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Common Fixed Point Theorem for Occasionally Weakly Compatible Mappings in Probabilistic Metric Spaces

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Abstract : The aim of this paper is to prove a common fixed point theorem for two pairs of single-valued and set-valued occasionally weakly compatible mappings in Menger spaces. An example is given to illustrate our main result.

Keywords : t-norm; Menger space; weakly compatible mappings; occasionally weakly compatible mappings; fixed point.
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1 Introduction

The notion of probabilistic metric spaces (briefly, PM-spaces), as a generalization of metric spaces, with non-deterministic distance, was introduced by Professor Karl Menger [1] in 1942. The study of these spaces received much attention after the pioneering work of Schweizer and Sklar [2, 3]. One of the simplest and most useful results in the fixed point theory is the Banach-Caccioppoli contraction mapping principle. This theorem provides a technique for solving a variety of applied

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problems in mathematical sciences and engineering. Banach contraction principle has been generalized in different spaces by mathematicians over the years. In 1972, Sehgal and Bharucha-Reid [4] initiated the study of contraction mappings in PM-spaces. For other related fixed point results in Menger spaces and their applications, we refer to [5].

Many mathematicians weakened the notion of commutativity by introducing the notions of weak commutativity [6], compatibility [7] and weak compatibility [8] in metric spaces and proved a number of fixed point theorems using these notions. In 2008, Al-Thagafi and Shahzad [9] gave a definition which is proper generalization of nontrivial weakly compatible mappings which have coincidence points. Jungck and Rhoades [10] studied fixed point results for occasionally weakly compatible mappings. Many authors exploited these concepts (see for example, [11–14] in framework of PM-spaces to obtain a number of common fixed point results.

In an interesting note, Dorić et al. [15] have shown that in respect of single-valued mappings, the condition of occasionally weak compatibility reduces to weak compatibility in the presence of a unique point of coincidence (or a unique common fixed point) of the given pair of mappings. Thus, no generalization can be obtained by replacing weak compatibility with occasionally weak compatibility.

In 1976, Caristi [16] proved a fixed point theorem. Since the Caristi's fixed point theorem does not require the continuity of the mappings, it has applications in many fields. In 1993, Zhang et al. [17] proved set-valued Caristi's theorem in probabilistic metric spaces. Chuan [18] brought forward the concept of Caristi type hybrid fixed point in Menger spaces. In 2006, Chen and Chang [19] proved a common fixed point theorem for four single-valued and two set-valued mappings in a complete Menger spaces by using the notion of compatibility. Further, Pant et al. [20] proved common fixed point theorems for single-valued and set-valued mappings in Menger spaces using implicit relation. More recently, Pant et al. [21] improved the results of Chen and Chang [19] by using the notion of occasionally weak compatible mappings. Several interesting results for multi-valued mappings are also appeared in [22–24].

In the present paper, we prove a common fixed point theorem for single-valued and set-valued occasionally weakly compatible mappings in Menger spaces. An example is furnished which demonstrates the validity of the hypotheses and degree of generality of our main result.

2 Preliminaries

Definition 2.1 ([3])**.** A triangular norm $\triangle$ (shortly t-norm) is a binary operation on the unit interval $[0,1]$ such that for all $a,b,c,d \in [0,1]$ and the following conditions are satisfied

1. $\triangle(a,1) = a$;
2. $\triangle(a,b) = \triangle(b,a)$;

3. $\triangle(a,b) \leq \triangle(c,d)$ whenever $a \leq c$ and $b \leq d$;
4. $\triangle(a, \triangle(b,c)) = \triangle(\triangle(a,b), c)$.

Examples of t-norms are $\triangle(a,b) = \min\{a,b\}$, $\triangle(a,b) = ab$ and $\triangle(a,b) = \max\{a+b-1, 0\}$.

Definition 2.2 ([3]). A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is said to be a *distribution function* if it is non-decreasing and left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in R\} = 1$.

We shall denote by $\Im$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \Im$ is called a probabilistic distance on X and $\mathcal{F}(x,y)$ is usually denoted by $F_{x,y}$.

Definition 2.3 ([3]). The ordered pair $(X, \mathcal{F})$ is called a *PM-space* if X is a non-empty set and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

1. $F_{x,y}(t) = H(t) \Leftrightarrow x = y$;
2. $F_{x,y}(t) = F_{y,x}(t)$;
3. $F_{x,z}(t) = 1, F_{z,y}(s) = 1 \Rightarrow F_{x,y}(t+s) = 1$.

The ordered triple $(X, \mathcal{F}, \triangle)$ is called a Menger space if $(X, \mathcal{F})$ is a PM-space, $\triangle$ is a t-norm and the following inequality holds:

$$F_{x,y}(t+s) \geq \triangle(F_{x,z}(t), F_{z,y}(s)),$$

for all $x, y, z \in X$ and $t, s > 0$.

Every metric space (X, d) can always be realized as a PM-space by considering $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x,y))$ for all $x, y \in X$. So PM-spaces offer a wider framework than that of metric spaces and are better suited to cover even wider statistical situations.

Throughout this paper, $\mathcal{B}(X)$ will denote the family of non-empty bounded subsets of a Menger space $(X, \mathcal{F}, \triangle)$. For all $A, B \in \mathcal{B}(X)$ and for every $t > 0$, we define

$$_D F_{A,B}(t) = \sup\{F_{a,b}(t); a \in A, b \in B\} \tag{2.1}$$

and

$$_\delta F_{A,B}(t) = \inf\{F_{a,b}(t); a \in A, b \in B\}. \tag{2.2}$$

If the set A consists of a single point a, we write

$$_{\delta}F_{A,B}(t) = {}_{\delta}F_{a,B}(t).$$

If the set B also consists of a single point b, we write

$$_{\delta}F_{A,B}(t) = F_{a,b}(t).$$

It follows immediately from the definition that

$$\begin{aligned} &_{\delta}F_{A,B}(t) = {}_{\delta}F_{B,A}(t) \geq 0, \\ &_{\delta}F_{A,B}(t) = 1 \Leftrightarrow A = B = \{a\}, \end{aligned}$$

for all $A, B \in \mathcal{B}(X)$.

Recall that $x \in X$ is called a *coincidence point* (respectively, *common fixed point*) of $S : X \to X$ and $A : X \to \mathcal{B}(X)$ if $Sx \in Ax$ (respectively, $x = Sx \in Ax$).

Definition 2.4 ([8])**.** Mappings $S : X \to X$ and $A : X \to \mathcal{B}(X)$ are said to be weakly compatible if $SAx = ASx$ whenever $Sx \in Ax$.

Example 2.5. *Let $X = [0, \infty)$ with usual metric. Define the mappings $S : X \to X$ and $A : X \to \mathcal{B}(X)$ as: $S(x) = x^2$ for all $x \in X$ and*

$$A(x) = \begin{cases} \{x\}, & \text{if } 0 \leq x \leq 1; \\ (1, x), & \text{if } 1 < x < \infty. \end{cases}$$

Then the mappings S and A are weakly compatible at their coincidence points.

Definition 2.6 ([25])**.** Mappings $S : X \to X$ and $A : X \to \mathcal{B}(X)$ are said to be *occasionally weakly compatible* if and only if there exists some point x in X $Sx \in Ax$ and $SAx \subseteq ASx$.

From the following example, it is clear that the notion of occasionally weakly compatible mappings is more general than weak compatibility.

Example 2.7. *In the setting of Example 2.5, replace the mappings S and A by the following, besides retaining the rest:*

$$S(x) = \begin{cases} 0, & \text{if } 0 \leq x < 2; \\ x + 2, & \text{if } 2 \leq x < \infty. \end{cases} \quad A(x) = \begin{cases} x, & \text{if } 0 \leq x < 2; \\ [2, x + 3], & \text{if } 2 \leq x < \infty. \end{cases}$$

Here, it can be easily verified that $x = 0, 2$ are the coincidence points of S and A, but S and A are not weakly compatible at $x = 2$ that is $AS(2) = [2, 7] \neq SA(2) = [4, 7]$. Hence S and A are not compatible. However, the pair (S, A) is occasionally weakly compatible, since the pair (S, A) is weakly compatible at $x = 0$.

3 Main Result

Theorem 3.1. *Let $(X, \mathcal{F}, \triangle)$ be a Menger space with continuous t-norm. Further, let $S, T : X \to X$ be single-valued and $A, B : X \to \mathcal{B}(X)$ be two set-valued mappings such that the pairs (S, A) and (T, B) are each occasionally weakly compatible satisfying*

$$_{\delta}F_{Ax,By}(t) \geq \phi\left(F_{Sx,Ty}(t)\right) \tag{3.1}$$

for all $x, y \in X$, where $\phi : [0,1] \to [0,1]$ is a continuous function such that $\phi(t) > t$ for each $0 < t < 1$, $\phi(0) = 0$ and $\phi(1) = 1$. Then A, B, S and T have a unique common fixed point.

Proof. Since the pairs (S, A) and (T, B) are each occasionally weakly compatible, there exist points $x, y \in X$ such that $Sx \in Ax$, $SAx \subseteq ASx$ and $Ty \in By$, $TBy \subseteq BTy$. Now we claim that $Sx = Ty$. For if $Sx \neq Ty$, then there exists a positive real number t such that $F_{Sx,Ty}(t) < 1$. Using inequality (3.1) and condition (2.2), we get

$$\begin{aligned} F_{Sx,Ty}(t) &\geq {}_{\delta}F_{Ax,By}(t) \\ &\geq \phi\left(F_{Sx,Ty}(t)\right) > F_{Sx,Ty}(t), \end{aligned}$$

a contradiction. Hence, $Sx = Ty$. Since $Sx \in Ax$, therefore $SSx \in SAx \subseteq ASx$. Also, from condition (2.2), we get $F_{SSx,Sx}(t) \geq {}_{\delta}F_{ASx,By}(t)$. Next we claim that $Sx = SSx$. For if $Sx \neq SSx$, then there exists a positive real number t such that $F_{Sx,SSx}(t) < 1$. Using inequality (3.1) and condition (2.2), we have

$$\begin{aligned} F_{SSx,Sx}(t) &\geq {}_{\delta}F_{ASx,By}(t) \\ &\geq \phi\left(F_{SSx,Ty}(t)\right) \\ &= \phi\left(F_{SSx,Ty}(t)\right) \\ &> F_{SSx,Sx}(t), \end{aligned}$$

which contradicts. Hence the claim follows. Similarly, it can be shown that $Ty = TTy$ which proves that Sx is a common fixed point of A, B, S and T. The uniqueness of common fixed point is an easy consequence of inequality (3.1). □

The following example illustrates Theorem 3.1.

Example 3.2. *Let $X = [0, \infty)$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$, define*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, \triangle)$ be a Menger space, with t-norm $\triangle$ is defined by $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. Define the mappings $S, T : X \to X$ and $A, B : X \to \mathcal{B}(X)$ by

$$A(x) = \begin{cases} \{x\}, & \text{if } 0 \leq x < 1; \\ [1, x+2], & \text{if } 1 \leq x < \infty. \end{cases} \quad B(x) = \begin{cases} \{0\}, & \text{if } 0 \leq x < 1; \\ [1, x+1], & \text{if } 1 \leq x < \infty. \end{cases}$$

$$S(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ x+1, & \text{if } 1 \leq x < \infty. \end{cases} \quad T(x) = \begin{cases} \frac{x}{2}, & \text{if } 0 \leq x < 1; \\ 2x+3, & \text{if } 1 \leq x < \infty. \end{cases}$$

Let $\phi : [0,1] \to [0,1]$ be defined by $\phi(t) = \sqrt{t}$ for $0 < t \leq 1$. Then $\phi(t) > t$ for each $0 < t < 1$ and ${}_\delta F_{Ax,By}(t) \geq \phi\left(F_{Sx,Ty}(t)\right)$ for all $x, y \in X$. Then A, B, S and T satisfy all the conditions of Theorem 3.1, i.e., $0 = S(0) \in A(0)$, $SA(0) = \{0\} = AS(0)$ and $0 = T(0) \in B(0)$, $TB(0) = \{0\} = BT(0)$. Also S and A as well as T and B are occasionally weakly compatible mappings. Hence, 0 is the unique common fixed point of A, B, S and T. On the other hand, it is clear to see that the mappings A, B, S and T are discontinuous at 0.

On taking $A = B$ and $S = T$ in Theorem 3.1, we get the following natural result.

Corollary 3.3. *Let $(X, \mathcal{F}, \triangle)$ be a Menger space with continuous t-norm. Further, let $S : X \to X$ be a single-valued and $A : X \to \mathcal{B}(X)$ be a set-valued mappings such that the pair (S, A) is occasionally weakly compatible satisfying condition*

$$ {}_\delta F_{Ax,Ay}(t) \geq \phi\left(F_{Sx,Sy}(t)\right) \tag{3.2}$$

for all $x, y \in X$, where $\phi : [0,1] \to [0,1]$ is a continuous function such that $\phi(t) > t$ for each $0 < t < 1, \phi(0) = 0$ and $\phi(1) = 1$. Then A and S have a unique common fixed point.

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