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Properties of (x(yz))z with Opposite Loop and Reverse Arc Graph Varieties of Type (2,0)[1]

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Abstract : Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies a term equation $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$. A class of graph algebras $\mathcal{V}$ is called a graph variety if $\mathcal{V} = Mod_g\Sigma$ where Σ is a subset of $T(X) \times T(X)$. A graph variety $\mathcal{V}' = Mod_g\Sigma^{'}$ is called an $(x(yz))z$ with opposite loop and reverse arc graph variety if $\Sigma^{'}$ is a set of $(x(yz))z$ with opposite loop and reverse arc term equations.

In this paper, we characterize all $(x(yz))z$ with opposite loop and reverse arc graph varieties.

Keywords : varieties; $(x(yz))z$ with opposite loop and reverse arc graph varieties; term; binary algebra; graph algebras.
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1 Introduction

Graph algebras have been invented in [1] to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph

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with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra* $\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and with two basic operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if} \quad (u,v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

In [2] graph varieties had been investigated for finite undirected graphs in order to get graph theoretic results (structure theorems) from universal algebra via graph algebras. In [3] these investigations are extended to arbitrary (finite) directed graphs where the authors ask for a graph theoretic characterization of graph varieties, i.e., of classes of graphs which can be defined by term equations for their corresponding graph algebras. The answer is a theorem of **Birkhoff-type**, which uses graph theoretic closure operations. *A class of finite directed graphs is equational* (i.e., a graph variety) *if and only if it is closed with respect to finite restricted pointed subproducts and isomorphic copies.*

In [4] Krapeedaeng and Poomsa-ard characterized biregular graph varieties and in [5] Anantpinitwatna and Poomsa-ard characterized $(x(yz))z$ with loop graph varieties.

A graph variety $\mathcal{V}' = Mod_g\Sigma^{'}$ is called a $(x(yz))z$ with opposite loop and reverse arc graph variety if $\Sigma^{'}$ is a set of $(x(yz))z$ with opposite loop and reverse arc term equations. In this paper, we characterize all $(x(yz))z$ with opposite loop and reverse arc graph varieties.

2 Terms and Graph Varieties

In [6], Pöschel was introduced terms for graph algebras, the underlying formal language has to contain a binary operation symbol (juxtaposition) and a symbol for the constant ∞ (denoted by ∞, too).

Definition 2.1. The set $T(X)$ of all terms over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...,$ and ∞ are terms;

(ii) if t_1 and t_2 are terms, then t_1t_2 is a term.

$T(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps. Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $T(X_2)$. The leftmost variable of a term t is denoted by $L(t)$. A term, in which the symbol ∞ occurs is called a *trivial term.*

Definition 2.2. For each non-trivial term t of type $\tau = (2,0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$$

where $t = t_1t_2$ is a compound term.

$L(t)$ is called the *root* of the graph $G(t)$, and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, we assign the empty graph ϕ to every trivial term t.

Definition 2.3. A non-trivial term t of type $\tau = (2,0)$ is called $(x(yz))z$ with opposite loop and reverse arc term if and only if $G(t)$ is a graph with $V(t) = \{x, y, z\}$ and $E(t) = E \cup (\cup_{X \in E'} X)$, where $E = \{(x, y), (x, z), (y, z)\}$, $E' \subseteq \{U, V, W\}$, $E' \neq \phi$ and $U = \{(x, x), (z, y)\}$, $V = \{(y, y), (z, x)\}$, $W = \{(z, z), (y, x)\}$. A term equation $s \approx t$ is called $(x(yz))z$ with opposite loop and reverse arc equation if s and t are $(x(yz))z$ with opposite loop and reverse arc terms

Definition 2.4. We say that a graph $G = (V, E)$ *satisfies a term equation* $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$ (i.e., we have $s = t$ for every assignment $V(s) \cup V(t) \to V \cup \{\infty\}$), and in this case, we write $G \models s \approx t$. Given a class $\mathcal{G}$ of graphs and a set Σ of term equations (i.e., $\Sigma \subset T(X) \times T(X)$) we introduce the following notation:

$G \models \Sigma$ if $G \models s \approx t$ for all $s \approx t \in \Sigma$,
$\mathcal{G} \models s \approx t$ if $G \models s \approx t$ for all $G \in \mathcal{G}$,
$\mathcal{G} \models \Sigma$ if $G \models \Sigma$ for all $G \in \mathcal{G}$,
$Id\mathcal{G} = \{s \approx t \mid s, t \in T(X),\ \mathcal{G} \models s \approx t\}$,
$Mod_g\Sigma = \{G \mid G \text{ is a graph and } G \models \Sigma\}$,
$\mathcal{V}_g(\mathcal{G}) = Mod_g Id\mathcal{G}$.

$\mathcal{V}_g(\mathcal{G})$ is called the *graph variety generated by* $\mathcal{G}$ and $\mathcal{G}$ is called *graph variety* if $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$. $\mathcal{G}$ is called *equational* if there exists a set Σ' of term equations such that $\mathcal{G} = \text{Mod}_g\Sigma'$. Obviously $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$ if and only if $\mathcal{G}$ is an equational class.

In [6], Pöschel showed that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Now, we want to describe how to construct the normal form term. Let t be a non-trivial term. The *normal form term of* t is the term $NF(t)$ constructed by the following algorithm:

(i) Construct $G(t) = (V(t), E(t))$.

(ii) Construct for every $x \in V(t)$ the list $l_x = (x_{i_1}, ..., x_{i_{k(x)}})$ of all out-neighbors (i.e., $(x, x_{i_j}) \in E(t), 1 \leq j \leq k(x)$) ordered by increasing indices $i_1 \leq ... \leq i_{k(x)}$ and let s_x be the term $(...((xx_{i_1})x_{i_2})...x_{i_{k(x)}})$.

(iii) Starting with $x := L(t), Z := V(t), s := L(t)$, choose the variable $x_i \in Z \cap V(s)$ with the least index i, substitute the first occurrence of x_i by the term s_{x_i}, denote the resulting term again by s and put $Z := Z \setminus \{x_i\}$. Continue this procedure while $Z \neq \phi$. The resulting term is the normal form $NF(t)$.

The algorithm stops after a finite number of steps, since $G(t)$ is a rooted graph. Without difficulties one shows $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

Definition 2.5. Let $G = (V, E)$ and $G^{'} = (V^{'}, E^{'})$ be graphs. A *homomorphism* h from G into G' is a mapping $h : V \to V^{'}$ carrying edges to edges ,that is, for which $(u, v) \in E$ implies $(h(u), h(v)) \in E^{'}$.

In [7], Kiss et al. proved the following proposition:

Proposition 2.6. *Let $G = (V, E)$ be a graph and let $h : X \cup \{\infty\} \longrightarrow V \cup \{\infty\}$ be an evaluation of the variables such that $h(\infty) = \infty$. Consider the canonical extension of h to the set of all terms. Then there holds: if t is a trivial term then $h(t) = \infty$. Otherwise, if $h : G(t) \longrightarrow G$ is a homomorphism of graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of graphs, then $h(t) = \infty$.*

Further in [4] Krapeedang and Poomsa-ard proved the following proposition:

Proposition 2.7. *Let $G = (V, E)$ be a graph s and t be non-trivial terms. Then $G \models s \approx t$ if and only if $G \models NF(s) \approx NF(t)$.*

3 $(x(yz))z$ with Opposite Loop and Reverse Arc Graph Varieties

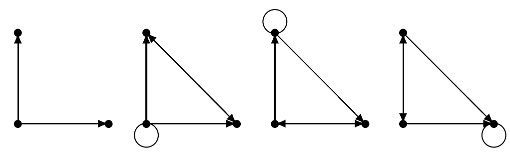
We see that if s and t are terms such that $V(s) \neq V(t)$ or $L(s) \neq L(t)$, then there exists a complete graph (a complete graph with more than one vertices) which does not belong to the graph variety $Mod_g\{s \approx t\}$. In this study, we will investigate only the graph varieties which contain all complete graphs. By Proposition 2.7, we see that for any $\Sigma \subseteq T(X) \times T(X)$ and Σ' is the set of term equations $NF(s) \approx NF(t)$, if $s \approx t \in \Sigma$. Then, $Mod_g\Sigma$ and $Mod_g\Sigma'$ are the same graph variety. Hence, if we want to find properties of all $(x(yz))z$ with opposite loop and reverse arc graph varieties, then it is enough to find the properties of all graph varieties $Mod_g\Sigma^{'}$ with $\Sigma^{'}$ is any subset of $T' \times T'$, where $T' = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ and $s_1 = (x((yx)(zz)))z$, $s_2 = ((xx)(y(zy)))z$, $s_3 = (x((yy)(zx)))z$, $s_4 = ((xx)((yx)((zy)z)))z$, $s_5 = (x(((yx)y)((zx)z)))z$, $s_6 = ((xx)((yy)((zx)y)))z$, $s_7 = ((xx)(((yx)y)(((zx)y)z)))z$. Clearly, for each $i = 1, 2, 3, ..., 7$, $\mathcal{K}_0 = Mod_g\{s_i \approx s_i\}$ is the set of all graph algebras.

In [7], Kiss et al. proved:

Proposition 3.1. *Let s and t be non-trivial terms from $T(X)$ with variables $V(s) = V(t) = \{x_0, x_1, ..., x_n\}$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if the graph algebra $\underline{A(G)}$ has the following property:*

A mapping $h : V(s) \longrightarrow V$ is a $\overline{\text{homomorphism}}$ from $G(s)$ into G iff it is a homomorphism from $G(t)$ into G.

For convenient to referent, we collect the following graphs:



G_1 G_2 G_3 G_4

Next, we will find all $(x(yz))z$ with opposite loop and reverse arc graph varieties. At first we will find all $(x(yz))z$ with opposite loop and reverse arc graph varieties of the form $Mod_g\{s_i \approx s_j\}, i \neq j$. Since $T^{'}$ has 7 elements, so there are at most 21 $(x(yz))z$ with opposite loop and reverse arc graph varieties of these forms.

Let

$$\mathcal{K}_1 = Mod_g\{((xx)(y(zy)))z \approx (x((yy)(zx)))z\},$$

$$\mathcal{K}_2 = Mod_g\{((xx)(y(zy)))z \approx (x((yx)(zz)))z\},$$

$$\mathcal{K}_3 = Mod_g\{(x((yy)(zx)))z \approx (x((yx)(zz)))z\}.$$

Theorem 3.2. *Let $G = (V, E)$ be a graph. Then we have the following:*

(i) $G \in \mathcal{K}_1$ if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(a, a), (c, b) \in E$ if and only if $(b, b), (c, a) \in E$.

(ii) $G \in \mathcal{K}_2$ if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(a, a), (c, b) \in E$ if and only if $(c, c), (b, a) \in E$.

(iii) $G \in \mathcal{K}_3$ if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(b, b), (c, a) \in E$ if and only if $(c, c), (b, a) \in E$.

Proof. Let $G = (V, E)$ be a graph. Suppose that $G \in \mathcal{K}_1$ and for any $a, b, c \in V$, $(a, b), (b, c), (a, c), (a, a), (c, b) \in E$. Let s and t be non-trivial terms such that $s = ((xx)(y(zy)))z$ and $t = (x((yy)(zx)))z$ and let $h : V(s) \to V$ be a function such that $h(x) = a,\ h(y) = b$ and $h(z) = c$. We see that h is a homomorphism from $G(s)$ into G. By Proposition 3.1, we have h is a homomorphism from $G(t)$ into G. Since $(y, y), (z, x) \in E(t)$, we have $(h(y), h(y)) = (b, b) \in E$ and $(h(z), h(x)) = (c, a) \in E$. In the same way, we can prove that if $(a, b), (b, c), (a, c), (b, b), (c, a) \in E$, then $(a, a), (c, b) \in E$.

Conversely, suppose that $G = (V, E)$ be a graph which has property that, for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(a, a), (c, b) \in E$ if and only if $(b, b), (c, a) \in E$. Let s and t be non-trivial terms such that $s = ((xx)(y(zy)))z$ and $t = (x((yy)(zx)))z$ and let $h : V(s) \to V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G. Since $(x, y), (y, z), (x, z), (x, x), (z, y) \in E(s)$, we have $(h(x), h(y)), (h(y), h(z)), (h(x), h(z)), (h(x), h(x)), (h(z), h(y)) \in E$. By assumption, we get $(h(y), h(y)), (h(z), h(x)) \in E$. Hence, h is a homomorphism from $G(t)$ into G. In the same way, we can prove that if h is a homomorphism from

$G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Then, by Proposition 3.1 we get $\overline{A(G)}$ satisfies $s \approx t$.

The proof of (ii) and (iii) are similar as the proof of (i). $\square$

Let

$$\mathcal{K}_4 = Mod_g\{((xx)(y(zy)))z \approx ((xx)((yy)((zx)y)))z\},$$
$$\mathcal{K}_5 = Mod_g\{(x((yy)(zx)))z \approx ((xx)((yy)((zx)y)))z\},$$
$$\mathcal{K}_6 = Mod_g\{(x((yx)(zz)))z \approx ((xx)((yx)((zy)z)))z\}.$$

Theorem 3.3. *Let $G = (V, E)$ be a graph. Then we have the following:*

(i) $G \in \mathcal{K}_4$ *if and only if for any* $a, b, c \in V$ *if* $(a, b), (b, c), (a, c), (a, a), (c, b) \in E$, *then* $(b, b), (c, a) \in E$.

(ii) $G \in \mathcal{K}_5$ *if and only if for any* $a, b, c \in V$ *if* $(a, b), (b, c), (a, c), (b, b), (c, a) \in E$, *then* $(a, a), (c, b) \in E$.

(iii) $G \in \mathcal{K}_6$ *if and only if for any* $a, b, c \in V$ *if* $(a, b), (b, c), (a, c), (c, c), (b, a) \in E$, *then* $(a, a), (c, b) \in E$.

Proof. Let $G = (V, E)$ be a graph. Suppose that $G \in \mathcal{K}_4$ and for any $a, b, c \in V$ $(a, b), (b, c), (a, c), (a, a), (c, b) \in E$. Let s and t be non-trivial terms such that $s = ((xx)(y(zy)))z$ and $t = ((xx)((yy)((zx)y)))z$ and let $h : V(s) \to V$ be a function such that $h(x) = a, h(y) = b$ and $h(z) = c$. We see that h is a homomorphism from $G(s)$ into G. By Proposition 3.1, we have h is a homomorphism from $G(t)$ into G. Since $(y, y), (z, x) \in E(t)$, we have $(h(y), h(y)) = (b, b) \in E$ and $(h(z), h(x)) = (c, a) \in E$.

Conversely, suppose that $G = (V, E)$ be a graph which has property that, for any $a, b, c \in V$ if $(a, b), (b, c), (a, c), (a, a), (c, b) \in E$, then $(b, b), (c, a) \in E$. Let s and t be non-trivial terms such that $s = ((xx)(y(zy)))z$ and $t = ((xx)((yy)((zx)y)))z$ and let $h : V(s) \to V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G. Since $(x, y), (y, z), (x, z), (x, x), (z, y) \in E(s)$, we have $(h(x), h(y)), (h(y), h(z)), (h(x), h(z)), (h(x), h(x)), (h(z), h(y)) \in E$. By assumption, we get $(h(y), h(y)), (h(z), h(x)) \in E$. Hence, h is a homomorphism from $G(t))$ into G. Clearly, if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Then, by Proposition 3.1 we get $\overline{A(G)}$ satisfies $s \approx t$.

The proof of (ii) and (iii) are similar as the proof of (i). $\square$

Theorem 3.4. *Let $G = (V, E)$ be a graph and $\mathcal{K}_7 = Mod_g\{((xx)((yy)((zx)y)))z \approx ((xx)(((yx)y)(((zx)y)z)))z\}$. Then $G \in \mathcal{K}_7$ if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c), (a, a), (c, b), (b, b), (c, a) \in E$, then $(c, c), (b, a) \in E$.*

Proof. Let $G = (V, E)$ be a graph. Suppose that $G \in \mathcal{K}_7$ and for any $a, b, c \in V$, $(a, b), (b, c), (a, c), (a, a), (c, b), (b, b), (c, a) \in E$. Let s and t be non-trivial terms such that $s = ((xx)((yy)((zx)y)))z$ and $t = ((xx)(((yx)y)(((zx)y)z)))z$ and let $h : V(s) \to V$ be a function such that $h(x) = a, h(y) = b$ and $h(z) = c$. We see that h is a homomorphism from $G(s)$ into G. By Proposition 3.1, we have

h is a homomorphism from $G(t)$ into G. Since $(z,z),(y,x) \in E(t)$, we have $(h(z),h(z)) = (c,c) \in E$ and $(h(y),h(x)) = (b,a) \in E$.

Conversely, suppose that $G = (V,E)$ be a graph which has property that, for any $a,b,c \in V$ if $((a,b),(b,c),(a,c),(a,a),(c,b),(b,b),(c,a) \in E$, then $(c,c),(b,a) \in E$. Let s and t be non-trivial terms such that $s = ((xx)((yy)((zx)y)))z$ and $t = ((xx)(((yx)y)(((zx)y)z)))z$ and let $h : V(s) \to V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G. Since $(x,y),(y,z),(x,z),(x,x),(z,y),(y,y),(z,x) \in E(s)$, we have $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)),(h(x),h(x)),(h(z),h(y)),(h(y),h(y)),(h(z),h(x)) \in E$. By assumption, we get $(h(z),h(z)),(h(y),h(x)) \in E$. Hence, h is a homomorphism from $G(t))$ into G. Clearly, if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Then by Proposition 3.1, we get $\underline{A(G)}$ satisfies $s \approx t$. $\square$

Let

$$\mathcal{K}^{'} = Mod_g\{((xx)(y(zy)))z \approx ((xx)((yx)((zy)z)))z\},$$

$$\mathcal{K}^{''} = Mod_g\{((xx)(y(zy)))z \approx (x(((yx)y)((zx)z)))z\},$$

$$\mathcal{K}^{'''} = Mod_g\{((xx)(y(zy)))z \approx ((xx)(((yx)y)(((zx)y)z)))z\}.$$

By using the Proposition 3.1 to check the graphs in $\mathcal{K}_4$ and the graphs in $Mod_g\{((xx)(y(zy)))z \approx ((xx)((yx)((zy)z)))z\}$, we found that they are the same graph variety. After that we use the Proposition 3.1 recheck again as the following theorem:

Theorem 3.5. *Let* $\mathcal{K}^{'} = Mod_g\{((xx)(y(zy)))z \approx ((xx)((yx)((zy)z)))z\}$. *Then,* $\mathcal{K}_4 = \mathcal{K}^{'}$.

Proof. Let $G = (V,E)$ be a graph. Suppose that $G \in \mathcal{K}_4$. Let $s = ((xx)(y(zy)))z$, $t = ((xx)((yy)((zx)y)))z$, $t^{'} = ((xx)((yx)((zy)z)))z$ and let $h : V(s) \longrightarrow V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G. Since $(x,y),(y,z),(x,z),(x,x),(z,y) \in E(s)$, we have $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)),(h(x),h(x)),(h(z),h(y)) \in E$. Since $G \in \mathcal{K}_4$, by Theorem 3.3(1), we have $(h(y),h(y)),(h(z),h(x)) \in E$. Consider for $(h(x),h(z)),(h(z),h(y)),(h(x),h(y)),(h(x),h(x)),(h(y),h(z)) \in E$. Since $G \in \mathcal{K}_4$, we have $(h(z),h(z)),(h(y),h(x)) \in E$. So, h is a homomorphism from $G(t^{'})$ into G. Clearly, if h is a homomorphism from $G(t^{'})$ into G, then it is a homomorphism from $G(s)$ into G. By Proposition 3.1, we have $G \in \mathcal{K}^{'}$.

Let $G \in \mathcal{K}^{'}$. Suppose that h is a homomorphism from $G(s)$ into G. Since $(x,y),(y,z),(x,z),(x,x),(z,y) \in E(s)$, we have $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)),(h(x),h(x)),(h(z),h(y)) \in E$. Since $G \in \mathcal{K}^{'}$, by Proposition 3.1, we get h is a homomorphism from $G(t^{'})$ into G. Since $(z,z),(y,x) \in E(t^{'})$, we have $(h(z),h(z)),(h(y),h(x)) \in E$. Let $h^{'} : V(s) \to V$ such that $h^{'}(x) = h(x)$, $h^{'}(y) = h(z)$ and $h^{'}(z) = h(y)$. We have $h^{'}$ is a homomorphism from $G(s)$ into G. By Proposition 3.1, we get $h^{'}$ is a homomorphism from $G(t^{'})$ into G. Hence, $(h^{'}(z),h^{'}(z)) = (h(y),h(y)) \in E$ and $(h^{'}(y),h^{'}(x)) = (h(z),h(x)) \in E$. Therefore, h is a homomorphism from $G(t)$ into G. Clearly, if h is a homomorphism from

$G(t)$ into G, then it is a homomorphism from $G(s)$ into G. By Proposition 3.1, we have $G \in \mathcal{K}_4$. We get $\mathcal{K}' = \mathcal{K}_4$. $\square$

In the similar way, we have the following:

$$\begin{aligned}\mathcal{K}_4 &= Mod_g\{((xx)(y(zy)))z \approx ((xx)((yy)((zx)y)))z\}\\ &= Mod_g\{((xx)(y(zy)))z \approx ((xx)((yx)((zy)z)))z\}\}\\ &= Mod_g\{((xx)(y(zy)))z \approx ((xx)((yx)y)(((zx)y)z)))z\}\\ &= Mod_g\{((xx)(y(zy)))z \approx (x(((yx)y)((zx)z)))z\}.\end{aligned}$$

$$\begin{aligned}\mathcal{K}_5 &= Mod_g\{(x((yy)(zx)))z \approx ((xx)((yy)((zx)y)))z\}\\ &= Mod_g\{(x((yy)(zx)))z \approx (x(((yx)y)((zx)z)))z\}\\ &= Mod_g\{(x((yy)(zx)))z \approx ((xx)((yx)y)(((zx)y)z)))z\}\\ &= Mod_g\{(x((yy)(zx)))z \approx ((xx)((yx)((zx)z)))z\}.\end{aligned}$$

$$\begin{aligned}\mathcal{K}_6 &= Mod_g\{(x((yx)(zz)))z \approx ((xx)((yx)((zy)z)))z\}\\ &= Mod_g\{(x((yx)(zz)))z \approx (x(((yx)y)((zx)z)))z\}\\ &= Mod_g\{(x((yx)(zz)))z \approx ((xx)((yx)y)(((zx)y)z)))z\}\\ &= Mod_g\{(x((yx)(zz)))z \approx ((xx)((yy)((zx)y)))z\}.\end{aligned}$$

$$\begin{aligned}\mathcal{K}_7 &= Mod_g\{((xx)((yy)((zx)y)))z \approx ((xx)(((yx)y)(((zx)y)z)))z\}\\ &= Mod_g\{((xx)((yx)((zy)z)))z \approx ((xx)(((yx)y)(((zx)y)z)))z\}\\ &= Mod_g\{(x(((yx)y)((zx)z)))z \approx ((xx)(((yx)y)(((zx)y)z)))z\}\\ &= Mod_g\{((xx)((yy)((zx)y)))z \approx ((xx)((yx)(((zy)z)))z\}\\ &= Mod_g\{((xx)((yy)((zx)y)))z \approx (x(((yx)y)((zx)z)))z\}\\ &= Mod_g\{((xx)((yx)((zy)z)))z \approx (x(((yx)y)((zx)z)))z\}.\end{aligned}$$

Hence, there are only seven $(x(yz))z$ with opposite loop and reverse arc graph varieties $Mod_g\{s_i \approx s_j\}$ with $i \neq j$ for all $i = 1, 2, 3, ..., 7$, $j = 1, 2, 3, ..., 7$. Since for any $\Sigma \subseteq T' \times T'$ the $(x(yz))z$ with opposite loop and reverse arc graph varieties $Mod_g\Sigma = \bigcap_{s\approx t\in\Sigma} Mod_g\{s \approx t\}$. Therefore, we can find all $(x(yz))z$ with opposite loop and reverse arc graph varieties in the following way:

At first let $\mathcal{A} = \{\mathcal{K}_0, \mathcal{K}_1, ..., \mathcal{K}_7\}$. Find the intersection between two elements of $\mathcal{A}$. If there are some of them do not belong to $\mathcal{A}$, then add them into $\mathcal{A}$ to get the set $\mathcal{A}_1$. Do the same manner for $\mathcal{A}_1$ to get $\mathcal{A}_2$. Continue this process until to get the set $\mathcal{A}_n$ such that all intersection between two elements of $\mathcal{A}_n$ belong to itself. Then, $\mathcal{A}_n$ close under inclusion.

Let $\mathcal{K}_8 = \mathcal{K}_1 \cap \mathcal{K}_2$. By Theorem 3.2, we have $G_1 \in \mathcal{K}_8$, $G_4 \in \mathcal{K}_1$,$G_3 \in \mathcal{K}_2$, $G_2 \in \mathcal{K}_3$, $G_2, G_3 \notin \mathcal{K}_1$, $G_2, G_4 \notin \mathcal{K}_2$ and $G_3, G_4 \notin \mathcal{K}_3$. Hence, $\mathcal{K}_8 \neq \phi$, $\mathcal{K}_1 \neq \mathcal{K}_2$, $\mathcal{K}_2 \neq \mathcal{K}_3$ and $\mathcal{K}_3 \neq \mathcal{K}_1$. By Theorem 3.3, we have $G_3, G_4 \in \mathcal{K}_4$, $G_2, G_4 \in \mathcal{K}_5$,

$G_2, G_3 \in \mathcal{K}_6$, $G_2 \notin \mathcal{K}_4$, $G_3 \notin \mathcal{K}_5$ and $G_4 \notin \mathcal{K}_6$. Hence, $\mathcal{K}_4 \neq \mathcal{K}_5$, $\mathcal{K}_5 \neq \mathcal{K}_6$ and $\mathcal{K}_6 \neq \mathcal{K}_4$.

Further, we have the following theorems:

Theorem 3.6. $\mathcal{K}_2 \cap \mathcal{K}_3 = \mathcal{K}_3 \cap \mathcal{K}_1 = \mathcal{K}_8$.

Proof. To show that $\mathcal{K}_2 \cap \mathcal{K}_3 = \mathcal{K}_8$. Let $G = (V, E)$ be a graph. Suppose that $G \in \mathcal{K}_2 \cap \mathcal{K}_3$. We have $G \in \mathcal{K}_2$. For any $a, b, c \in V$, suppose that $(a, b), (b, c), (a, c), (a, a), (c, b) \in E$. Since $G \in \mathcal{K}_2$ and $G \in \mathcal{K}_3$, we have $(c, c), (b, a) \in E$ and $(b, b), (c, a) \in E$. For any $a, b, c \in V$, suppose that $(a, b), (b, c), (a, c), (b, b), (c, a) \in E$. Since $G \in \mathcal{K}_3$ and $G \in \mathcal{K}_2$, we have $(c, c), (b, a) \in E$ and $(a, a), (c, b) \in E$. Hence, $G \in \mathcal{K}_1$ and thus $G \in \mathcal{K}_8$. Suppose that $G \in \mathcal{K}_1 \cap \mathcal{K}_2 = \mathcal{K}_8$. We have $G \in \mathcal{K}_2$. For any $a, b, c \in V$, suppose that $(a, b), (b, c), (a, c), (b, b), (c, a) \in E$. Since $G \in \mathcal{K}_1$ and $G \in \mathcal{K}_2$, we have $(a, a), (c, b) \in E$ and $(c, c), (b, a) \in E$. For any $a, b, c \in V$, suppose that $(a, b), (b, c), (a, c), (c, c), (b, a) \in E$. Since $G \in \mathcal{K}_2$ and $G \in \mathcal{K}_1$, we have $(a, a), (c, b) \in E$ and $(b, b), (c, a) \in E$. Hence, $G \in \mathcal{K}_3$ and thus $G \in \mathcal{K}_2 \cap \mathcal{K}_3$. Therefore, $\mathcal{K}_2 \cap \mathcal{K}_3 = \mathcal{K}_8$.

The proof of $\mathcal{K}_3 \cap \mathcal{K}_1 = \mathcal{K}_8$ is similar as the proof of $\mathcal{K}_2 \cap \mathcal{K}_3 = \mathcal{K}_8$. □

Theorem 3.7. (*i*) $\mathcal{K}_4 \cap \mathcal{K}_5 = \mathcal{K}_1$. (*ii*) $\mathcal{K}_5 \cap \mathcal{K}_6 = \mathcal{K}_3$. (*iii*) $\mathcal{K}_6 \cap \mathcal{K}_4 = \mathcal{K}_2$.

Proof. Let $G = (V, E)$ be a graph. Suppose that $G \in \mathcal{K}_1$ and for any $a, b, c \in V$, $(a, b), (b, c), (a, c), (a, a), (c, b) \in E$. Since $G \in \mathcal{K}_1$, we have $(b, b), (c, a) \in E$. Hence, $G \in \mathcal{K}_4$. For any $a, b, c \in V$, suppose that $(a, b), (b, c), (a, c), (b, b), (c, a) \in E$. Since $G \in \mathcal{K}_1$, we have $(a, a), (c, b) \in E$. Hence, $G \in \mathcal{K}_5$. Then, we get $G \in \mathcal{K}_4 \cap \mathcal{K}_5$.

Suppose that $G \in \mathcal{K}_4 \cap \mathcal{K}_5$ and for any $a, b, c \in V$, $(a, b), (b, c), (a, c), (a, a), (c, b) \in E$. Since $G \in \mathcal{K}_4$, we have $(b, b), (c, a) \in E$. For any $a, b, c \in V$, suppose that $(a, b), (b, c), (a, c), (b, b), (c, a) \in E$. Since $G \in \mathcal{K}_5$, we have $(a, a), (c, b) \in E$. Hence, $G \in \mathcal{K}_1$. Then, we get $G \in \mathcal{K}_4 \cap \mathcal{K}_5 = \mathcal{K}_1$.

The proof of (*ii*) and (*iii*) are similar as the proof of (*i*). □

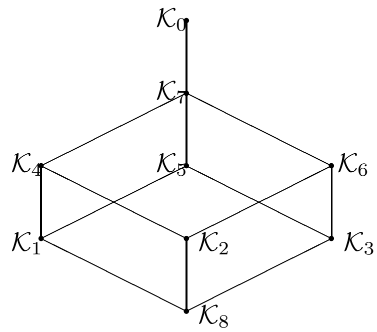
Theorem 3.8. (*i*) $\mathcal{K}_4 \subseteq \mathcal{K}_7$. (*ii*) $\mathcal{K}_5 \subseteq \mathcal{K}_7$. (*iii*) $\mathcal{K}_6 \subseteq \mathcal{K}_7$.

Proof. Let $G = (V, E)$ be a graph. Suppose that $G \in \mathcal{K}_4$. For any $a, b, c \in V$, suppose that $(a, b), (b, c), (a, c), (a, a), (c, b), (b, b), (c, a) \in E$. Since $(a, c), (c, b), (a, b), (a, a), (b, c) \in E$ and $G \in \mathcal{K}_4$, we have $(c, c), (b, a) \in E$. Hence, $G \in \mathcal{K}_7$.

The proof of (*ii*) and (*iii*) are similar as the proof of (*i*), □

From our results, we found that the set $\mathcal{K} = \{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, ..., \mathcal{K}_8\}$ close under inclusion. Hence, $\mathcal{K}$ is the set of all $(x(yz))z$ with opposite loop and reverse arc graph varieties. Since the intersection of any family of $(x(yz))z$ with opposite loop and reverse arc graph varieties is again an $(x(yz))z$ with opposite loop and reverse arc graph variety, we get $\mathcal{K}$ forms a poset under inclusion, in which two elements have a greatest lower bound (their intersection) and least upper bound (the intersection of all $(x(yz))z$ with opposite loop and reverse arc graph varieties which contain both of them), Thus we have a lattice $(\mathcal{K}; \wedge, \vee)$, which we call the

$(x(yz))z$ *with opposite loop and reverse arc lattice* of graph varieties. The following is the Hasse diagram of this lattice.



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