such that $x \in B \subseteq intclB \subseteq A$. A subset A of a space X is called θ-semi-open [9] (resp., semi-θ-open [10] if for each $x \in A$, there exists a semi-open set G such that $x \in G \subseteq clG \subseteq A$ (resp., $x \in G \subseteq sclG \subseteq A$). A function $f : X \to Y$ is a semi-continuous if the inverse image of each open subset of Y is semi-open in X Also f is said to be δ-continuous if for each x in X and each open set V of Y containing $f(x)$ there exists an open set U of X such that $f(intclU) \subseteq intclf(V)$.

Definition 1.1 ([4])**.** A topological space (X, τ) is said to be

1. *Extremally disconnected* if $clV \in \tau$ for every $V \in \tau$.
2. *Locally indiscrete* if every open subset of X is closed.
3. *Hyperconnected* if every nonempty open subset of X is dense.

Lemma 1.2.

1. *If X is a locally indiscrete space, then each semi-open subset of X is closed and hence each semi-closed subset of X is open [11].*
2. *A topological space (X, τ) is hyperconnected if and only if $RO(X) = \{\phi, X\}$ [12].*

Theorem 1.3 ([13])**.** *Let (X, τ) be a topological space, then:*

1. *$SO(X, \tau) = SO(X, \alpha O(X))$.*
2. *$\beta C(X, \tau) = \beta C(X, \alpha O(X))$.*

Theorem 1.4 ([1])**.**

1. *Let A be any subset of a space X. Then $A \in SO(X, \tau)$ if and only if $clA = clintA$.*
2. *If $\{A_\gamma : \gamma \in \Gamma\}$ is a collection of semi-open sets in a topological space (X, τ), then $\cup\{A_\gamma : \gamma \in \Gamma\}$ is semi-open.*
3. *Let (Y, τ_Y) be a subspace of a space (X, τ). If $A \subseteq Y$ and $A \in SO(X)$, then $A \in SO(Y, \tau_Y)$.*

Theorem 1.5.

1. *If Y is a semi-open subspace of a space X, then a subset A of Y is a semi-open set in X if and only if A is semi-open set in Y [14].*
2. *Let (Y, τ_Y) be a subspace of a space (X, τ). If $A \in SO(Y, \tau_Y)$ and $Y \in SO(X, \tau)$, then $A \in SO(X, \tau)$ [15].*
3. *If F is an closed subset of a space X and $B \in \beta c(X)$, then $F \cup B \in \beta c(X)$ [7].*

Theorem 1.6 ([16])**.** *Let (X, τ) be a topological space. If $A \in \tau$ and $B \in SO(X)$, then $A \cap B \in SO(X)$.*

Theorem 1.7. *For any spaces X and Y. If $A \subseteq X$ and $B \subseteq Y$, then:*

1. *$\beta cl_{X \times Y}(A \times B) = \beta cl_X(A) \times \beta cl_Y(B)$ [7].*
2. *$sint_{X \times Y}(A \times B) = sint_X(A) \times sint_Y(B)$ [15].*

Definition 1.8. A subset A of a space X is said to be $S_p - open$ [3] (resp., $S_c - open$ [17]) if for each $x \in A$ there exists a pre-closed (resp., closed) set F such that $x \in F \subseteq A$.

Proposition 1.9 ([16])**.** *LetA be any subset of a space X.Then $A \in SC(X)$if and only if $intclA \subseteq A$.*

Theorem 1.10 ([14])**.** *A subset A of a space X is dense in X if and only if A is semi-dense in X.*

Theorem 1.11 ([7])**.** *The intersection of a β-open set and an α-open set is β-open.*

Theorem 1.12 ([7])**.** *Let(Y, τ_Y) be a subspace of a space(X, τ), and $Y \in \alpha C(X)$, then $A \in \beta C(X)$ If and only if $A \in \beta C(Y)$.*

Theorem 1.13 ([18])**.** *A space X is extremely disconnected if and only if $RO(X) = RC(X)$.*

2 S_β-Open Sets

In this section, we introduce and study the concept of S_β-open sets in topological spaces and give some basic properties of this set.

Definition 2.1. A semi open subset A of a topological space (X, τ) is said to be S_β-open if for each $x \in A$ there exists a β-closed set F such that $x \in F \subseteq A$. A subset B of a topological space X is S_β-closed, if $X \setminus B$ is S_β-open.

The family of S_β-open subsets of X is denoted by $S_\beta O(X)$.

Proposition 2.2. *A subset A of a topological space (X, τ) is S_β-open set if and only if A is semi open and it is a union of β-closed sets.*

Proof. Obvious. □

The following result shows that any union of S_β-open sets in a topological space is S_β-open.

Theorem 2.3. *Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of S_β-open sets in a topological space (X, τ). Then $\bigcup_{\alpha\in\Delta} A_\alpha$ is an S_β-open set.*

Proof. The union of an arbitrary semi open sets is semi open Theorem 1.4. Suppose that $x \in \bigcup_{\alpha\in\Delta} A_\alpha$, this implies that there exists $\alpha_0 \in \Delta$ such that $x \in A_{\alpha 0}$ and since $A_{\alpha 0}$ is an S_β-open set, so there exists a β-closed set F in X such that $x \in F \subseteq A_{\alpha 0} \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$. Therefore, $\bigcup_{\alpha\in\Delta} A_\alpha$ is an S_β-open set. □

From Theorem 2.3 it is clear that any intersection of S_β-closed sets of a topological space (X, τ) is S_β-closed. The following example shows that the intersection of two S_β-open sets is not an S_β-open set.

Example 2.4. *Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then $\{a, c\}$ and $\{b, c\}$ are S_β-open sets in X but $\{a, c\} \cap \{b, c\} = \{c\}$ is not S_β-open sets.*

Proposition 2.5. *A subset G in the space X is S_β-open, if and only if for each $x \in G$ there exists an S_β-open set H such that $x \in H \subseteq G$.*

Proof. Let G be an S_β-open set in X, then for each $x \in G$, we have G is an S_β-open set containing x such that $x \in G \subseteq G$.

Conversely, suppose that for each $x \in G$ there exists an S_β-open set H such that $x \in H \subseteq G$, then G is a union of S_β-open sets, hence by Theorem 2.3, G is S_β-open. □

Proposition 2.6. *Every semi-θ-open subset A of X is S_β-open.*

Proof. Let $A \in S_\theta O(X)$, then for each $x \in A$ there exists a semi-open set G such that $x \in G \subseteq sclG \subseteq A$, So A is semi open and moreover, $sclG$ is semi closed and hence it is β-closed. Therefore, by Proposition 2.5 $A \in S_\beta(O(X))$. □

Proposition 2.7. *A subset A of a topological space X is regular β open if A is $S_\beta O(X)$.*

Proof. First if $A \in S_\beta O(X))$ then A is semi-open and for each $x \in A$ there exist a β-closed set F such that $x \in F \subseteq A$, therefore $x \in F = \beta clF \subseteq A$, so we get $x \in \beta clF \subseteq A$,since $A \in SO(X)$, then $A \in \beta O(X)$ and $x \in \beta clF \subseteq A$, it follows that A is regular $\beta O(X)$ □

Corollary 2.8. *Every $\theta SO(X), \theta(OX), RSO(X)$ and $\delta(OX)$ are $S_\beta(OX)$.*

Proof. Since each of $\theta SO(X), \theta(OX), RSO(X)$ and $\delta(OX)$ are $S\theta O(X)$. □

Proposition 2.9.

1. *Every S_P-open set is S_β-open*
2. *S_β-open set is regular β-open set.*
3. *Regular closed set is S_β-open set.*
4. *Every Regular open sets is S_β-closed set.*

Proof. Obvious. □

In general, the converse of above proposition not true in general as shown in the following examples.

Example 2.10. *Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a, b\}, X\}$, then $S_\beta(O(X)) = \{\phi, X\}$ and $R\beta(O(X)) = PO(X) \setminus \{\{a, b\}, \{c\}\}$.*

Example 2.11. *Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$, then $SO(X) = \beta O(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, c, d\}, \{a, b, c\}, \{a, b, d\}, X\}$ so $S_\beta O(X) = \{\phi, \{b\}, \{a, c, d\}, X\}$ but $S\theta O(X) = SO(X)$.*

Example 2.12. *Let $X = \{a, b, c, d\}$ and let $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ then $\beta O(X) = SO(X) = \setminus\{\{c\}, \{d\}, \{c, d\}\}$ and $PO(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, X\}$ we get that $S_\beta O(X) = SO(X)$ but $S_P O(X) = \{\phi, X\}$.*

Proposition 2.13. *If a space X is T_1-space, then $S_\beta O(X) = SO(X)$.*

Proof. Since every closed set is β-closed and every singleton set is closed. Hence $SO(X) = S_\beta O(X)$. □

Proposition 2.14. *If the family of all semi-open subsets of a topological space is a topology on X then the family of all $S_\beta O(X)$ is also a topology on X.*

Proof. Obvious. □

Proposition 2.15. *Let (X, τ) be a topological space and if X is extremally disconnected then $S_\beta O(X)$ form a topology on X.*

Proof. Obvious. □

Proposition 2.16. *If a space X is hyperconnected, then the only S_β- open sets in X are ϕ, and X.*

Proof. Suppose that $A \subseteq X$ such that A is S_β- open sets in X. If $A = X$, then there is nothing to prove. If $A \neq X$, then we must prove that $A = \phi$, since A is S_β- open sets in X then by definition 2.1, for each $x \in A$ there exist $F \in \beta C(X)$ such that $x \in F \subseteq A$,therefore $X \setminus A \subseteq X \setminus F$, but $X \setminus A$ is semi closed, then by Preposition 1.9 implies that $intclX \setminus A \subseteq (X \setminus A)$. Since X is hyperconnected, then by definition 1.1 and Theorem 1.10 $Scl(int(cl(X \setminus A)) = X \subseteq (X \setminus A)$. Thus $X \setminus A = X$ this implies that $A = \phi$. Hence the only S_β-open sets of X are ϕ and X. □

If (Y, τ_Y) is a subspace of the space (X, τ) and if a subset A is S_β- open set relative to Y, then A may not be S_β- open set in X, as shown in the following example:

Example 2.17. *Let $X = \{a, b, c, d\}$ and let $\tau = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, c, d\}, \{a, b, c\}, \{a, b, d\}, X\}$. So we obtain that $S_\beta O(X) = \{\phi, \{b\}, \{c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, X\}$. let $Y = \{b, c, d\}$, then $\tau_Y = \{\phi, \{d\}, \{b\}, \{c\}, \{c, d\}, \{b, d\}, \{b, c\}, Y\}$ is relative topology on Y, $S_\beta O(Y) = P(X)$ then $\{a\}$ is S_β-open set on Y, but $\{a\}$ is not S_β-open set in X.*

Also if Y is a subspace of a space X, and if A is S_β-open set in X, then $A \cup Y$ may not be S_β-open set in Y. As shown in the following example.

Example 2.18. *Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\} = S_\beta O(X)$, let $Y = \{a, c, d\}$,then $\tau_Y = \{\phi, \{a\}, \{c\}, \{a, c\}, \{a, d\}, Y\}$ is relative topology on Y, and $S_\beta O(Y) = \{\phi, \{c\}, \{a, d\}, Y\}$, but $\{a, b\} \in S_\beta O(X)$ and $\{a, b\} \cap Y = \{a\} \notin S_\beta O(Y)$.*

Proposition 2.19. *If a topological space X is locally indiscrete, then every semi-open set is S_β-open.*

Proof. Let A be a semi-open set in X, then $A \subseteq (intclA)$. Since X is locally indiscrete, then $intA$ is closed and hence $intA = clintA$, which implies that A is regular closed in X. Therefore by Proposition 2.9, A is S_β-open. □

Remark 2.20. *Since every open set is semi-open set, it follows that if a topological space (X, τ) is T_1 or Locally indiscrete then $\tau \subseteq S_\beta O(X)$.*

Lemma 2.21. *If B is clopen subset of a space X and A is S_β-open set in X, then $A \cap B \in S_\beta O(X)$.*

Proof. Let A be S_β-open set, so A is semi-open and B is open and closed in X, then by Theorem 1.6 $A \cap B$ is semi-open in X, let $x \in A \cap B$, this implies that $x \in A$ and $x \in B$, since A is S_β-open there exist a β-closed set F in X such that $x \in F \subseteq A$, also B is closed then B is β-closed,hence $B \cap F$ is β-closed set, therefore $A \cap B$ isS_β-open set in X. □

Proposition 2.22. *Let (X, τ) be a locally indiscrete topological space, and $A, B \subseteq X$. If $A \in S_\beta O(X)$ and B is open, then $A \cap B$ is S_β-open set in X.*

Proof. Follows from Lemma 2.21. □

Proposition 2.23. *Let (X, τ) be an extremally disconnected topological space and, $A, B \subseteq X$. If $A \in S_\beta O(X)$ and $B \in RO(X)$, then $A \cap B$ is S_β-open set in X.*

Proof. Let $A \in S_\beta O(X)$ and $B \in RO(X)$ so A is semi-open set. Then by Theorem 1.6. $A \cap B \in SO(X)$. Now let $x \in A \cap B$, this implies that $x \in A$ and $x \in B$, since A is S_β-open there exist a β-closed set F in X such that $x \in F \subseteq A$. Since X is extremally disconnected, then by Theorem 1.13 B is a regular closed set. This implies that $B \cap F$ is β-closed set, therefore $x \in (F \cap B) \subseteq (A \cap B)$, so $A \cap B$ is S_β-open set in X. □

Lemma 2.24. *Let $A \subseteq Y \subseteq X$, and $A \in S_\beta O(X)$, If Y is α-open set in X, then $A \in S_\beta O(Y)$.*

Proof. Let $A \in S_\beta O(X)$ then $A \in SO(X)$ for $A \subseteq Y \subseteq X$, and $A \in \alpha O(X)$, then by Theorem 1.4, $A \in SO(Y)$ and for each $x \in A$ there exists a β-closed set F in X such that $x \in F \subseteq A$. Since F is β-closed, then $X \setminus F$ is β-open in X and since Y is an α-open set in X, then by Theorem 1.11, $(X \setminus F) \cap Y = Y \setminus F$ is β-open in X and since $Y \setminus F \subseteq Y \subseteq X$, by Theorem 1.12, $Y \setminus F$ is β-open in Y. This implies that F is β-closed set in Y, thus $A \in S_\beta O(Y)$. □

Corollary 2.25. *Let Y be a subspace of the space X, and A be a subset of Y. If A is S_β-open set in X, and Y is open set in X, then A is S_β-open set in Y.*

Proof. follows from Lemma 2.24. □

Lemma 2.26. *Let $A \subseteq Y \subseteq X$, and $A \in S_\beta O(Y, \tau_Y)$, If Y is regular-closed set in X, then $A \in S_\beta O(X, \tau)$.*

Proof. Let $A \in S_\beta O(Y, \tau_Y)$ then $A \in SO(Y, \tau_Y)$ and for each $x \in A$ there exists a β-closed set F in X such that $x \in F \subseteq A$. Since $Y \in RC(X)$ then Y is semi-open in X and since $A \in S_\beta O(Y, \tau_Y)$, then by Theorem 1.5 $A \in SO(X, \tau)$. Again Since $Y \in RC(X)$, then $Y \in \alpha c(X)$, since F is β-closed in Y, then by Theorem 1.12, F is β-closed in X. Hence $A \in S_\beta O(X, \tau)$. □

Corollary 2.27. *Let Y be a subspace of the space X, and A be a subset of Y. If A is S_β-open set in Y, and Y is clopen set in X, then A is S_β-open set in X.*

Proof. Follows from Lemma 2.26. □

Corollary 2.28. *Let $A \subseteq Y \subseteq X$, if $A \in S_\beta O(X)$ and Y is clopen subset of X, then $A \cap Y \in S_\beta O(Y)$.*

Proof. Follows from Lemma 2.21 and Corollary 2.25. □

Proposition 2.29. *If a topological space X is locally indiscrete, then every semi-open set is S_β-open set.*

Proof. Let A be a semi-open set in X, then $A \subseteq intclA$, since X is locally indiscrete, then $intA$ is closed and hence $intA = clintA$, which implies that A is regular closed, therefore by Proposition 2.9, A is S_β-open set. □

Corollary 2.30. *For any space X, $S_\beta O(X, \tau) = S_\beta O(X, \tau_\alpha)$.*

Proof. Let A be any subset of a space X and $A \in S_\beta O(X, \tau)$. If $A = \phi$, then $A \in S_\beta O(X, \tau_\alpha)$. If $A \neq \phi$, and since $A \in S_\beta O(X, \tau)$, so $A \in SO(X, \tau)$ and $A = \cup F_k$ where F_k is β-closed for each k. Since $A \in SO(X, \tau)$, then by Theorem 1.3 $A \in SO(X, \tau_\alpha)$. Since F_k is β-closed in (X, τ) for each k, then by Theorem 1.3, F_k is β-closed in (X, τ_α) for each k, therefore by Proposition 2.2 $A \in S_\beta O(X, \tau_\alpha)$, So $S_\beta O(X, \tau_\alpha) \subseteq S_\beta O(X, \tau)$. By the same way we can prove $S_\beta O(X, \tau) \subseteq S_\beta O(X, \tau_\alpha)$. Hence we get $S_\beta O(X, \tau) = S_\beta O(X, \tau_\alpha)$. □

Theorem 2.31. *Let X, Y be two topological spaces and $X \times Y$ be the topological product. If $A_1 \in S_\beta O(X)$, and $A_2 \in S_\beta(Y)$, then $(A_1 \times A_2) \in S_\beta O(X \times Y)$.*

Proof. Let $(x, y) \in A_1 \times A_2$. Then $x \in A_1$ and $y \in A_2$. Since $A_1 \in S_\beta O(X)$ and $A_2 \in S_\beta O(Y)$, then $A_1 \in SO(X)$ and $A_2 \in SO(Y)$, and there exist $F_1 \in \beta C(X)$ and $F_2 \in \beta C(Y)$ such that $x \in F_1 \subseteq A_1$ and $y \in F_2 \subseteq A_2$. Therefore,$(x, y) \in F_1 \times F_2 \subseteq A_1 \times A_2$, and since $A_1 \in SO(X)$ and $A_2 \in SO(Y)$, then by Theorem 1.7, $A_1 \times A_2 = Sint_X A_1 \times Sint_Y A_2 = Sint_X \times Y(A_1 \times A_2)$, so $A_1 \times A_2 \in SO(X \times Y)$. Since $F_1 \in \beta C(X)$ and $F_2 \in \beta C(Y)$, then by Theorem 1.7 part(4) we get $F_1 \times F_2 = \beta cl_X F_1 \times \beta cl_Y F_2 = \beta cl_X \times Y(F_1 \times F_2)$, so $F_1 \times F_2$ is β-closed in $X \times Y$. Therefore, $(A_1 \times A_2) \in S\beta O(X \times Y)$. □

3 S_β-Operations

Definition 3.1. A subset N of a topological space(X, τ) is called *S_β- neighborhood of a subset A of X*, if there exists an S_β- open set U such that $A \subseteq U \subseteq N$. When $A = \{x\}$, we say that N is S_β- neighborhood of x.

Definition 3.2. A point $x \in X$ is said to be *an S_β- interior point of A*, if there exists an S_β- open set U containing x such that $x \in U \subseteq A$. The set of all S_β-interior points of A is said to be S_β- interior of A and it is denoted by $S_\beta intA$.

Proposition 3.3. *Let A be any subset of a topological space X. If a point x is in the S_β- interior of A, then there exists a semi-closed set F of X containing x such that $F \subseteq A$.*

Proof. Suppose that $x \in S_\beta int(A)$. Then there exists an S_β- open set U of X containing x such that $U \subseteq A$. Since U is an S_β-open set, so there exists an β-closed set F containing x such that $F \subseteq U \subseteq A$, hence $x \in F \subseteq A$. □

Here we give some properties of S_β-interior operator on a set.

Proposition 3.4. *For any subset A of a topological space X, the following statements are true:*

1. *the S_β-interior of A is the union of all S_β-open sets contained in A.*
2. *$S_\beta intA$ is the largest S_β-open set contained in A.*
3. *A is S_β-open set if and only if $A = S_\beta intA$.*

Finally from 3, we get that $S_\beta intAS_\beta intA = S_\beta intA$.

Proposition 3.5. *If A and B are any subsets of a topological space X, then:*

1. *$S_\beta int(\phi)$=ϕ, and $S_\beta intX$=X .*
2. *$S_\beta intA \subseteq A$.*
3. *If $A \subseteq B$,then $S_\beta intA \subseteq S_\beta intB$.*

4. $S_\beta intA \cup S_\beta intB \subseteq S_\beta int(A \cup B)$.
5. $S_\beta int(A \cap B) \subseteq S_\beta intA \cap S_\beta intB$.
6. $S_\beta int(A \setminus B) \subseteq S_\beta intA \setminus S_\beta intB$.
7. *A is S_β-open at $x \in X$ if and only if $x \in S_\beta intA$.*

Proof. Straight forward. □

Definition 3.6. Intersection of all S_β-closed sets containing F is called the S_β-closure of F and is denoted by $S_\beta clF$.

Corollary 3.7. *Let A be a set in a topological space X. A point $x \in X$ is in S_β-closure of A if and only if $A \cap U \neq \phi$, for every S_β- open set U containing x.*

Proof. Obvious. □

Proposition 3.8. *Let A be any subset of a space X. If $A \cap F \neq \phi$ for every β-closed set F of X containing x, then the point x is in the S_β-closure of A.*

Proof. Suppose that U is any S_β-open set containing x, then by Definition 1.1 there exists β-closed set F such that $x \in F \subseteq U$. So by hypothesis $A \cap F \neq \phi$ which implies that $A \cap U \neq \phi$ for every S_β-open set U containing x, therefore $x \in S_\beta clA$ by Corollary 3.7. □

Here we give some properties of.S_β-closure of a set:

Theorem 3.9. *For any subset F of a topological space X, the following statements are true:*

1. *$S_\beta clF$ is the intersection of all S_β- closed set in X containing F.*
2. *$S_\beta clF$ is the smallest S_β-closed set containing F.*
3. *F is S_β-closed set if and only if $F = S_\beta clF$.*

Proof. Obvious. □

Proposition 3.10. *Let A be any subset of a space X. If a point x is in the S_β-closure of A, then $A \cap F \neq \phi$ for every β- closed set F of X containing x.*

Proof. Suppose that $x \in S_\beta clA$, then by Corollary 3.7, $A \cap U \neq \phi$ for every S_β-open set U containing x. Since U is S_β-open set, so there exists a β- closed set F containing x such that $x \in F \subseteq U$. Hence $A \cap F \neq \phi$. □

Theorem 3.11. *If F and E are any subsets of a topological space X, then*

1. *$S_\beta cl(\phi)$=ϕ, and $S_\beta clX$=X.*
2. *for any subset $F$of X, $F \subseteq S_\beta(clF)$.*
3. *If $F \subseteq E$,then $S_\beta clF \subseteq S_\beta clE$.*

4. $S_\beta clF \cup S_\beta clE) \subseteq S_\beta cl(f \cup E)$.

5. $S_\beta cl(F \cap E) \subseteq S_\beta clF \cap S_\beta clE$.

Proof. Obvious. □

In general $S_\beta clF \cup S_\beta clE \neq S_\beta cl(f \cup E)$ and $S_\beta cl(F \cap E) \neq S_\beta clF \cap S_\beta clE$, as it is shown in the following example:

Example 3.12. *Considering a space* $X = \{a, b, c, d\}$ *and* $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$, *then* $SO(X) = S_\beta O(X) = PO(X) \setminus \{d, \{c, d\}\}$, *if we take* $F = \{b, d\}$ *and* $E = \{a, b, c\}$ *then* $S_\beta clF = F = \{b, d\}$, *and* $S_\beta clE = X$, *and* $S_\beta clF \cap S_\beta clE = S_\beta clF = \{b, d\}$, *and* $S_\beta cl(F \cap E) = \{b\}$. *It follow that* $S_\beta clF \cup S_\beta clE \neq S_\beta cl(f \cup E)$. *Again if we take* $F = \{a\}$ *and* $E = \{b\}$, *we get* $S_\beta clF = F = \{a\}$ *and* $S_\beta clE = E = \{b\}$, *then* $S_\beta cl(F \cup E) = \{a\} \cup \{b\} = \{a, b\}$, *but* $S_\beta clF \cup E) = \{a, b, c\}$, *so* $S_\beta cl(F \cap E) \neq S_\beta clF \cap S_\beta clE$.

Corollary 3.13. *For any subset* A *of topological space* X. *Then the following statements are true:*

1. $X \setminus S_\beta clA = S_\beta int(X \setminus A)$.

2. $X \setminus S_\beta intA = S_\beta cl(X \setminus A)$.

3. $S_\beta intA = X \setminus S_\beta cl(X \setminus A)$.

Proof. Obvious. □

Proposition 3.14. *Let* A *and* Y *be subsets of a topological space* X *such that* $A \subseteq Y \subseteq X$. *If* Y *is clopen, then* $S_\beta clA \cap Y = S_\beta cl_Y(A)$.

Proof. Let $x \in (S_\beta clA \cap Y)$ and $V \in S_\beta O(Y)$ containing x. Since Y clopen, then, $Y \in RC(X)$, by Lemma 2.26, $V \in S_\beta O(X)$ containing x, and hence $V \cap A \neq \phi$. Therefore, $x \in S_\beta cl_Y(A)$. Hence $S_\beta clA \cap Y \subseteq S_\beta cl_Y(A))$. On the other hand, let $x \in S_\beta cl_Y(A)$, and $V \in S_\beta O(X)$ containing x. Then $x \in V \cap Y$. Since Y clopen, then by Corollary 2.28, we have $x \in V \cap Y \in S_\beta O(Y)$ and hence $\phi \neq (A \cap (V \cap Y)) \subseteq (A \cap V)$. Therefore, we obtain $x \in S_\beta clA \cap Y$. Hence $S_\beta cl_Y(A) \subseteq S_\beta clA \cap Y$, therefore $S_\beta clA \cap Y \subseteq S_\beta cl_Y(A)$. □

Definition 3.15. Let A be a subset of a topological space X. A point $x \in X$ is said to be S_β*-limit point of* A if for each S_β-open set U containing $x, U \cap (A \setminus \{x\}) \neq \phi$. The set of all S_β-limit point of A is called S_β- derived set of A and is denoted by $S_\beta D(A)$.

Proposition 3.16. *Let* A *be any subset of* X. *If* $F \cap (A \setminus \{x\}) \neq \phi$, *for every* β*-closed set* F *containing* x, *then* $x \in S_\beta D(A)$.

Proof. Let U be any S_β-open containing x. Then there exists β-closed set F such that $x \in F \subseteq U$. By hypothesis, we have $F \cap (A \setminus \{x\}) \neq \phi$, hence $U \cap (A \setminus \{x\}) \neq \phi$. Therefore, a point $x \in S_\beta D(A)$. □

Proposition 3.17. *If a subset A of a topological space X is S_β-closed, then A contains the set of all of it's S_β-limit point.*

Proof. Suppose that A is S_β- closed set, then $X \setminus A$ is S_β- open set, thus A is S_β-closed set if and only if each point of $X \setminus A$ has S_β- neighborhood contained in $X \setminus A$ if and only if no point of $X \setminus A$ is S_β-limit point of A, or equivalently that A contains each of its S_β-limit points. □

Proposition 3.18. *Let F and E be subsets of a topological space X. If $F \subseteq E$, then $S_\beta D(F) \subseteq S_\beta D(E)$.*

Proof. Obvious. □

Some properties of S_β-derived set are mentioned in the following result:

Theorem 3.19. *Let A and B be subsets of a topological space X. Then we have the following properties:*

1. $S_\beta D(\phi) = \phi$.
2. $x \in S_\beta D(A)$ *implies* $x \in S_\beta D(A \setminus X)$.
3. $S_\beta D(A) \cup S_\beta D(B)) \subseteq S_\beta D(A \cup B)$.
4. $S_\beta D(A \cap B) \subseteq S_\beta D(A) \cap S_\beta D(B)$.
5. *If A is S_β-closed, then* $S_\beta D(A) \subseteq A$.

Proof. Obvious. □

Theorem 3.20. *Let X be a topological space and A be a subset of X, then:*

1. $A \cup S_\beta D(A))$ *is S_β- closed.*
2. $S_\beta D(S_\beta D(A)) \setminus A \subseteq S_\beta D(A)$.
3. $S_\beta D(A \cup S_\beta D(A)) \subseteq A \cup\ S_\beta D(A)$.

Proof.

1- Let $x \notin A \cup S_\beta D(A)$. Then $x \notin A$ and $x \notin D(A)$ this implies that there exists anS_β- open set N_x in X which contain no point of A other than x. But $x \notin A$, so N_x contains no point of A, which implies that $N_x \subseteq X \setminus A$, again N_x is an S_β- open set, it is a neihgbourhood of each of its points, but N_x does not contain any point of A, no point of N_x can be S_β-limit of A. Therefore no point of N_x can belong to $S_\beta D(A)$, this implies that $N_x \subseteq X \setminus S_\beta D(A)$, hence it follows that $x \in N_x \subseteq (X \setminus A) \cap (X \setminus S_\beta D(A) \subseteq X \setminus (A \cup S_\beta D(A))$. Therefore, $A \cup S_\beta D(A)$ is S_β-closed.

2- If $x \in S_\beta D S_\beta D(A) \setminus A$ and U is an S_β-open set containing x, then $U \cap (S_\beta D(A) \setminus \{x\}) \neq \phi$, let $y \in (U \cap S_\beta D(A) \setminus \{x\})$. Then $y \in U$ and $y \in S_\beta D(A)$, so $U \cap (A \setminus \{y\}) \neq \phi$, let $z \in (U \cap (A \setminus \{y\})$. Then $z \neq x$ for $z \in A$ and $x \notin A$, hence $U \cap (A \setminus \{x\}) \neq \phi$. Therefore,$x \in S_\beta D(A)$.

3- Let $x \in S_\beta D(A \cup S_\beta D(A))$. If $x \in A$, the result is obvious, let $x \in S_\beta D(A \cup S_\beta D(A)) \setminus A$, then for S_β-open set U containing x, $U \cap (A \cup S_\beta D(A)) \setminus \{x\})) \neq \phi$, thus $U \cap (A \setminus \{x\}) \neq \phi$, or $U \cap S_\beta D(A)\{x\} \neq \phi$. Now it follows similarly From 2 that $U \cap (A \setminus \{x\}) \neq \phi$. Hence $x \in S_\beta D(A)$, therefore $S_\beta D(A \cup S_\beta D(A)) \subseteq A \cup S_\beta D(A)$. □

Theorem 3.21. *Let A be a subset of a space X, then $S_\beta clA = A \cup S_\beta D(A)$.*

Proof. Since $S_\beta D(A) \subseteq S_\beta clA$ and $A \subseteq S_\beta clA$, we have $A \cup S_\beta D(A) \subseteq S_\beta clA$. Again since $S_\beta clA$ is the smallest S_β-closed set containing A, but by Proposition 2.2 $A \cup S_\beta D(A)$ is S_β-closed. Hence $S_\beta clA \subseteq A \cup S_\beta D(A)$. Thus $S_\beta clA = A \cup S_\beta D(A)$. □

Theorem 3.22. *Let X be any topological space and A be a subset of X. Then $S_\beta intA = A \setminus S_\beta D(X \setminus A)$.*

Proof. Obvious. □

Definition 3.23. If A is a subset of a topological space X, then S_β- boundary of A is $S_\beta clA \setminus S_\beta intA$, and is denoted by $S_\beta Bd(A)$.

Proposition 3.24. *For any subset A of a topological space X, the following statements are true:*

1. $S_\beta clA = S_\beta intA \cup S_\beta Bd(A)$.
2. $S_\beta intA \cap S_\beta Bd(A) = \phi$.
3. $S_\beta Bd(A) = S_\beta clA \cap S_\beta cl(X \setminus A)$.
4. $S_\beta Bd(A)$ *is* S_β*-closed.*

Proof. Obvious. □

Theorem 3.25. *For any subset A of a topological space X, the following statements are true:*

1. $S_\beta Bd(A) = S_\beta Bd(X \setminus A)$.
2. $A \in SO(X))$ *if and only if* $S_\beta Bd(A) \subseteq X \setminus A$,*that is* $A \cap S_\beta Bd(A) = \phi$.
3. $A \subseteq S_\beta C(X)$ *if and only if* $S_\beta Bd(A) \subseteq A$.
4. $S_\beta Bd(S_\beta(Bd(A)) \subseteq S_\beta Bd(A)$.
5. $S_\beta Bd(S_\beta intA) \subseteq S_\beta Bd(A)$.
6. $S_\beta Bd(S_\beta clA) \subseteq S_\beta Bd(A)$.
7. $S_\beta intA = A \setminus S_\beta Bd(A)$.

Proof. Obvious. □

Remark 3.26. *Let A be a subset of a topological space X, then $S_\beta Bd(A) = \phi$ if and only if A is both S_β-open and S_β- closed set.*

4 S_β-Continuous Functions

In this section, we introduce the concepts of S_β- continuity by using S_β-open sets. Several relations between these functions and other types of continuous functions and spaces are investigated.

Definition 4.1. A function $f : (X, \tau) \to (Y, \vartheta)$ is called *S_β-continuous at a point* $x \in X$, if for each open set V of Y containing $f(x)$, there exists an S_β-open set U in X containing x such that $f(U) \subseteq V$. If f is S_β- continuous at every point x of X, then it is called S_β- continuous.

Proposition 4.2. *A function $f : (X, \tau) \to (Y, \vartheta)$ is S_β- continuous if and only if the inverse image of every open set in Y is S_β- open set in X.*

Proof. Necessity. Let f be an S_β- continuous function and V be any open set in Y. To show that$f^{-1}(V)$ is S_β- open set in X, if $f^{-1}(V) = \phi$, implies that $f^{-1}(V)$ is S_β-open in X. If $f^{-1}(V) \neq \phi$, then there exists $x \in f^{-1}(V)$ which implies that $f(x) \in V$. Since f is S_β- continuous, so there exists an S_β- open set U in X containing x such that $f(U) \subseteq V$, this implies that $x \in U \subseteq f^{-1}(V)$, this shows that $f^{-1}(V)$is S_β- open in X.

Sufficiency. Let V be open set in Y, and its inverse is S_β- open set in X. Since, $f(x) \in V$, then $x \in f^{-1}(V)$ and by hypothesis $f^{-1}(V)$ is S_β- open set in X containing x, so $f(f^{-1}(V)) \subseteq V$. Therefore, f is S_β- continuous. □

Remark 4.3. *Every S_β- continuous function is semi- continuous.*

The converse of Remark 4.3 is not true in general as it is shown in the following example.

Example 4.4. *Let $X = \{a, b, c\}$, and take $\tau = \{\phi, \{a\}, X\}$, then $SO(X)$=$PO(X)$ and $S_\beta O(X) = \{\phi, X\}$, the identity function is semi-continuous but not S_β-continuous.*

Corollary 4.5. *If $f : (X, \tau) \to (Y, \vartheta)$ be semi-continuous function and (X, τ) is locally indiscrete, then f is S_β-continuous function.*

Proof. Let f be semi-continuous and X be locally indiscrete, and let V be any open subset in Y. Then $f^{-1}(V)$ is semi-open subset in X, and since X is locally indiscrete space then $f^{-1}(V) \in S_\beta O(X))$, thus by Proposition 4.2 f is S_β-continuous function. □

Remark 4.6. *Every S_p-continuous function is S_β-continuous, but the converse is not true in general. as shown in the following example.*

Example 4.7. *Let $X = \{a, b, c, d\}$, and, $\tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$ then $SO(X) = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}\} = S_\beta O(X)$ and $S_pO(X) = \{X, \phi, \{b, c\}, \{a, d\}, \{b, c, d\}\}$, also identity function is S_β-continuous which is not S_p-continuous function.*

Theorem 4.8. *Let $f : X \to Y$ be a function, then the following statements are equivalent:*

1. *f is S_β-continuous function.*
2. *The inverse image of every open set in Y is S_β-open set in X.*
3. *The inverse image of every closed set in Y is S_β-closed set in X.*
4. *For each $A \subseteq X$,$f(S_\beta cl(A)) \subseteq clf(A)$.*
5. *For each $A \subseteq X$, $intf(A) \subseteq f(S_\beta int(A))$.*
6. *For each$B \subseteq Y$, $S_\beta clf^{-1}(B) \subseteq f^{-1}(clB)$.*
7. *For each $B \subseteq Y$,$f^{-1}(intB) \subseteq S_\beta intf^{-1}(B)$.*

Proof. (1) $\Rightarrow$ (2) Follows from Proposition 4.2.

(2) $\Rightarrow$ (3) Let B be any closed subset of Y, then $Y \setminus B$ is open subset in Y, and hence$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ is S_β-open set in X. Thus $f^{-1}(B)$ is S_β-closed subset in X.

(3) $\Rightarrow$ (4) Let $A \subseteq X$, then $f(A) \subseteq Y$. But $f(A) \subseteq clf(A)$ and By (3) $f^{-1}(clA)$ is S_β-closed subset in X and $A \subseteq f^{-1}(clf(A))$, then $S_\beta clA \subseteq f^{-1}(clf(A))$. This implies that $f(S_\beta clA) \subseteq clf(A)$.

(4) $\Rightarrow$ (5) Let $A \subseteq X$, then $A \setminus X \subseteq X$ and then By (4) $f(S_b etacl(X \setminus A) \subseteq clf(X \setminus A)$. Therefore, $f(X \setminus S_\beta intA) \subseteq cl(Y \setminus f(A))$. This implies that $Y \setminus f(S_\beta intA \subseteq Y \setminus intf(A)$, thus $intf(A) \setminus f(S_\beta intA)$.

(5) $\Rightarrow$ (6) Let $B \subseteq Y$, then $f^{-1}(B) \subseteq X$ and then$X \setminus f^{-1}(B) \setminus X$. Therefore $intf(X \setminus f^{-1}(B) \subseteq f(S_\beta int(X \setminus f^{-1}(B))$, then $int(Y \setminus f(f^{-1}(B)) \subseteq f(X \setminus (S_\beta clf^{-1}(B))$, this implies that $int(Y \setminus B) \subseteq Y \setminus f(S_\beta clf^{-1}(B))$, then $Y \setminus clB \subseteq Y \setminus f(S_\beta clf^{-1}(B))$, that is $f(S_\beta clf^{-1}(B)) \subseteq clB$,hence $S_\beta clf^{-1}(B) \subseteq f^{-1}(clB)$.

(6) $\Rightarrow$ (7) Let $B \subseteq Y$, then $Y \setminus A \subseteq Y$. Therefore, by 6, $S_\beta clf^{-1}(Y \setminus B) \subseteq f^{-1}(cl(Y \setminus B)$, then $S_\beta cl(X \setminus f^{-1}(B)) \subseteq f^{-1}(Y \setminus intB)$, so we get $X \setminus S_\beta int(f^{-1}(B)) \subseteq X \setminus f^{-1}(intB)$, hence $f^{-1}(intB) \subseteq S_\beta int(f^{-1}(B))$.

(7) $\Rightarrow$ (1) Let $x \in X$ and U be any open subset of Y containing $f(x)$, then by (7) $f^{-1}(intU) \subseteq S_\beta intf^{-1}(U)$, this implies that $f^{-1}(U) \subseteq S_\beta f^{-1}(U)$. Hence $f^{-1}(U)$ is S_β-open set in X containing x such that $f(f^{-1}(U)) \subseteq U)$. Thus f is S_β-continuous function. □

Theorem 4.9. *Let $f : X \to Y$ be a subjective function, then the following statements are equivalent*

1. *f is S_β-continuous function.*
2. *For every $B \subseteq Y$, $intclf^{-1}(B) \subseteq f^{-1}(clB)$ and $f^{-1}(clB) = \bigcap_{i \in \Delta} V_i$ where $V_i \in \beta O(X)$.*

3. *For every* $B \subseteq Y$, $f^{-1}(intB) \subseteq clintf^{-1}(B)$, *and* $f^{-1}(intB) = \bigcup_{i\in\Delta} F_i$ *where* $F_i \in \beta C(X)$.

4. *For every* $A \subseteq X$, $f(intclA) \subseteq clf(A)$ *and* $f^{-1}(clf(A)) = \bigcap_{i\in\Delta} V_i$ *where* $V_i \in \beta O(X)$.

Proof. (1) $\Rightarrow$ (2) Let $B \subseteq Y$, then clB is closed subset in Y. Since f is S_β-continuous. Then by Theorem 4.8. $f^{-1}(clB)$ is S_β-closed in X. Therefore, by Proposition 2.2 $f^{-1}(clB)$ is semi-closed and $f^{-1}(clB)$=$\bigcap_{i\in\Delta} V_i$ where $V_i \in \beta O(X)$, thus $intclf^{-1}(clB) \subseteq f^{-1}(clB)$ and $f^{-1}(clB) = \bigcap_{i\in\Delta} V_i$ where $V_i \in \beta O(X)$. Hence $intclf^{-1}(B) \subseteq f^{-1}(clB)$ and $f^{-1}(clB) = \bigcap_{i\in\Delta} V_i$ where $V_i \in \beta O(X)$.

(2) $\Rightarrow$ (1) Let B be closed subset of Y, then By (2), $intclf^{-1}(B) \subseteq f^{-1}(clB) = f^{-1}(B)$ and $f^{-1}(B) = \bigcap_{i\in\Delta} V_i$ where $V_i \in \beta O(X)$. This implies that $f^{-1}(B) \subseteq Sc(X)$, and $f^{-1}(B) = \bigcap_{i\in\Delta} V_i$ where $V_i \in \beta O(X)$. Thus by Proposition 2.2 $f^{-1}(B)$ is S_β -closed in X. Hence by Theorem 4.8 f is S_β-continuous function.

(1) $\Rightarrow$ (3) Let $B \subseteq Y$, then $intB$ is open subset in Y, since f is S_β-continuous. Therefore, $f^{-1}(intB)$ is S_β-open in X. This implies that $f^{-1}(intB) \in SO(X)$, and $f^{-1}(intB) = \bigcup_(i \in \Delta)F_i$, where $F_i \in \beta C(X)$, therefore $f^{-1}(intB) \subseteq clintf^{-1}(B)$,and $f^{-1}(intB) = \bigcup_{i\in\Delta} F_i$, where $F_i \in \beta C(X)$.

(3) $\Rightarrow$ (1) Let B be open subset of Y, then $intB = B$ and thus by (3), $f^{-1}(B) \subseteq clintf^{-1}(B)$ and $f^{-1}(B) = \bigcup_{i\in\Delta} F_i$, where $F_i \in \beta C(X)$, this implies that $f^{-1}(B) \in S_\beta O(X)$. Hence f is S_β-continuous.

(2) $\Rightarrow$ (4) Let $A \subseteq X$, then $f(A) \subseteq Y$ and then by(2), $intclf^{-1}(f(A)) \subseteq f^{-1}(clf(A))$ and $f^{-1}(clf(A)) = \bigcap_{i\in\Delta} V_i$ where $V_i \in \beta O(X)$, therefore $intclA \subseteq f^{-1}(clf(A))$ and $f^{-1}(clf(A)) = \bigcap_{i\in\Delta} V_i$ where $V_i \in \beta O(X)$. Thus $f(intclA) \subseteq clf(A)$ and $f^{-1}(clf(A)) = \bigcap_{i\in\Delta} V_i$ where $V_i \in \beta O(X)$.

(4) $\Rightarrow$ (2) Let $B \subseteq Y$, then $f^{-1}(B) \subseteq X$. Therefore by (4), $f(intclf^{-1}(B)) \subseteq clf(f^{-1}(B)) \subseteq clB$ and $f^{-1}(clf(f^{-1}(B))) = \bigcap_{i\in\Delta} V_i$ where $V_i \in \beta O(X)$. This implies that $intclf^{-1}(B) \subseteq f^{-1}(clB)$ and $f^{-1}(cl(B)) = \bigcap_{i\in\Delta} V_i$ where $V_i \in \beta O(X)$. □

Proposition 4.10. *Let* $f : X \to Y$ *be* S_β*-continuous function and* $A \subseteq X$ *such that* A *is clopen, then the restriction function* $f|A : A \to Y$ *is* S_β*-continuous.*

Proof. Let B be any open subset of Y, since f is S_β- continuous, then By Proposition 4.2 $f^{-1}(B) \in S_\beta O(X)$, but A is clopen then $A \in \alpha O(X)$ By Lemma 2.21, $f^{-1}(B)\bigcap A \in S_\beta O(X)$, since A is open then $A \in \alpha O(X)$ so $f^{-1}(B)\bigcap A = (f|A)^{-1}(B) \in S_\beta O(A)$. Hence $f|A$ is S_β-continuous. □

Theorem 4.11. *Let* $f : X \to Y$ *be* S_β*-continuous function and let* $\{A_\lambda : \lambda \in \Delta\}$ *be regular closed cover of* X. *If the restriction* $f|A_\lambda : A_\lambda \to Y$ *is* S_β*-continuous for each* $\lambda \in \Delta$, *then* f *is* S_β*-continuous.*

Proof. Let $f|A_\lambda : A_\lambda \to Y$ be S_β-continuous for each $\lambda \in \Delta$, and let G be any open subset in Y, then by Proposition 4.2, $(f|A_\lambda)^{-1}(G) \in S_\beta O(A_\lambda)$ for each $\lambda \in \Delta$, but $(f|A_\lambda)^{-1}(G) = f^{-1}(G)\bigcap A_\lambda \in S_\beta O(A_\lambda)$ for each$\lambda \in \Delta$, Since for each for each $\lambda \in \Delta$, A_λ is regular closed, then by Proposition 2.9, $f^{-1}(G\bigcap A_\lambda) \in S_\beta O(A_\lambda)$ for each $\lambda \in \Delta$, but $\bigcup(f^{-1}(G\bigcap A_\lambda) \in S_\beta O(X)$, and $f^{-1}(G) = \bigcup_{\lambda\in\Delta}(f^{-1}(G\bigcap A_\lambda)) \in S_\beta O(X)$, then $f^{-1}(G) \in S_\beta O(X)$. Thus by Theorem 4.8, f is S_β-continuous. □

Theorem 4.12. *Let $f : X \to Y$ be a function. Let $\Im$ be any basis for σ in Y. Then f is S_β-continuous if and only if for each $B \in \Im$, $f^{-1}(B)$ is S_β-open subset of X.*

Proof. Suppose that f is S_β-continuous, since each $B \in \Im$ is open subset of Y and f is S_β-continuous. Then by Proposition 4.2, $f^{-1}(B)$ is S_β-open subset of X. Conversely; Let for each $B \in \Im$, is S_β-open subset of X. Let V be any open set in Y, then $V = \bigcup B_i : i \in \Delta$ where B_i is a member of $\Im$ and $\Im$ is a suitable index set. It follows that $f^{-1}(V) = f^{-1}\bigcup B_i : i \in \Delta\} = \bigcup f^{-1}(B_i : \{i \in \Delta\}$. But $f^{-1}(B_i)$ is an S_β-open subset in X for each $i \in \Delta$. Therefore $f^{-1}(V)$ is the union of a family of S_β-open sets of X and hence is a S_β-open set of X. Therefore, by Proposition 4.2, f is S_β-continuous function. □

References

[1] N. Levine, Semi open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963) 36–41.

[2] M.E. Abd El-Monsef, S.N. El-Deeb, R.A. Mahmoud, β-open sets and β-continuous mappings, Bull. Fac. Sci. Assiut Univ. 12 (1983) 77–90.

[3] A.H. Shareef, S_pre-open sets, S_pre-continuity and S_pre-compactness in topological spaces, M.Sc. Thesis, College of Science, Sulaimani Univ., 2007.

[4] A.S. Mashhour, M.E. Abd El-monsef, S.N. El.Deeb, On pre-continuous and week pre-continuous mappings, Proc. Math. Phys. Soc. Egypt 53 (1982) 47–53.

[5] O. Njstad, On some classes of nearly open sets, Pacific J. Math. 15 (3) (1965) 961–970.

[6] L.A. Steen, J.A. Seebach Jr., Counterexamples in topology, Holt, Rinehart and Winston, Inc., New York, 1970.

[7] R.H. Yunis, Regular β-open sets, Zonco J. Pure and Applied Science 16 (3) (2004) 79–83.

[8] N.V. Velicko, H-closed topological spaces, Amer. Math. Soc. Transl. 78 (2) (1992) 103–118.

[9] J.E. Joseph, M.H. Kwach, On S-closed spaces, Proc. Amer. Math Soc. 80 (2) (1980) 341–348.

[10] G. Dimaio, T. Noiri, On S-closed spaces, Indian J. Pure Apple .Math. 18 (3) (1987) 226–233.

[11] B.A. Asaad, Utilization of some types of pre-open sets in topological space, M.Sc. Thesis, College of Science, Dohuk Univ., 2007.

[12] K. Dlaska, M. Ganster, S-sets and Co-S-closed topologies, Indian J. Pure Apple. Math. 23 (10) (1992) 731–737.

[13] D.E. Cameron, Properties of S-closed spaces, Proc. Amer. Math. Soc. 72 (1978) 581–586.

[14] T. Noiri, On semi continuous mapping, Accad. Naz. Lincei. Rend. CLSci. Fis. Mat. Natur. 54 (8) (1973) 210–214.

[15] N.K. Ahmed, On some types of separation axioms, M.Sc. Thesis, College of Science, Salahaddin Univ., 1990.

[16] S.G. Crossely, S.K. Hildebrand, Semi closure, Texas. J. Sci. 22 (1971) 99–112.

[17] Z.A. Ameen, A new class of semi-open sets in topological spaces, M.Sc. Thesis, College of Science, Dohuk Univ., 2009.

[18] J. Dontchev, Survey on pre-open sets, The proceedings of Yatsushiro topological conference, (1998) 1–18.

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