Thai Journal of Mathematics
Volume 4 (2006) Number 1 : 73–84

Strong Maximum Principles for Implicit Parabolic Functional-Differential Problems Together with Nonstandard Inequalities with Integrals

L. Byszewski

Abstract : The aim of the paper is to give strong maximum principles for implicit parabolic functional-differential problems together with nonstandard inequalities with integrals in relatively arbitrary $(n+1)$ - dimensional time-space sets more general than the cylindrical domain. The results obtained can be applied in the theory of diffusion and in the theory of heat conduction.

Keywords : Strong maximum principles, implicit nonlinear systems, parabolic systems, functional-differential systems, nonstandard inequalities with integrals.
2000 Mathematics Subject Classification : 35B50, 35R45, 35K20, 35K60, 35K99.

1 Introduction

In this paper we consider implicit diagonal systems of nonlinear parabolic functional-differential inequalities of the form

$$\begin{aligned} &F^i(t,x,u(t,x),u_t^i(t,x),u_x^i(t,x),u_{xx}^i(t,x),u) \\ &\geq F^i(t,x,v(t,x),v_t^i(t,x),v_x^i(t,x),v_{xx}^i(t,x),v), \qquad (i=1,\ldots,m) \end{aligned} \tag{1.1}$$

for $(t,x)=(t,x_1,\ldots,x_n)\in D$, where $D\subset(t_0,t_0+T]\times\mathbb{R}^n$ is one of five relatively arbitrary sets more general than the cylindrical domain $(t_0,t_0+T]\times D_0\subset\mathbb{R}^{n+1}$. The symbol $w(=u$ or $v)$ denotes the mapping

$$w:\tilde{D}\ni(t,x)\to w(t,x)=(w^1(t,x),\ldots,w^m(t,x))\in\mathbb{R}^m,$$

where $\tilde{D}$ is an arbitrary set contained in $(-\infty,t_0+T]\times\mathbb{R}^n$ such that $\bar{D}\subset\tilde{D}$; $F^i(i=1,...,m)$ are functionals of $w; w_x^i(t,x)=grad_x w^i(t,x)(i=1,...,m)$ and $w_{xx}^i(t,x)(i=1,\ldots,m)$ denote the matrices of second order derivatives with respect to x of $w^i(t,x)$ $(i=1,...,m)$. We give a theorem on strong maximum principles for problems with inequalities of types (1.1) and with the nonstandard inequalities

$$\left[\, u^j(t_0,x) - K^j\right] + \sum_{i\in I_*} h_i(x)\left[\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(\tau,x)d\tau - K^j\right] \le 0$$

for $x \in S_{t_0} (j = 1,\dots,m)$, where

$$S_{t_0} := int\{x \in \mathbb{R}^n : (t_0,x) \in \bar{D}\},$$

K^j $(j = 1,\dots,m)$ are some constants, I_* is a subset of a countable set I of natural indices, $t_0 < T_{2i-1} < T_{2i} \le t_0 + T (i \in I)$ and $h_i : S_{t_0} \to (-\infty, 0]$ $(i \in I_*)$ are some functionals.

The results obtained are a generalization of some results given by J. Chabrowski [6], R. Redheffer and W. Walter [7], J. Szarski [9], P. Besala [1], W. Walter[11], N. Yoshida [12], the author [3], [4] and [5], and base on those results. To prove the results of this paper we use the theorem on a strong maximum principle form [5].

Partial differential problems together with nonstandard ("nonlocal") conditions were also studied in paper [2]. Since, in the theory of differential equations, the name "the nonlocal condition" is sometimes confusing then the name "the nonlocal condition" (used for example in [6] and [4]) has been changed to the name "the nonstandard condition" in paper [2] and next consequently used in the changed terminology "nonstandard inequalities" in this paper.

2 Preliminaries

The notation and definitions given in the section are valid throughout this paper. Some of them are similar to those applied by J. Szarski [8]-[10], R. Redheffer and W. Walter [7], P. Besala [1], N. Yoshida [12], J. Chabrowski [6] and the author [4], [5].

We use the following notation:

$$\mathbb{R} = (-\infty,\infty),\ \mathbb{N} = \{1,2,...\},\ x = (x_1,\dots,x_n)\ (n \in \mathbb{N}).$$

For vectors $z = (z^1,\dots,z^m) \in \mathbb{R}^m,\ \tilde{z} = (\tilde{z}^1,\dots,\tilde{z}^m) \in \mathbb{R}^m$ we write

$$z \le \tilde{z} \text{ if } z^i \le \tilde{z}^i\ (i = 1,\dots,m).$$

Let t_0 be a real number and let $0 < T < \infty$. A set $D \subset \{(t,x) : t > t_0, x \in \mathbb{R}^n\}$ (bounded or unbounded) is called a *set of type* (P) if

(i) The projection of the interior of D on the t-axis is the interval $(t_0, t_0 + T)$.

(ii) The projection of the interior of D on the t-axis is the interval $(t_0, t_0 + T)$.

$$\{(t,x) : (t-\tilde{t})^2 + \sum_{i=1}^{n}(x_i - \tilde{x}_i)^2 < r,\ t < \tilde{t}\} \subset D.$$

(iii) All the boundary points $(\tilde{t},\tilde{x})$ of D for which there is a positive number r such that

$$\{(t,x) : (t-\tilde{t})^2 + \sum_{i=1}^{n}(x_i-\tilde{x}_i)^2 < r,\ t \le \tilde{t}\} \subset D$$

belong to D.

For any $t \in [t_0, t_0+T]$ we define the following sets:

$$S_t = \begin{cases} int\{x \in \mathbb{R}^n : (t_0,x) \in \bar{D}\}, & \text{for} \quad t = t_0, \\ \{x \in \mathbb{R}^n : (t,x) \in D\}, & \text{for} \quad t \ne t_0, \end{cases}$$

and

$$\sigma_t = \begin{cases} int[\bar{D} \cap (\{t_0\} \times \mathbb{R}^n)], & \text{for} \quad t = t_0, \\ D \cap (\{t\} \times \mathbb{R}^n), & \text{for} \quad t \ne t_0. \end{cases}$$

Let $\tilde{D}$ be a set contained in $(-\infty, t_0+T] \times \mathbb{R}^n$ such that $\bar{D} \subset \tilde{D}$. We introduce the following sets:

$$\partial_p D := \tilde{D} \backslash D \quad \text{and} \quad \Gamma := \partial_p D \backslash \sigma_{t_0}.$$

For an arbitrary fixed point $(\tilde{t},\tilde{x}) \in D$, we denote by $S^-(\tilde{t},\tilde{x})$ the set of points $(t,x) \in D$ that can be joined to $(\tilde{t},\tilde{x})$ by a polygonal line contained in D along which the t-coordinate is weakly increasing from (t,x) to $(\tilde{t},\tilde{x})$.

Let $Z_m(\tilde{D})$ denote the space of mappings

$$w : \tilde{D} \ni (t,x) \to w(t,x) = (w^1(t,x), \dots, w^m(t,x)) \in \mathbb{R}^m$$

continuous in $\bar{D}$.

In the set of mappings bounded from above in $\tilde{D}$ and belonging to $Z_m(\tilde{D})$ we define the functional

$$[w]_t = \max_{i=1,\dots,m} sup\{0, w^i(\tilde{t},x) : (\tilde{t},x) \in \tilde{D}, \tilde{t} \le t\}, \quad \text{where} \quad t \le t_0+T.$$

By $M_{n\times n}(\mathbb{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n\times n}$.

A mapping $w \in Z_m(\tilde{D})$ is called *regular* in D if $w^i_t, w^i_x = grad_x w^i, w^i_{xx} = [w^i_{x_j x_k}]_{n\times n}$ $(i=1,\dots,m)$ are continuous in D.

Let the mappings

$$F^i : D \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times M_{n\times n}(\mathbb{R}) \times Z_m(\tilde{D}) \ni (t,x,z,p,q,r,w)$$

$$\to F^i(t,x,z,p,q,r,w) \in \mathbb{R} \ \ (i=1,\dots,m)$$

be given and let for an arbitrary regular in D function $w \in Z_m(\tilde{D})$

$$F^i\big[t,x,w\big] := F^i(t,x,w(t,x),w^i_t(t,x),w^i_x(t,x),w^i_{xx}(t,x),w),$$

$$(t,x) \in D \ (i = 1, \ldots, m).$$

Each two regular in D mappings $u, v \in Z_m(\tilde{D})$ are said to be *solutions* of the system

$$F^i\big[t,x,u\big] \geq F^i\big[t,x,v\big] \ (i = 1, \ldots, m) \tag{2.1}$$

in D if they satisfy (2.1) for all $(t,x) \in D$.

For a set $A \subset \tilde{D}$ and for a function $w \in Z_m(\tilde{D})$ we apply the notation:

$$\max_{(t,x)\in A} w(t,x) := \left(\max_{(t,x)\in A} w^1(t,x), \ldots, \max_{(t,x)\in A} w^m(t,x) \right).$$

Let $I = \mathbb{N}$ or I be a finite set of mutually different natural numbers.

Define the set

$$S = \bigcup_{i \in I} \Big(\sigma_{T_{2i-1}} \cup \sigma_{T_{2i}} \Big),$$

where, in the case if $I = \mathbb{N}$, the following conditions are satisfied:

(i) $t_0 < T_{2i-1} < T_{2i} \leq t_0 + T$ for $i \in I$ and $T_{2i-1} \neq T_{2j-1}$, $T_{2i} \neq T_{2j}$ for $i, j \in I$, $i \neq j$,

(ii) $T_0 := \inf\{T_{2i-1} : i \in I\} > t_0$,

(iii) $S_t \supset S_{t_0}$ for every $t \in \bigcup_{i\in I} \big[T_{2i-1}, T_{2i}\big]$,

(iv) $S_t \supset S_{t_0}$ for every $t \in \big[T_0, t_0 + T\big]$,

and, in the case if I is a finite set of mutually different natural numbers, conditions (i), (iii) are satisfied.

An unbounded set D of type (P) is called a *set of type* $(P_{S\Gamma})$ if :

(a) $S \neq \phi$,

(b) $\Gamma \cap \bar{\sigma}_{t_0} \neq \phi$.

Let S_* denote a non-empty subset od S. We define the following set

$$I_* = \{i \in I : \sigma_{T_i} \subset S_*\}.$$

A bounded set D of type (P) satisfying condition (a) of the definition of a set of type $(P_{S\Gamma})$ is called a *set of type* (P_{SB}).

It is easy to see that if D is a set of type (P_{SB}) then D satisfies condition (b) of the definition of a set of type $(P_{S\Gamma})$. Moreover, it is obvious that if D_0 is a bounded subset [D_0 is an unbounded proper subset] of $\mathbb{R}^n$ then $D = (t_0, t_0 + T] \times D_0$ is a set of type (P_{SB})[$(P_{S\Gamma})$, respectively].

3 Strong Maximum Principles Together with Nonstandard Inequalities with Integrals in Sets of Types ($\mathbf{P_{S\Gamma}}$) or ($\mathbf{P_{SB}}$).

Now we shall prove the following theorem on strong maximum principles together with nonstandard inequalities in sets of types $(P_{S\Gamma})$ and (P_{SB}) :

Theorem 3.1 *Assume that:*

(1) *D is a set of type $(P_{S\Gamma})$ or (P_{SB}).*

$(2)_1$ *The mappings F^i $(i = 1, \ldots, m)$ are weakly increasing with respect to $z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^m$ $(i = 1, \ldots, m)$. Moreover, there is a constant $L > 0$ such that*

$$F^i(t, x, z, p, q, r, w) - F^i(t, x, \tilde{z}, p, \tilde{q}, \tilde{r}, \tilde{w})$$

$$\leq L\left(\max_{k=1,\ldots,m} |z^k - \tilde{z}^k| + |x| \sum_{j=1}^{n} |q^j - \tilde{q^j}| + |x|^2 \sum_{j,k=1}^{n} |r_{jk} - \tilde{r}_{jk}| + \left[w - \tilde{w}\right]_t \right)$$

for all $(t, x) \in D$, $z, \tilde{z} \in \mathbb{R}^m$, $p \in \mathbb{R}$, $q, \tilde{q} \in \mathbb{R}^n$, $r, \tilde{r} \in M_{n\times n}(\mathbb{R}), w, \tilde{w} \in Z_m(\tilde{D})$, where $\sup_{(t,x)\in\tilde{D}} (w(t, x) - \tilde{w}(t, x)) < \infty$ $(i = 1, \ldots, m)$.

$(2)_2$ *There are constants $C_i > 0$ $(i = 1, 2)$ such that*

$$F^i(t, x, z, p, q, r, w) - F^i(t, x, z, \tilde{p},\ q, r, w) < C_1(\tilde{p} - p)\ (i = 1, \ldots, m)$$

for all $(t, x) \in D$, $z \in \mathbb{R}^m$, $p > \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n\times n}(\mathbb{R})$, $w \in Z_m(\tilde{D})$ and

$$F^i(t, x, z, p, q, r, w) - F^i(t, x, z, \tilde{p},\ q, r, w) < C_2(\tilde{p} - p)\ (i = 1, \ldots, m)$$

for all $(t, x) \in D$, $z \in \mathbb{R}^m$, $p < \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n\times n}(\mathbb{R})$, $w \in Z_m(\tilde{D})$.

(3) *The mapping u belongs to $Z_m(\tilde{D})$ and the maximum of u on Γ is attained. Moreover,*

$$K = (K^1, \ldots, K^m) := \max_{(t,x)\in\Gamma} u(t, x). \tag{3.1}$$

(4) *The inequalities*

$$\left[u^j(t_0, x) - K^j\right] + \sum_{i\in I_*} h_i(x) \left[\frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(\tau, x)d\tau - K^j \right] \leq 0 \tag{3.2}$$

$$\text{for } x \in S_{t_0}\ (j = 1, .., m)$$

are satisfied, where $h_i : S_{t_0} \to (-\infty, 0)$ $(i \in I_*)$ are given functions such that $-1 \leq \sum_{i\in I_*} h_i(x) \leq 0$ for $x \in S_{t_0}$, and, additionally, if card $I_* = \aleph_0$, then the series $\sum_{i\in I_*} \frac{h_i(x)}{T_{2i}-T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(\tau, x)d\tau$ $(j = 1, \ldots, m)$ are convergent for $x \in S_{t_0}$.

(5) *The maximum of* u *in* $\tilde{D}$ *is attained. Moreover,*

$$M = (M^1, \ldots, M^m) := \max_{(t,x)\in\tilde{D}} u(t,x). \tag{3.3}$$

(6) *The mappings* u *and* $v = M$ *are solutions of the system*

$$F^i\big[t,x,u\big] \geq F^i\big[t,x,v\big] \ \ (i = 1,..,m)$$

in D.

(7) *The mappings* F^i $(i = 1, \ldots, m)$ *are parabolic with respect to* u *in* D *and uniformly parabolic with respect to* M *in any compact subset of* D *(see [3] or [5]).*

Then

$$\max_{(t,x)\in\tilde{D}} u(t,x) = \max_{(t,x)\in\Gamma} u(t,x). \tag{3.4}$$

Moreover, if there is a point $(\tilde{t},\tilde{x}) \in D$ *such that* $u(\tilde{t},\tilde{x}) = \max_{(t,x)\in\tilde{D}} u(t,x)$ *then*

$$u(t,x) = \max_{(t,x)\in\Gamma} u(t,x) \quad \textit{for} \quad (t,x) \in S^-(\tilde{t},\tilde{x}).$$

Proof. We shall prove Theorem 3.1 for a set of type $(P_{S\Gamma})$ only since the proof of this theorem for a set of type (P_{SB}) is analogous.

Since each set of type $(P_{S\Gamma})$ is a set of type (P_Γ) from [5], it follows that in the case where $\sum_{i\in I_*} h_i(x) = 0$ for $x \in S_{t_0}$, Theorem 3.1 is a consequence of Theorem 4.1 of [5]. Therefore, we shall prove Theorem 3.1 only in the case if

$$-1 \leq \sum_{i\in I_*} h_i(x) < 0 \quad \text{for} \quad x \in S_{t_0}. \tag{3.5}$$

Assume that (3.5) holds and, since we shall argue by contradiction, suppose

$$M \neq K. \tag{3.6}$$

Form (3.1) and (3.3) we have

$$K \leq M. \tag{3.7}$$

Consequently,

$$K < M. \tag{3.8}$$

Observe that from assumption (5)

$$\text{There is } \ (t^*,x^*) \in \tilde{D} \ \text{ such that } \ u(t^*,x^*) = M := \max_{(t,x)\in\tilde{D}} u(t,x). \tag{3.9}$$

By (3.9), by assumption (3) and by (3.8), we have

$$(t^*,x^*) \in \tilde{D} \setminus \Gamma = D \cup \sigma_{t_0}. \tag{3.10}$$

An argument analogous to the proof of Theorem 4.1 from [5] yields

$$(t^*, x^*) \notin D. \tag{3.11}$$

Conditions (3.10) and (3.11) give

$$(t^*, x^*) \in \sigma_{t_0}. \tag{3.12}$$

On the other hand, because of the definitions of sets I and I_*, we must consider the following cases:

(A) I_* is a finite set, i.e., without loss of generality there is a number $p \in \mathbb{N}$ such that $I_* = \{1, .., p\}$.

(B) card $I_* = \aleph_0$.

First we shall consider case (A). And so, by (3.2) and by the inequality

$$u(t, x^*) < u(t_0, x^*) \quad \text{for} \quad t \in \bigcup_{i=1}^{p} \big[T_{2i-1}, T_{2i}\big],$$

being a consequence of (3.9), (3.12), and of (a)(i), (a)(iii) of the definition of a set of type $(P_{S\Gamma})$, we have

$$\begin{aligned} 0 &\geq \big[u^j(t_0, x^*) - K^j\big] + \sum_{i=1}^{p} h_i(x^*) \left[\frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(\tau, x^*) d\tau - K^j\right] \\ &\geq \big[u^j(t_0, x^*) - K^j\big] + \sum_{i=1}^{p} h_i(x^*) \left[\frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(t_0, x^*) - K^j\right] \\ &= \big[u^j(t_0, x^*) - K^j\big] \cdot \left[1 + \sum_{i=1}^{p} h_i(x^*)\right] \ \ (j = 1, \dots, m). \end{aligned}$$

Hence

$$u(t_0, x^*) \leq K \quad \text{if} \quad 1 + \sum_{i=1}^{p} h_i(x^*) > 0. \tag{3.13}$$

Then, from (3.8) and (3.12), we obtain a contradiction of (3.13) with (3.9). Assume now

$$\sum_{i=1}^{p} h_i(x^*) = -1. \tag{3.14}$$

By the mean-value integral theorem we have that for every $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, p\}$ there is $\tilde{T}_i^j \in \big[T_{2i-1}, T_{2i}\big]$ such that

$$u^j\big(\tilde{T}_i^j, x^*\big) = \frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(\tau, x^*) d\tau. \tag{3.15}$$

Simultaneously, for every $j \in \{1, \dots, m\}$ there is a number $l_j \in \{1, \dots, p\}$ such that

$$u^j\big(\tilde{T}_{l_j}^j, x^*\big) = \max_{i=1,\dots,p} u^j\big(\tilde{T}_i^j, x^*\big). \tag{3.16}$$

Consequently, by (3.14), (3.16), (3.15) and (3.2), we obtain

$$\begin{aligned} u^j(t_0,x^*) - u^j\big(\tilde T^j_{l_j},x^*\big) &= \big[u^j(t_0,x^*)-K^j\big] - \big[u^j\big(\tilde T^j_{l_j},x^*\big)-K^j\big] \\ &= \big[u^j(t_0,x^*)-K^j\big] + \sum_{i=1}^{p} h_i(x^*)\big[u^j\big(\tilde T^j_{l_j},x^*\big)-K^j\big] \\ &\le \big[u^j(t_0,x^*)-K^j\big] + \sum_{i=1}^{p} h_i(x^*)\big[u^j\big(\tilde T^j_{i},x^*\big)-K^j\big] \\ &= \big[u^j(t_0,x^*)-K^j\big] \\ &\quad + \sum_{i=1}^{p} h_i(x^*)\Big[\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(\tau,x^*)d\tau - K^j\Big] \\ &\le 0 \ \ (j=1,\dots,m). \end{aligned}$$

Hence

$$u^j(t_0,x^*) \le u^j\big(\tilde T^j_{l_j},x^*\big) \ \ (j=1,\dots,m) \quad \text{if} \quad \sum_{i=1}^{p} h_i(x^*) = -1. \tag{3.17}$$

Since, by (a)(i) of the definition of a set of type $(P_{S\Gamma}), \tilde T^j_{l_j} > t_0$ $(j=1,\dots,m)$, we get from (3.12) that (3.17) is at a contradiction with (3.9). This completes the proof of (3.4) if I_* is a finite set.

It remains to investigate case (B). Analogously to the proof of (3.4) in case (A), by assumption (4) and by the inequality

$$u(t,x^*) < u(t_0,x^*) \quad \text{for} \quad t \in \bigcup_{i\in I_*}\big[T_{2i-1},T_{2i}\big],$$

being a consequence od (3.9), (3.12), and of (a)(i), (a)(iii) of the definition of a set of type $(P_{S\Gamma})$, we have

$$\begin{aligned} 0 &\ge \big[u^j(t_0,x^*)-K^j\big] + \sum_{i\in I_*} h_i(x^*)\Big[\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(\tau,x^*)d\tau - K^j\Big] \\ &\ge \big[u^j(t_0,x^*)-K^j\big] + \sum_{i\in I_*} h_i(x^*)\Big[\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(t_0,x^*)d\tau - K^j\Big] \\ &= \big[u^j(t_0,x^*)-K^j\big]\cdot\big[1+\sum_{i\in I_*} h_i(x^*)\big] \ (j=1,\dots,m). \end{aligned}$$

Hence

$$u(t_0,x^*) \le K \quad \text{if} \quad 1+\sum_{i\in I_*} h_i(x^*) > 0. \tag{3.18}$$

Then, from (3.8) and (3.12), we obtain a contradiction of (3.18) with (3.9). Assume now

$$\sum_{i\in I_*} h_i(x^*) = -1. \tag{3.19}$$

By the mean-value integral theorem we have that for every $j \in \{1, \ldots, m\}$ and $i \in I_*$ there is $\tilde{T}_i^j \in \left[T_{2i-1}, T_{2i}\right]$ such that

$$u^j\left(\tilde{T}_i^j, x^*\right) = \frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(\tau, x^*) d\tau. \tag{3.20}$$

Let

$$\tilde{T}_*^j = \inf_{i \in I_*} \tilde{T}_i^j \quad (j = 1, \ldots, m). \tag{3.21}$$

Since $u \in C(\bar{D})$ and since, by (3.12) and by (a)(iv), (a)(ii) of the definition of a set of type $(P_{S\Gamma})$, $x^* \in S_t$ for every $t \in \left[T_0, t_0 + T\right]$ if $I = \mathbb{N}$, it follows from (3.21) that for every $j \in \{1, \ldots, m\}$ there is a number $\hat{t}_j \in \left[\tilde{T}_*^j, t_0 + T\right]$ such that

$$u^j\left(\hat{t}_j, x^*\right) = \max_{t \in [\tilde{T}_*^j, t_0 + T]} u^j(t, x^*). \tag{3.22}$$

Consequently, by (3.19), (3.22), (3.20) and by assumption (4), we obtain

$$\begin{aligned}
u^j(t_0, x^*) - u^j(\hat{t}_j, x^*) &= \left[u^j(t_0, x^*) - K^j\right] - \left[u^j(\hat{t}_j, x^*) - K^j\right] \\
&= \left[u^j(t_0, x^*) - K^j\right] + \sum_{i \in I_*} h_i(x^*)\left[u^j(\hat{t}_j, x^*) - K^j\right] \\
&\leq \left[u^j(t_0, x^*) - K^j\right] + \sum_{i \in I_*} h_i(x^*)\left[u^j(\tilde{T}_i^j, x^*) - K^j\right] \\
&= \left[u^j(t_0, x^*) - K^j\right] \\
&\quad + \sum_{i \in I_*} h_i(x^*)\Big[\frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(\tau, x^*) d\tau - K^j\Big] \\
&\leq 0 \ (j = 1, \ldots, m).
\end{aligned}$$

Hence

$$u^j(t_0, x^*) \leq u^j(\hat{t}_j, x^*)(j = 1, \ldots, m) \quad \text{if} \quad \sum_{i \in I_*} h_i(x^*) = -1. \tag{3.23}$$

Since, by (a)(ii) of the definition of a set type $(P_{S\Gamma})$, $\hat{t}_j > t_0$ $(j = 1, \ldots, m)$, we get from(3.12) that (3.23) is at a contradiction with (3.9). This completes the proof of equality (3.4).

The second part of Theorem 3.1 is a consequence of (3.4) and Lemma 3.1 from [5]. Therefore, the proof of Theorem 3.1 is complete. □

4 Remarks

Remark 4.1 Observe, by the proof of Theorem 3.1 from this paper and by the proof of Theorem 4.1 from [5], that if the functions $h_i (i \in I_*)$ from assumption

(4) of Theorem 3.1 satisfy the condition

$$\left[\sum_{i\in I_*} h_i(x)=0\right] \quad -1<\sum_{i\in I_*} h_i(x)\le 0 \text{ for } x\in S_{t_0}$$

then it is sufficient to assume in this theorem that [D is only an unbounded set of type (P) satisfying condition (b) of the definition of a set of type ($P_{S\Gamma}$) or D is only a bounded set of type (P), i.e., according to the terminology of [5], D is a set of type (P_Γ) or (P_B), respectively] D is an unbounded set of type (P) satisfying conditions (a)(i), (a)(iii) and (b) of the definition of a set of type ($P_{S\Gamma}$) or D is a bounded set of type (P) satisfying conditions (a)(i) and (a)(iii) of the definition of a set of type ($P_{S\Gamma}$). Moreover, if I_* is a finite set and $-1\le \sum_{i\in I_*} h_i(x)\le 0$ for $x\in S_{t_0}$, then it is sufficient to assume in Theorem 3.1 that D is an unbounded set of type (P) satisfying conditions (a)(i), (a)(iii) and (b), or D is a bounded set of type (P) satisfying conditions (a)(i) and (a)(iii).

Remark 4.2 If D is a set of type (P_{SB}) and if $\tilde{D}=\bar{D}$ then the first part of assumption (3) of Theorem 3.1 relative to the maximum of u and the first part of assumption (5) of this theorem are trivially satisfied since $u\in C(\bar{D})$ and Γ is the bounded and closed set in this case.

Remark 4.3 If the mappings F^i ($i=1,\dots,m$) do not depend on the functional argument w then Theorem 3.1 reduces to the theorem on strong maximum principles for implicit parabolic differential problems together with nonstandard inequalities with integrals and in this case we can put $\tilde{D}=\bar{D}$.

Remark 4.4 The results obtained can be applied in the theory of diffusion and in the theory of heat conduction. For this purpose see [4].

5 Physical Interpretations of Problems Considered

Theorem 3.1 can be applied to descriptions of physical phenomena in which we can measure sums of mean temperatures of substances or sums of mean amounts of substances according to the following formulae :

$$u^j(t_0,x)+\sum_{i\in I_*}\frac{h_i(x)}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(\tau,x)d\tau \text{ for } x\in S_{t_0} \quad (j=1,\dots,m)$$

(h_i ($i\in I_*$) are known function). For example, Theorem 3.1 can be applied to the description of a diffusion phenomenon of a little amount of a gas in a transparent tube, under the assumption that the diffusion is observed by the surface of this tube. The measurement $u(t_0,x)$ ($m=1$) of small amount of the gas at the initial instant t_0 is usually less precise than the following measurement:

$$u(t_0,x)+\sum_{i\in I_*}\frac{h_i(x)}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}}u(\tau,x)d\tau \ \text{ for } \ x\in S_{t_0} \ \ (m=1),$$

where

$$\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}}u(\tau,x)d\tau \ \text{ for } \ x\in S_{t_0} \ \ (i\in I_*,\ m=1)$$

are the mean amounts of this gas on the intervals $[T_{2i-1},T_{2i}]$ $(i\in I_*)$, respectively. Therefore, Theorem 3.1 seems to be more useful in some physical applications than Theorem 4.1 from [5] on strong maximum principles with initial inequalities of the form

$$u(t_0,x)\le K \ \text{ for } \ x\in S_{t_0}.$$

If $I_*=1,\ T_1=t_0+T-\Delta t,\ 0<\Delta t<T,\ T_2=t_0+T, -1\le h_i(x)=-h(x)\le 0$ for $x\in S_{t_0}$ and $m=1$, then the nonstandard conditions

$$u^j(t_0,x)+\sum_{i\in I_*}\frac{h_i(x)}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}}u^j(\tau,x)d\tau=0 \ \text{ for } \ x\in S_{t_0} \ \ (j=1,\ldots,m)$$

are reduced to the following condition:

$$u(t_0,x)=\frac{h(x)}{\Delta t}\int_{t_0+T-\Delta t}^{t_0+T}u(\tau,x)d\tau \ \text{ for } \ x\in S_{t_0} \ \ (m=1) \tag{5.1}$$

and this condition can be used to the description of heat effects in atomic reactors. It is easy to see, by (5.1), that if $u(t_0,x)$ is interpreted as the given temperature in an atomic reactor at the initial instant t_0, then the atomic reaction is the safest for $1\simeq h(x)\le 1$ and this reaction in the most dangerous for $0<h(x)\simeq 0$. In the case if $h(x)=1$ for $x\in S_{t_0}$, formula (5.1) is reduced to the condition

$$u(t_0,x)=\frac{1}{\Delta t}\int_{t_0+T-\Delta t}^{t_0+T}u(\tau,x)d\tau \ \text{ for } \ x\in S_{t_0} \ \ (m=1)$$

which is the modification of the periodic condition

$$u(t_0,x)=u(t_0+T,x) \ \text{ for } \ x\in S_{t_0}\ (m=1).$$

References

[1] P. Besala, An extension of the strong maximum principle for parabolic equations, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, **19**(1971), 1003–1006.

[2] J. Brandys, L. Byszewski, Uniqueness of solutions to inverse parabolic problems, *Commentationes Mathematicae*, **42**(1)(2002), 17–30.

[3] L. Byszewski, Strong maximum principle for implicit nonlinear parabolic functional-differential inequalities in arbitrary domains, *Univ. Iagell. Acta Math.*, **24**(1)(1984), 327–339.

[4] L. Byszewski, Strong maximum principles for parabolic nonlinear problems with nonlocal inequalities together with integrals, *Journal of Applied Mathematics and Stochastic Analysis,* **3**(1)(1990), 65–79.

[5] L. Byszewski, Strong maximum principles for implicit parabolic functional -differential problems together with initial inequalities, *Annales Academiae Pedagogicae Cracoviensis*, *Studia Mathematica*, **23 (IV)**(2004), 9–16.

[6] J. Chabrowski, On the non-local problem with a functional for parabolic equation, *Funkcialaj Ekvacioj,* **27**(1984), 101–123.

[7] R. Redheffer, W. Walter, Das Maximumprinzip in unbeschränkten Gebieten für parabolische Ungleichungen mit Funktionalen, *Math. Ann.,* **226**(1977), 155–170.

[8] J. Szarski, *Differential Inequalities*, PWN, Warszawa, 1967.

[9] J. Szarski, Strong maximum principle for non-linear parabolic differential - functional inequalities, *Ann. Polon. Math.,* **29** (1974), 207–217.

[10] J. Szarski, Inifinite systems for parabolic differential - functional inequalities, *Bull. Acad. Polon. Sci. Sér. Sci. Math.,* **28**(1980),471–481.

[11] W. Walter, *Differential and Integral Inequalities*, Springer - Verlag, Berlin, Heidelberg, New York, 1970.

[12] N. Yoshida, Maximum principles for implicit parabolic equations, *Proc. Japan Acad.,* **49**(1973), 785–788.

(Received 15 November 2005)

L. Byszewski
Institute of Mathematics
Cracow University of Technology
Warszawska 24, 31-155 Cracow, Poland.
e-mail : `lbyszews@usk.pk.edu.pl`