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Approximation Method for Fixed Point Problem and Variational Inequality Problem without Assumption on the Set of Fixed Pint and the Set of Variational Inequality

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Abstract : The purpose of this paper is to prove a necessary and sufficient condition to obtain strong convergence theorem for finding a common element of the set of fixed point problem of nonexpansive mapping and the set of variational inequality problem without assumption $F(T)\bigcap VI(C,A) \neq \emptyset$ in framework of Hilbert space.

Keywords : nonexpansive mappings; strongly positive operator; fixed point.
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1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a nonlinear mapping. A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$). It is known that $F(T)$ is always closed convex and also nonempty provided T has a bounded trajectory.

A bounded linear operator A on H is called *strongly positive* with coefficient

$\bar{\gamma}$ if there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2.$$

The variational inequality problem is to find a point $u \in C$ such that

$$\langle v-u, Au\rangle \geq 0 \quad \text{for all } v \in C. \tag{1.1}$$

The set of solutions of the variational inequality is denoted by $VI(C,A)$.

Variational inequalities were introduced and studied by Stampacchia [1] in 1964. It is now well-known that a wide classes of problems arising in various branches of pure and applied sciences can be reduce to find element of (1.1). Many author has been studied strong convergence theorem for finding a common element of the set of fixed point problem and the set of variational inequality problem by using condition $F(T)\bigcap VI(C,A) \neq \emptyset$. (see, e.g., [2, 3])

In this paper, we prove strong convergence theorem for finding a common element of the set of fixed point problem of nonexpansive mapping and the set of variational inequality problem without assumption $F(T)\bigcap VI(C,A) \neq \emptyset$.

2 Preliminaries

In this section, we provide the well-known lemmas are needed to prove our main result.

Let C be closed convex subset of a real Hilbert space H, let P_C be the metric projection of H onto C i.e., for $x \in H$, P_Cx satisfies the property

$$\|x - P_Cx\| = \min_{y\in C}\|x-y\|.$$

Lemma 2.1 (See [4]). *Let H be a Hilbert space, let C be nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u \in VI(C,A) \Leftrightarrow u = P_C(I-\lambda A)u$$

where P_C is the metric projection of H onto C.

Lemma 2.2 (See [5]). *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Lemma 2.3 (See [6]). *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

3 Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping, $A : C \to H$ a strongly positive* *linear bounded operator with coefficient* $\overline{\gamma} > 0$. *Let* $\{x_n\}$ *be the sequence generated by* $x_0 \in C$ *and*

$$x_{n+1} = \alpha T x_n + (1-\alpha) P_C (I - pA) x_n, \quad \forall n = 0, 1, 2, \ldots \tag{3.1}$$

where $\alpha \in (0,1)$, $0 < p \leq \frac{1}{\|A\|}$ *and* $p\overline{\gamma} < 1$. *Then the following are equivalent.*

(i) The sequence $\{x_n\}$ *defined by (3.1) converges strongly to* $x^* \in F(T) \bigcap VI(C, A)$;

(ii) $\lim_{n\to\infty} \|x_n - T x_n\| = 0$.

Proof. $(i) \Rightarrow (ii)$ Let condition (i) holds. Since $x^* \in F(T) \bigcap VI(C, A)$, we have

$$\|x_n - T x_n\| \leq \|x_n - x^*\| + \|T x^* - T x_n\| \leq 2\|x_n - x^*\|.$$

It implies that $\lim_{n\to\infty} \|x_n - T x_n\| = 0$.

Conversely, $(ii) \Rightarrow (i)$, let condition (ii) holds. First we show that $I - pA$ is contraction mapping. Let $x, y \in C$. Since A is a strongly positive linear bounded operator and Lemma 2.3, we have

$$\begin{aligned} \|(I - pA)x - (I - pA)y\| &= \|(I - pA)(x - y)\| \\ &\leq (1 - p\overline{\gamma})\|x - y\|. \end{aligned}$$

Since $0 < p\overline{\gamma} < 1$, we have $I - pA$ is a contraction mapping with coefficient $1 - p\overline{\gamma}$. For every $n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha\|T x_n - T x_{n-1}\| + (1-\alpha)\|P_C(I - pA)x_n - P_C(I - pA)x_{n-1}\| \\ &\leq \alpha\|x_n - x_{n-1}\| + (1-\alpha)\|(I - pA)x_n - (I - pA)x_{n-1}\| \\ &= \alpha\|x_n - x_{n-1}\| + (1-\alpha)\|(I - pA)(x_n - x_{n-1})\| \\ &\leq \alpha\|x_n - x_{n-1}\| + (1-\alpha)(1 - p\overline{\gamma})\|x_n - x_{n-1}\| \\ &\leq \big(1 - p\overline{\gamma}(1-\alpha)\big)\|x_n - x_{n-1}\| \\ &= a\|x_n - x_{n-1}\| \\ &\leq a\big(a\|x_{n-1} - x_{n-2}\|\big) \\ &\leq a^2\|x_{n-1} - x_{n-2}\| \\ &\;\;\vdots \\ &\leq a^n\|x_1 - x_0\| \end{aligned} \tag{3.2}$$

where $a = \big(1 - p\overline{\gamma}(1-\alpha)\big) \in (0,1)$.

For any number $n, k \in \mathbb{N}, k > 0$ and (3.2), we have

$$\begin{aligned} \|x_{n+k} - x_n\| &\leq \sum_{j=n}^{n+k-1} \|x_{j+1} - x_j\| \\ &\leq \sum_{j=n}^{n+k-1} a^j\|x_1 - x_0\| \\ &\leq \frac{a^n}{1-a}\|x_1 - x_0\|. \end{aligned} \tag{3.3}$$

Since $\lim_{n\to\infty} a^n = 0$ and (3.3), we have $\{x_n\}$ is a Cauchy sequence. Since H is a Hilbert space, we get $\{x_n\}$ converges to x^* i.e.,

$$\lim_{n\to\infty} x_n = x^*. \tag{3.4}$$

Since C is closed, so we get $x^* \in C$. By $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, (3.4) and Lemma 2.2, we have $x^* \in F(T)$. Since $\lim_{n\to\infty} x_n = x^*$ and definition of x_n, we have

$$\begin{aligned} x^* &= \alpha T x^* + (1-\alpha) P_C (I - pA) x^* \\ &= \alpha x^* + (1-\alpha) P_C (I - pA) x^*, \end{aligned} \tag{3.5}$$

it implies that $x^* \in F(P_C(I - pA))$. From Lemma 2.1, we have $x^* \in VI(C, A)$. Hence, the sequence $\{x_n\}$ converges strongly to $x^* \in F(T) \bigcap VI(C, A)$. Complete the proof. □

The following result can be obtain from Theorem 3.1, then we, therefore, omit the prove.

Corollary 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H and let $A : C \to H$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Let $\{x_n\}$ be the sequence generated by $x_0 \in C$ and*

$$x_{n+1} = \alpha x_n + (1-\alpha) P_C (I - pA) x_n, \quad \forall n = 0, 1, 2, \ldots \tag{3.6}$$

where $\alpha \in (0, 1)$, $0 < p \leq \frac{1}{\|A\|}$ and $p\overline{\gamma} < 1$. Then the sequence $\{x_n\}$ defined by (3.6) converges strongly to $x^* \in VI(C, A)$.

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