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Asymptotically and Statistically Equivalent Functions Defined on Amenable Semigroups[1]

Fatih Nuray[†,2] **and B.E. Rhoades**[‡]

[†]Department of Mathematics
Afyon Kocatepe University, Afyonkarahisar, Turkey
e-mail : fnuray@aku.edu.tr

[‡]Department of Mathematics
Indiana University, Bloomington, IN., U.S.A.
e-mail : rhoades@indiana.edu

Abstract : In this study we define the notions of asymptotically, statistically, almost statistically and strong almost asymptotically equivalent functions defined on discrete countable amenable semigroups. In addition to these definitions, we give some inclusion theorems. Also we prove that the strong almost asymptotically equivalence of the functions $f(g)$ and $h(g)$ defined on discrete countable amenable semigroups does not depend on the particular choice of the Folner sequence.

Keywords : Folner sequence; amenable group; statistical convergence; equivalent function.
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1 Introduction and Background

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold, and $w(G)$ and $m(G)$ denote the spaces of all real valued functions and all bounded real functions f on G respectively. $m(G)$ is a Banach space with the supremum norm $\|f\|_\infty = \sup\{|f(g)| : g \in G\}$. Nomika

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[2]Corresponding author.

[1] showed that, if G is countable amenable group, there exists a sequence $\{S_n\}$ of finite subsets of G such that

(1) $G = \cup_{n=1}^{\infty} S_n$,

(2) $S_n \subset S_{n+1}$, $n = 1, 2, 3, ...$,

(3)
$$\lim_{n\to\infty} \frac{|S_n g \cap S_n|}{|S_n|} = 1, \quad \lim_{n\to\infty} \frac{|gS_n \cap S_n|}{|S_n|} = 1$$

for all $g \in G$. Here $|A|$ denotes the number of elements in the finite set A.

Any sequence of finite subsets of G satisfying (1), (2) and (3) is called a Folner sequence for G.

The sequence $S_n = \{0, 1, 2, ..., n-1\}$ is a familiar Folner sequence giving rise to the classical Cesaro method of summability.

In [2], Fast introduced the concept of statistical convergence for real number sequence; in [3], Zygmund called it "almost convergence" and established a relation between it and strong summability. Also statistical convergence was studied in [4] and [5]. Amenable semigroups were studied in [6] and [7]. The concept of summability in amenable semigroups introduced in [8–10]. In [10], Douglas extended the notion of arithmetic mean to amenable semigroups and obtained a characterization for almost convergence in amenable semigroups. In [11], we introduced notion of convergence and statistical convergence in amenable semigroups. In [12], we introduced notion statistical convergence in amenable semigroups. In [13], Marouf introduced asymptotically equivalent sequence and asymptotic regular matrices. Patterson [14] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Asymptotically and σ-asymptotically lacunary statistically equivalent sequences were studied in [15] and [16] respectively. The purpose of this study is to extend the notions of asymptotically and statistically equivalent sequence to functions defined on discrete countable amenable semigroups. When the amenable semigroup is the additive positive integers, our definitions and theorems yield those results of [14].

2 Statistical Convergence

Definition 2.1. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be *convergent to* s, for any Folner sequence $\{S_n\}$ for G, if for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $|f(g) - s| < \epsilon$ for all $m > k_0$ and $g \in G \setminus S_m$.

Definition 2.2. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be a *Cauchy sequence*, for any Folner sequence $\{S_n\}$ for G, if for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $|f(g) - f(h)| < \epsilon$ for all $m > k_0$ and $g, h \in G \setminus S_m$.

Definition 2.3. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be *strongly summable to* s, for any Folner sequence $\{S_n\}$ for G, if

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{g\in S_n} |f(g) - s| = 0.$$

where $|S_n|$ denotes the cardinality of the set S_n.

Definition 2.4. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold and $0 < p < \infty$. $f \in w(G)$ is said to be *strongly* p*-summable to* s, for any Folner sequence (S_n) for G, if

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{g\in S_n} |f(g) - s|^p = 0.$$

The set of all strongly p-summable functions f will be denoted $w_p(G)$,

The upper and lower Folner densities of a set $S \subset G$ are defined by

$$\overline{\delta}(S) = \limsup_{n\to\infty} \frac{1}{|S_n|} |\{g \in S_n : g \in S\}|$$

and

$$\underline{\delta}(S) = \liminf_{n\to\infty} \frac{1}{|S_n|} |\{g \in S_n : g \in S\}|$$

respectively. When the above limits exist and equal, then we say that S has Folner density and denote $\delta(S)$. We shall be particularly concerned with sets having Folner density zero. To facilitate this we introduce the following notation: If f is function such that $f(g)$ satisfies property P for all g expect a set of Folner density zero, then we say that $f(g)$ satisfies P for "almost all g", and we abbreviate this by "$a.a.g$".

Definition 2.5 ([11])**.** Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be *statistically convergent to* s, for any Folner sequence $\{S_n\}$ for G, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{|S_n|} |\{g \in S_n : |f(g) - s| \geq \epsilon\}| = 0$$

i.e., $|f(g) - s| < \epsilon$ $a.a.g$. The set of all statistically convergent functions will be denoted $S(G)$.

Definition 2.6 ([12])**.** Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be *almost statistically convergent to* s, for any Folner sequence $\{S_n\}$ for G, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{|S_n|} |\{g \in S_n : |f(gm) - s| \geq \epsilon\}| = 0$$

and

$$\lim_{n\to\infty} \frac{1}{|S_n|} |\{g \in S_n : |f(mg) - s| \geq \epsilon\}| = 0$$

uniformly in $m \in G$. The set of all almost statistically convergent functions will be denoted $\hat{S}(G)$.

3 Asymptotically Equivalence

Definition 3.1. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. Two nonnegative functions $f, h \in w(G)$ are said to be *asymptotically equivalent*, for any Folner sequence $\{S_n\}$ for G, if for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\left|\frac{f(g)}{h(g)} - 1\right| < \epsilon$$

for all $m > k_0$ and $g \in G \setminus S_m$. It will be denoted by $f \sim h$.

Definition 3.2. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. Two nonnegative functions $f, h \in w(G)$ are said to be *strong asymptotically equivalent*, for any Folner sequence $\{S_n\}$ for G, if

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{g \in S_n} \left|\frac{f(g)}{h(g)} - 1\right| = 0.$$

It will be denoted by $f \sim^w h$.

4 Statistically Equivalence

Definition 4.1. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. Two nonnegative functions $f, h \in w(G)$ are said to be *statistically equivalent*, for any Folner sequence $\{S_n\}$ for G, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{|S_n|} \left|\left\{g \in S_n : \left|\frac{f(g)}{h(g)} - 1\right| \geq \epsilon\right\}\right| = 0.$$

It will be denoted by $f \sim^S h$.

Definition 4.2. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold . Two nonnegative functions $f, h \in w(G)$ are said to be *almost statistically equivalent*, for any Folner sequence $\{S_n\}$ for G, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{|S_n|} \left|\left\{g \in S_n : \left|\frac{f(gm)}{h(gm)} - 1\right| \geq \epsilon\right\}\right| = 0$$

and

$$\lim_{n\to\infty}\frac{1}{|S_n|}\left|\left\{g\in S_n:\left|\frac{f(mg)}{h(mg)}-1\right|\geq\epsilon\right\}\right|=0$$

uniformly in $m\in G$. It will be denoted by $f\sim^{\hat{S}} h$.

Definition 4.3. Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. Two nonnegative functions $f,h\in w(G)$ are said to be *strong almost asymptotically equivalent*, for any Folner sequence $\{S_n\}$ for G, if

$$\lim_{n\to\infty}\frac{1}{|S_n|}\sum_{g\in S_n}\left|\frac{f(gm)}{h(gm)}-1\right|=0$$

and

$$\lim_{n\to\infty}\frac{1}{|S_n|}\sum_{g\in S_n}\left|\frac{f(mg)}{h(mg)}-1\right|=0$$

uniformly in $m\in G$. It will be denoted by $f\sim^{\hat{w}} h$.

The finite subsets $A_1,A_2,...,A_k\subseteq G$ are called *δ-disjoint* if there are subsets $B_1,B_2,...,B_k\subseteq G$ such that

(i) $B_i\subseteq A_i$ for any $i=1,2,3,...,k$.

(ii) $\frac{|B_i|}{|A_i|}>1-\delta$.

(iii) $B_i\cap B_j=\emptyset$ if $i\neq j$.

We say that the finite subsets $A_1,A_2,...,A_k\subseteq G$ $(1-\delta)$-cover a set $A\subseteq G$ if $\frac{|A\cap(\cup_{i=1}^k A_i)|}{|A|}\geq 1-\delta$.

The finite subsets $A_1,A_2,...,A_k\subseteq G$ δ-quasitile the set A, if there are finite sets $C_1,C_2,...,C_k\subseteq G$ such that

(iv) $A_iC_i\subseteq A$ for any $i=1,2,...,k$.

(v) $A_iC_i\cap A_jC_j=\emptyset$ if $i\neq j$

(vi) $\{A_ic:c\in C_i\}$ form an δ-disjoint family.

(vii) $\{A_iC_i:i=1,2,...,k\}$ form a $(1-\delta)$ cover A.

The sets C_i are called the tiling center (see, e.g., [17]).

The following result is due to Ornstein and Weiss [18] (see also [17]).

Lemma 4.4. *Let $S_1\subseteq S_2\subseteq\cdots$ and $T_1\subseteq T_2\subseteq\cdots$ be two Folner sequences. Let $\delta\in(0,\frac{1}{4})$ and $N\in\mathbb{N}$. Then there exist integers $n_1,n_2,...,n_k$ with $N\leq n_1<n_2<\cdots<n_k$ such that $S_{n_1},...,S_{n_k}$ δ-quasitile T_m when m is large enough.*

Theorem 4.5. *The strong almost asymptotically equivalence of the $f(g)$ and $h(g)$ does not depend on the particular choice of the Folner sequence.*

Proof. Let the functions $f(g)$ and $h(g)$ be strong almost asymptotically equivalent with respect to the Folner sequence $\{S_n\}$. We shall show that the function $f(g)$ and $h(g)$ are strong almost asymptotically equivalent with respect to the Folner sequence $\{T_n\}$ too. Since $f(g)$ and $h(g)$ are strong almost asymptotically equivalent with respect to the Folner sequence $\{S_n\}$ we can write

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{g\in S_n} \left|\frac{f(gm)}{h(gm)} - 1\right| = 0$$

uniformly in m. This means that

$$\lim_{n\to\infty} \frac{1}{|S_n|} \sup_{m\in G} \sum_{g\in S_n} \left|\frac{f(gm)}{h(gm)} - 1\right| = 0.$$

Let us consider the Folner sequence $\{T_n\}$. Let $\delta \in (0, \frac{1}{4})$ and $N \in \mathbb{N}$. By Lemma 4.4, there exist integers $n_1, n_2, ..., n_k$ with $N \le n_1 < n_2 < \cdots < n_k$ such that when n is large enough $S_{n_1}, ..., S_{n_k}$ δ-quasitile T_n with tiling center $C_1^n, C_2^n, ..., C_k^n$. Thus when n is large enough then $T_n \supseteq \cup_{i=1}^k S_{n_i} C_i^n$ and

$$|\cup_{i=1}^k S_{n_i} C_i^n| \ge \max\left\{(1-\delta)|T_n|, (1-\delta)\sum_{i=1}^k |C_i^n||S_{n_i}|\right\}$$

which implies

$$\begin{aligned}
\frac{1}{|T_n|} \sup_{m\in G} \sum_{g\in S_n} \left|\frac{f(gm)}{h(gm)} - 1\right| &\le \frac{1}{|T_n|} \sup_{m\in G} \sum_{g\in T_n \setminus \cup_{i=1}^k S_{n_i} C_i^n} \left|\frac{f(gm)}{h(gm)} - 1\right| \\
&\quad + \frac{1}{|T_n|} \sup_{m\in G} \sum_{g\in \cup_{i=1}^k S_{n_i} C_i^n} \left|\frac{f(gm)}{h(gm)} - 1\right| \\
&\le \frac{1}{|T_n|} \sup_{m\in G} \sum_{g\in T_n \setminus \cup_{i=1}^k S_{n_i} C_i^n} \left|\frac{f(gm)}{h(gm)} - 1\right| \\
&\quad + \frac{1}{|\cup_{i=1}^k S_{n_i} C_i^n|} \sup_{m\in G} \sum_{g\in \cup_{i=1}^k S_{n_i} C_i^n} \left|\frac{f(gm)}{h(gm)} - 1\right| \\
&\le \delta + \frac{1}{|\cup_{i=1}^k S_{n_i} C_i^n|} \sup_{m\in G} \sum_{g\in \cup_{i=1}^k S_{n_i} C_i^n} \left|\frac{f(gm)}{h(gm)} - 1\right| \\
&\le \delta + \sum_{i=1}^k \frac{|C_i^n| \sup_{m\in G} \sum_{g\in S_{n_i}} \left|\frac{f(gm)}{h(gm)} - 1\right|}{(1-\delta)\sum_{i=1}^k |C_i^n||S_{n_i}|} \\
&\le \delta + \frac{1}{1-\delta} \max_{1\le i\le k} \frac{\sup_{m\in G} \sum_{g\in S_{n_i}} \left|\frac{f(gm)}{h(gm)} - 1\right|}{|S_{n_i}|}
\end{aligned}$$

$$\leq \delta + \frac{1}{1-\delta} \sup_{l \geq N} \frac{\sup_{m\in G} \sum_{g\in S_l} \left|\frac{f(gm)}{h(gm)} - 1\right|}{|S_l|}.$$

Now letting $\delta \to 0^+$ and $N \to +\infty$, we have

$$\frac{1}{|T_n|} \sup_{m\in G} \sum_{g\in T_n} \left|\frac{f(gm)}{h(gm)} - 1\right| \leq \frac{1}{|S_n|} \sup_{m\in G} \sum_{g\in S_n} \left|\frac{f(gm)}{h(gm)} - 1\right|$$

and

$$\lim_{n\to\infty} \frac{1}{|T_n|} \sup_{m\in G} \sum_{g\in T_n} \left|\frac{f(gm)}{h(gm)} - 1\right| = 0.$$

Therefore

$$\lim_{n\to\infty} \frac{1}{|T_n|} \sum_{g\in T_n} \left|\frac{f(gm)}{h(gm)} - 1\right| = 0$$

uniformly in m. A similar argument holds for the other limit. □

Theorem 4.6. *Let $f, h \in w(G)$ two nonnegative functions. Then*

(a) f is statistically equivalent to h, for the Folner sequence (S_n) for G, if it strong asymptotically equivalent to h, for the Folner sequence $\{S_n\}$ for G,

(b) $f, h \in m(G)$ and f is statistically equivalent to h for the Folner sequence $\{S_n\}$ for G, then f strong asymptotically equivalent to h for the Folner sequence $\{S_n\}$ for G.

Proof. (a) If $\epsilon > 0$ and $f \sim^w h$, then

$$\sum_{g\in S_n} \left|\frac{f(g)}{h(g)} - 1\right| \geq \left|\left\{g \in S_n : \left|\frac{f(g)}{h(g)} - 1\right| \geq \epsilon\right\}\right| \epsilon$$

it follows that if $f \sim^w h$, for any Folner sequence (S_n) for G, then f is statistically equivalent to h for the Folner sequence (S_n) for G.

(b) Let $f, h \in m(G)$ be statistically equivalent for the Folner sequence $\{S_n\}$ for G. Since $f, h \in m(G)$, they are bounded, assume that$|\frac{f(g)}{h(g)} - 1| \leq K$ for all g. Let $\epsilon > 0$ be given and set $L_n = \{g \in S_n : |\frac{f(g)}{h(g)} - 1| \geq \epsilon\}$.

$$\frac{1}{|S_n|} \sum_{g\in S_n} \left|\frac{f(g)}{h(g)} - 1\right| = \frac{1}{|S_n|} \left(\sum_{g\in L_n} \left|\frac{f(g)}{h(g)} - 1\right| + \sum_{g\in S_n\setminus L_n} \left|\frac{f(g)}{h(g)} - 1\right| \right)$$
$$\leq \frac{K}{|S_n|} \left|\left\{g \in S_n : \left|\frac{f(g)}{h(g)} - 1\right| \geq \epsilon\right\}\right| + \epsilon.$$

Hence f is strong asymptotically equivalent to h for the Folner sequence $\{S_n\}$ for G. □

Theorem 4.7. *Let $f, h \in w(G)$ two nonnegative functions. Then*

(a) f is almost statistically equivalent to h, for the Folner sequence (S_n) for G, if it strong almost asymptotically equivalent to h, for the Folner sequence $\{S_n\}$ for G,

(b) $f, h \in m(G)$ and f is almost statistically equivalent to h for the Folner sequence $\{S_n\}$ for G, then f strong almost asymptotically equivalent to h for the Folner sequence $\{S_n\}$ for G.

Proof. Proof is similar to the proof of Theorem 4.6. □

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