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Explicit Eigenvectors Formulae for Lower Doubly Companion Matrices

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Abstract : Butcher and Wright [1] used doubly companion matrices as a tool to analyze numerical methods and some general linear methods property. In this paper, we considered a lower doubly companion matrix and prove that any lower doubly companion matrix (LDCM) are similar to a companion matrix. We obtain the LDCM, the explicit form of eigenvectors of the sum of two LDCMs, the explicit form of the upper doubly companion matrix (UDCM), and the explicit form of the sum of two UDCMs.

Keywords : companion matrix; doubly companion matrix; eigenvalue; eigenvector; lower doubly companion matrix; upper doubly companion matrix; Toeplitz matrix.
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1 Introduction

Let $\mathbb{C}$ be the field of complex numbers. The set of all polynomials in x over $\mathbb{C}$ is denoted by $\mathbb{C}[x]$. For a positive integer n, let M_n be the set of all $n \times n$ matrices over $\mathbb{C}$. The set of all complex vectors, or $n \times 1$ matrices over $\mathbb{C}$ is denoted by $\mathbb{C}^n$. A nonzero vector $\mathbf{v} \in \mathbb{C}^n$ is called an eigenvector of $A \in M_n$ corresponding to a scalar $\lambda \in \mathbb{C}$ if $A\mathbf{v} = \lambda\mathbf{v}$, and the scalar λ is an eigenvalue of the matrix A. The set of eigenvalues of A is call the spectrum of A and is denoted by $\sigma(A)$. Eigenvectors and eigenvalues are used widely in science and engineering.

Butcher and Wright [1, p. 363] defined a doubly companion matrix for the pair of polynomials $\alpha(x) = x^n - \alpha_1 x^{n-1} - \alpha_2 x^{n-2} - \cdots - \alpha_n$ and $\beta(x) = x^n - \beta_1 x^{n-1} - \beta_2 x^{n-2} - \cdots - \beta_n$, as $C \in M_n$ given by

$$C = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{n-1} & -\alpha_n - \beta_n \\ 1 & 0 & 0 & \dots & 0 & -\beta_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -\beta_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -\beta_2 \\ 0 & 0 & 0 & \dots & 1 & -\beta_1 \end{bmatrix}, \tag{1.1}$$

that is, a $n \times n$ matrix C with $n > 1$ is called a *doubly companion matrix* if its entries c_{ij} satisfy $c_{ij} = 1$ for all entries in the sub-maindiagonal of C and else $c_{ij} = 0$ for $i \neq 1$ and $j \neq n$, which is a special case of unreduced upper Hessenberg matrix. Butcher and Wright in [1, pp. 363–364] used the doubly companion matrices as a tool for analyzing various extension of classical methods with inherent Runge-Kutta stability. The doubly companion matrices is important for application in some certain matrix equations, numerical and linear methods. The author in [2] proved that the doubly companion matrix also the sum of two doubly companion matrices are nonderogatory, and also obtained the explicit form of its minimal polynomials.

In this papers, we obtain the eigenvectors formulas for the lower doubly companion matrix, the sum of two lower doubly companion matrices, the upper doubly companion matrix, and the sum of two upper doubly companion matrices, respectively.

2 Preliminaries

Let $\alpha(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ and $\beta(x) = x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$ be two monic polynomials over complex numbers, we prefer to consider the corresponding *lower doubly companion matrix* of $\alpha(x)$ and $\beta(x)$ as,

$$L(\alpha, \beta) = \begin{bmatrix} -b_{n-1} & 1 & \dots & 0 & 0 \\ -b_{n-2} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -b_1 & 0 & \dots & 0 & 1 \\ -b_0 - a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}. \tag{2.1}$$

Malešević, Todorić, Jovović, and Telebaković in [3, Lemma 3.3] studied the sum of its principal minors of order k containing the first column $(1 \leq k \leq n)$ of the lower doubly companion matrix for using in the second step of reduction process for linear system of first order operator equations.

We define the corresponding upper doubly companion matrix of $\alpha(x)$ and $\beta(x)$ as,

$$U(\alpha,\beta) = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -a_0 - b_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}. \tag{2.2}$$

From (2.1), if $b_0 = b_1 = \cdots = b_{n-2} = b_{n-1} = 0$ then the lower doubly companion matrix is become a companion matrix of the form,

$$L(\alpha) = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \tag{2.3}$$

and, if $a_0 = a_1 = \cdots = a_{n-2} = a_{n-1} = 0$ then the matrix in (2.1) is become a companion matrix of another form,

$$L(\beta) = \begin{bmatrix} -b_{n-1} & 1 & \dots & 0 & 0 \\ -b_{n-2} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -b_1 & 0 & \dots & 0 & 1 \\ -b_0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{2.4}$$

It is well known that the last two of these companion matrices are nonderogatory. The matrix $U(\alpha,\beta)$ is also nonderogatory, that is the characteristic polynomial $c_{U(\alpha,\beta)}$ is equal to the minimal polynomial $m_{U(\alpha,\beta)}$, see [2] for more details. We shall abbreviate "the lower doubly companion matrix" into "LDCM," and abbreviate "the upper doubly companion matrix" into "UDCM."

We recall some well-known results from linear algebra.

Theorem 2.1 ([4, Theorem 7.12(1)])**.** *Let $\alpha(x) \in F[x]$. A companion matrix $A := L(\alpha)$ is nonderogatory; in fact, $c_A(x) = m_A(x) = \alpha(x)$.*

Theorem 2.2 ([5, Theorem 3.3.15])**.** *A matrix $A \in M_n$ is similar to the companion matrix of its characteristic polynomial if and only if the minimal and characteristic polynomial of A are identical.*

Theorem 2.3 ([5, Theorem 1.4.8])**.** *Let $A, B \in M_n$, if $\mathbf{x} \in \mathbb{C}^n$ is an eigenvector corresponding to $\lambda \in \sigma(B)$ and B is similar to A via S, then $S\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ.*

3 Main Results

According to any companion matrix $C(\alpha)$ is a nonderogatory with the characteristic and minimal polynomial both equal to $\alpha(x)$, by Theorem 2.1. Firstly, we shown that the LDCM as in (2.1) were similar to a companion matrix in the form (2.3) and produced an explicit eigenvectors formula of the LDCM. Secondly, we prove that the sum of two LDCM of the same size are also similar to a companion matrix. We wish to find the explicit formula of eigenvector for the matrix in this case. Finally we should able to apply the result of LDCM to find the explicit eigenvectors formula of the UDCM, and find the explicit formula of eigenvector for the sum of two UDCM, respectively.

Theorem 3.1. *The LDCM $L(\alpha,\beta)$ in (2.1) is similar to a companion matrix.*

Proof. Let

$$L(\alpha,\beta)=\begin{bmatrix} -b_{n-1} & 1 & \dots & 0 & 0 \\ -b_{n-2} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -b_1 & 0 & \dots & 0 & 1 \\ -b_0-a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}.$$

To show that $L := L(\alpha,\beta)$ is similar to a companion matrix. We shall prove by explicit construction the existence of an invertible matrix M such that $M^{-1}LM$ is a companion matrix. Now, chosen an upper triangular matrix of size $n\times n$,

$$M=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ b_{n-1} & 1 & 0 & \ddots & \vdots \\ b_{n-2} & b_{n-1} & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ b_1 & \dots & b_{n-2} & b_{n-1} & 1 \end{bmatrix}.$$

The matrix M is an lower triangular Toeplitz matrix with diagonal-constant 1. Then M is nonsingular matrix, it is obtained that

$$M^{-1}=\begin{bmatrix} \mathbf{e}_1 & L\mathbf{e}_1 & L^2\mathbf{e}_1 & \dots & L^{n-1}\mathbf{e}_1 \end{bmatrix}^T,$$

where $\mathbf{e}_1=\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T \in \mathbb{C}^n$ is the unit column vector. Computation shows that

$$M^{-1}LM=\begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -\gamma_0 & -\gamma_1 & \dots & -\gamma_{n-2} & -\gamma_{n-1} \end{bmatrix},$$

where

$$\begin{aligned}
\gamma_0 &= \sum_{i+j=n} a_i b_j + a_0 + b_0,\\
\gamma_1 &= \sum_{i+j=n+1} a_i b_j + a_1 + b_1,\\
&\vdots\\
\gamma_{n-2} &= \sum_{i+j=n+n-2} a_i b_j + a_{n-2} + b_{n-2},\\
\gamma_{n-1} &= a_{n-1} + b_{n-1}.
\end{aligned} \tag{3.1}$$

The matrix $L(\sigma) := M^{-1}LM$ is the desired companion matrix, where

$$\begin{aligned}
\sigma(x) &= x^n + \gamma_{n-1}x^{n-1} + \gamma_{n-2}x^{n-2} + \cdots + \gamma_1 x + \gamma_0\\
&= x^n + (a_{n-1} + b_{n-1})x^{n-1} + (a_{n-1}b_{n-1} + a_{n-2} + b_{n-2})x^{n-2} + \cdots\\
&\quad + \left(\sum_{i+j=n+1} a_i b_j + a_1 + b_1\right) x + \left(\sum_{i+j=n} a_i b_j + a_0 + b_0\right).
\end{aligned}$$

□

Since the LDCM $L(\alpha, \beta)$ is similar to the companion matrix $L(\sigma)$, therefore Theorem 2.2 and Theorem 2.1, asserted that $L(\alpha, \beta)$ is a nonderogatory matrix, and the characteristic polynomial also the minimal polynomial of $L(\alpha, \beta)$ is the polynomial $\sigma(x)$ appears above.

Corollary 3.2. *The LDCM $L(\alpha, \beta)$ in (2.1) is a nonderogatory matrix.*

Now analogous as eigenvector of a companion matrix in [6, pp. 630–631] and in [7, p. 6], we obtain.

Theorem 3.3 (Eigenvector for LDCM)**.** *Let λ be an eigenvalue of a lower doubly companion matrix $L(\alpha, \beta) \in M_n$ defined in (2.1). Then*

$$\mathbf{v} = \begin{bmatrix} 1 \\ b_{n-1} + \lambda \\ b_{n-2} + b_{n-1}\lambda + \lambda^2 \\ \vdots \\ b_1 + b_2\lambda + \cdots + b_{n-1}\lambda^{n-2} + \lambda^{n-1} \end{bmatrix}$$

is an eigenvector of $L(\alpha, \beta)$ corresponding to the eigenvalue λ.

Proof. Let $c_{L(\sigma)}(t) = \det(tI_n - L(\sigma))$ be the characteristic polynomial of the companion matrix $L(\sigma)$, where I_n be the identity matrix, we have $c_{L(\sigma)}(x) = \sigma(x)$.

From Theorem 3.1, $L(\alpha,\beta)$ is similar to the companion matrix $L(\sigma)$. Then they have the same eigenvalues in common. Let λ be an eigenvalue of $L(\alpha,\beta)$, then λ also an eigenvalue of $L(\sigma)$. Since λ is a root of the characteristic polynomial $c_{L(\sigma)}(x)$, we have

$$\begin{aligned} c_{L(\sigma)}(\lambda) &= \sigma(\lambda) \\ &= \lambda^n + (a_{n-1}+b_{n-1})\lambda^{n-1} + (a_{n-1}b_{n-1}+a_{n-2}+b_{n-2})\lambda^{n-2} \\ &\quad + \cdots + \left(\sum_{i+j=n+1} a_ib_j + a_1 + b_1\right)\lambda + \left(\sum_{i+j=n} a_ib_j + a_0 + b_0\right) = 0, \end{aligned}$$

that is,

$$c_{L(\sigma)}(\lambda) = \lambda^n + \gamma_{n-1}\lambda^{n-1} + \gamma_{n-2}\lambda^{n-2} + \cdots + \gamma_1\lambda + \gamma_0 = 0,$$

where γ_i, $i = 0, 1, \ldots, n-1$, as in (3.1). Then, we obtain

$$\begin{aligned} L(\sigma)\mathbf{u} &= \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -\gamma_0 & -\gamma_1 & \ldots & -\gamma_{n-2} & -\gamma_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ -\gamma_0 - \gamma_1\lambda - \cdots - \gamma_{n-2}\lambda^{n-2} - \gamma_{n-1}\lambda^{n-1} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ \lambda^n \end{bmatrix} = \lambda\mathbf{u}. \end{aligned}$$

Since $M^{-1}L(\alpha,\beta)M = L(\sigma)$ by Theorem 3.1, implies $L(\alpha,\beta)M = ML(\sigma)$. Analogous as in Theorem 2.3, we obtain $M\mathbf{u}$ is an eigenvector corresponding to λ of the LDCM $L(\alpha,\beta)$. Hence, the explicit form of an eigenvector corresponding to an eigenvalue λ of the matrix $L(\alpha,\beta)$ is

$$\begin{aligned} \mathbf{v} := M\mathbf{u} &= \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ b_{n-1} & 1 & 0 & \ddots & \vdots \\ b_{n-2} & b_{n-1} & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_1 & b_2 & \ldots & b_{n-1} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ b_{n-1} + \lambda \\ b_{n-2} + b_{n-1}\lambda + \lambda^2 \\ \vdots \\ b_1 + b_2\lambda + \cdots + b_{n-1}\lambda^{n-2} + \lambda^{n-1} \end{bmatrix}, \end{aligned}$$

such that

$$L(\alpha,\beta)\mathbf{v} = \lambda\mathbf{v},$$

it is easy to see that the first component in the vector $\mathbf{v}$ cannot be zero, which proves the assertion. $\square$

The following corollary is a particular case of Theorem 3.3, when $a_0 = a_1 = \cdots = a_{n-1} = 0$. But there is a direct proof as follows.

Corollary 3.4. *Let λ be an eigenvalue of the companion matrix $L(\beta)$ defined in (2.4). Then*

$$\mathbf{v} = \begin{bmatrix} 1 \\ b_{n-1}+\lambda \\ b_{n-2}+b_{n-1}\lambda+\lambda^2 \\ \vdots \\ b_1+b_2\lambda+\cdots+b_{n-1}\lambda^{n-2}+\lambda^{n-1} \end{bmatrix}$$

is an eigenvector of $L(\beta)$ corresponding to the eigenvalue λ.

Proof. Let M be the lower triangular Toeplitz matrix with diagonal-constant 1 in Theorem 3.1. A calculation shows that

$$L(\beta)' := M^{-1}L(\beta)M = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -b_0 & -b_1 & \dots & -b_{n-2} & -b_{n-1} \end{bmatrix}.$$

Let λ be an eigenvalue of $L(\beta)$, then λ also an eigenvalue of $L(\beta)'$. Now, we have

$$c_{L(\beta)'}(\lambda) = \lambda^n + b_{n-1}\lambda^{n-1} + b_{n-2}\lambda^{n-2} + \cdots + b_1\lambda + b_0 = 0,$$

and

$$L(\beta)'\mathbf{u} = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -b_0 & -b_1 & \dots & -b_{n-2} & -b_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix} = \lambda\mathbf{u}.$$

Therefore, the explicit form of an eigenvector corresponding to an eigenvalue λ of the matrix $L(\beta)$ is

$$\mathbf{v} := M\mathbf{u} = \begin{bmatrix} 1 \\ b_{n-1}+\lambda \\ b_{n-2}+b_{n-1}\lambda+\lambda^2 \\ \vdots \\ b_1+b_2\lambda+\cdots+b_{n-1}\lambda^{n-2}+\lambda^{n-1} \end{bmatrix},$$

yield the required eigenvector. ◻

We would like to prove that the sum of two LDCMs is similar to a LDCM, and construct the explicit eigenvector formula of the sum of two LDCMs.

Theorem 3.5. *Let $L(\alpha,\beta)$ and $L(\gamma,\delta)$ be two lower doubly companion matrices as the same type of (2.1) and of the same size. Then $L(\alpha,\beta)+L(\gamma,\delta)$ is similar to a lower doubly companion matrix.*

Proof. Let $\alpha(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$, $\beta(x)=x^n+b_{n-1}x^{n-1}+\cdots+b_2x^2+b_1x+b_0$, $\gamma(x)=x^n+c_{n-1}x^{n-1}+\cdots+c_2x^2+c_1x+c_0$, and $\delta(x)=x^n+d_{n-1}x^{n-1}+\cdots+d_2x^2+d_1x+d_0$ are in $\mathbb{C}[x]$. Then

$$\begin{aligned}Z &:= L(\alpha,\beta)+L(\gamma,\delta)\\ &= \begin{bmatrix} -b_{n-1}-d_{n-1} & 2 & \dots & 0 & 0\\ -b_{n-2}-d_{n-2} & 0 & \ddots & \ddots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ -b_1-d_1 & 0 & \dots & 0 & 2\\ -b_0-a_0 & -a_1-c_1 & \dots & -a_{n-2}-c_{n-2} & -a_{n-1}-c_{n-1}\end{bmatrix}.\end{aligned}$$

Let $D=Diag(1,\frac{1}{2},\frac{1}{2^2},\dots,\frac{1}{2^{n-1}})$. Then $D^{-1}=Diag(1,2,2^2,\dots,2^{n-1})$. A calculation shows that

$$\begin{aligned}&D^{-1}ZD\\ &= \begin{bmatrix} 1 & 0 & 0 & \dots & 0\\ 0 & 2 & 0 & \dots & 0\\ 0 & 0 & 2^2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 2^{n-1}\end{bmatrix}\\ &\quad\times \begin{bmatrix} -b_{n-1}-d_{n-1} & 2 & \dots & 0 & 0\\ -b_{n-2}-d_{n-2} & 0 & \ddots & \ddots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ -b_1-d_1 & 0 & \dots & 0 & 2\\ -b_0-a_0 & -a_1-c_1 & \dots & -a_{n-2}-c_{n-2} & -a_{n-1}-c_{n-1}\end{bmatrix}\\ &\quad\times \begin{bmatrix} 1 & 0 & 0 & \dots & 0\\ 0 & \frac{1}{2} & 0 & \dots & 0\\ 0 & 0 & \frac{1}{2^2} & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & \frac{1}{2^{n-1}}\end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} -(b_{n-1}+d_{n-1}) & 1 & \dots & 0 & 0 \\ -2(b_{n-2}+d_{n-2}) & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -2^{n-2}(b_1+d_1) & 0 & \dots & 0 & 1 \\ -2^{n-1}(a_0+b_0+c_0+d_0) & -2^{n-2}(a_1+c_1) & \dots & -2(a_{n-2}+c_{n-2}) & -(a_{n-1}+c_{n-1}) \end{bmatrix}.$$

Therefore, $Z = L(\alpha, \beta) + L(\gamma, \delta)$ is similar to a LDCM. □

Let $L(\alpha+\gamma, \beta+\delta) := D^{-1}ZD$, Theorem 3.5 shows that the sum of two LDCMs $L(\alpha, \beta)$ and $L(\gamma, \delta)$ is similar to the LDCM, $L(\alpha+\gamma, \beta+\delta)$. Then we have the following corollary.

Corollary 3.6. *Let $L(\alpha, \beta)$ and $L(\gamma, \delta)$ be two lower doubly companion matrices as in Theorem 3.5 and let μ be an eigenvalue of the matrix $L(\alpha, \beta)+L(\gamma, \delta)$. Then*

$$\mathbf{w} = \begin{bmatrix} 1 \\ \frac{1}{2}[(b_{n-1}+d_{n-1})+\mu] \\ \frac{1}{2^2}[2(b_{n-2}+d_{n-2})+(b_{n-1}+d_{n-1})\mu+\mu^2] \\ \vdots \\ \frac{1}{2^{n-1}}[2^{n-2}(b_1+d_1)+2^{n-3}(b_2+d_2)\mu+\cdots+(b_{n-1}+d_{n-1})\mu^{n-2}+\mu^{n-1}] \end{bmatrix}$$

is an eigenvector of $L(\alpha, \beta)+L(\gamma, \delta)$ corresponding to the eigenvalue μ.

Proof. Since the matrix $L(\alpha, \beta)+L(\gamma, \delta)$ is similar to the LDCM $L(\alpha+\gamma, \beta+\delta)$, then by hypothesis μ is also the eigenvalue of $L(\alpha+\gamma, \beta+\delta)$. Apply Theorem 3.3 to the LDCM $L(\alpha+\gamma, \beta+\delta)$, then we have

$$\mathbf{u}' := \begin{bmatrix} 1 \\ (b_{n-1}+d_{n-1})+\mu \\ 2(b_{n-2}+d_{n-2})+(b_{n-1}+d_{n-1})\mu+\mu^2 \\ \vdots \\ 2^{n-2}(b_1+d_1)+2^{n-3}(b_2+d_2)\mu+\cdots+(b_{n-1}+d_{n-1})\mu^{n-2}+\mu^{n-1} \end{bmatrix},$$

is an eigenvector corresponding to μ of the matrix $L(\alpha+\gamma, \beta+\delta)$. But $D^{-1}ZD = L(\alpha+\gamma, \beta+\delta)$; hence by Theorem 2.3 we have

$$\mathbf{w} := D\mathbf{u}'$$

$$= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2^{n-1}} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \\ (b_{n-1}+d_{n-1})+\mu \\ 2(b_{n-2}+d_{n-2})+(b_{n-1}+d_{n-1})\mu+\mu^2 \\ \vdots \\ 2^{n-2}(b_1+d_1)+2^{n-3}(b_2+d_2)\mu+\cdots+(b_{n-1}+d_{n-1})\mu^{n-2}+\mu^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \frac{1}{2}[(b_{n-1}+d_{n-1})+\mu] \\ \frac{1}{2^2}[2(b_{n-2}+d_{n-2})+(b_{n-1}+d_{n-1})\mu+\mu^2] \\ \vdots \\ \frac{1}{2^{n-1}}[2^{n-2}(b_1+d_1)+2^{n-3}(b_2+d_2)\mu+\cdots+(b_{n-1}+d_{n-1})\mu^{n-2}+\mu^{n-1}] \end{bmatrix}$$

is an eigenvector corresponding to μ of the matrix $L(\alpha,\beta)+L(\gamma,\delta)$. It is easy to see that the first component in the vector $\mathbf{w}$ cannot be zero, which proves the assertion. □

4 Some Applications

Consider, each UDCM is similar to a LDCM via the *backward identity* matrix of order $n \times n$ (or *reversal* matrix of order $n \times n$), J $(= J^{-1})$, which showing that the UDCM is similar to its transpose [5, pp. 207–208], as follows,

$$\begin{aligned} &J^{-1}U(\alpha,\beta)J \\ &= \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -a_0-b_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix} \\ &\times \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -b_{n-1} & 1 & \dots & 0 & 0 \\ -b_{n-2} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -b_1 & 0 & \dots & 0 & 1 \\ -a_0-b_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} =: L(\alpha,\beta). \end{aligned}$$

According to Theorem 3.3 and Theorem 2.3 we have the following results:

Theorem 4.1 (Eigenvector of UDCM)**.** *Let λ be an eigenvalue of a upper doubly companion matrix $U(\alpha,\beta) \in M_n$ defined in (2.2). Then*

$$\mathbf{v} := \begin{bmatrix} b_1+b_2\lambda+\cdots+b_{n-1}\lambda^{n-2}+\lambda^{n-1} \\ \vdots \\ b_{n-2}+b_{n-1}\lambda+\lambda^2 \\ b_{n-1}+\lambda \\ 1 \end{bmatrix}$$

is an eigenvector of $U(\alpha,\beta)$ *corresponding to the eigenvalue* λ.

Proof. Form the previous matrix equation we see that $U(\alpha,\beta)$ is similar to $L(\beta,\alpha)$, if λ is an eigenvalue of the UDCM

$$U(\alpha,\beta)=\begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -a_0-b_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}$$

then λ is also an eigenvalue of the LDCM,

$$L(\alpha,\beta)=\begin{bmatrix} -b_{n-1} & 1 & \dots & 0 & 0 \\ -b_{n-2} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -b_1 & 0 & \dots & 0 & 1 \\ -a_0-b_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}.$$

Form Theorem 3.3 we known that the formula of a eigenvector for $L(\alpha,\beta)$ is

$$\mathbf{u}=\begin{bmatrix} 1 \\ b_{n-1}+\lambda \\ b_{n-2}+b_{n-1}\lambda+\lambda^2 \\ \vdots \\ b_1+b_2\lambda+\cdots+b_{n-1}\lambda^{n-2}+\lambda^{n-1} \end{bmatrix}.$$

Since $J^{-1}U(\alpha,\beta)J = L(\alpha,\beta)$, by Theorem 3.1, implies $U(\alpha,\beta)J = JL(\alpha,\beta)$. Analogous as in Theorem 2.3, we obtain $J\mathbf{u}$ is an eigenvector corresponding to λ of the UDCM $U(\alpha,\beta)$. Hence, the explicit form of an eigenvector corresponding to an eigenvalue λ for the matrix $U(\alpha,\beta)$ is

$$\begin{aligned}\mathbf{v}:=J\mathbf{u}&=\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ b_{n-1}+\lambda \\ b_{n-2}+b_{n-1}\lambda+\lambda^2 \\ \vdots \\ b_1+b_2\lambda+\cdots+b_{n-1}\lambda^{n-2}+\lambda^{n-1} \end{bmatrix}\\ &=\begin{bmatrix} b_1+b_2\lambda+\cdots+b_{n-1}\lambda^{n-2}+\lambda^{n-1} \\ \vdots \\ b_{n-2}+b_{n-1}\lambda+\lambda^2 \\ b_{n-1}+\lambda \\ 1 \end{bmatrix},\end{aligned}$$

such that

$$U(\alpha,\beta)\mathbf{v} = \lambda\mathbf{v},$$

it is easy to see that the last component in the vector $\mathbf{v}$ cannot be zero, which proves the assertion. $\square$

The sum of two UDCMs is similar via $D = Diag(1, 2, 2^2, \ldots, 2^{n-1})$ to a UDCM can be found in [2, Theorem 3.4]. Let

$$\begin{aligned} Z' &:= U(\alpha,\beta) + U(\gamma,\delta) \\ &= \begin{bmatrix} -b_{n-1} - d_{n-1} & -b_{n-2} - d_{n-2} & \ldots & -b_1 - d_1 & -a_0 - b_0 - c_0 - d_0 \\ 2 & 0 & \ldots & 0 & -a_1 - c_1 \\ 0 & 2 & \ldots & 0 & -a_2 - c_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & -a_{n-2} - c_{n-2} \\ 0 & 0 & \ldots & 2 & -a_{n-1} - c_{n-1} \end{bmatrix}, \end{aligned}$$

we have

$$\begin{aligned} &D^{-1}Z'D \\ &= \begin{bmatrix} -(b_{n-1} + d_{n-1}) & -2(b_{n-2} + d_{n-2}) & \ldots & -2^{n-2}(b_1 + d_1) & -2^{n-1}(a_0 + b_0 + c_0 + d_0) \\ 1 & 0 & \ldots & 0 & -2^{n-2}(a_1 + c_1) \\ 0 & 1 & \ldots & 0 & -2^{n-3}(a_2 + c_2) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & -2(a_{n-2} + c_{n-2}) \\ 0 & 0 & \ldots & 1 & -(a_{n-1} + c_{n-1}) \end{bmatrix}. \end{aligned}$$

Then by Theorem 4.1 above we obtain the following corollary.

Corollary 4.2. *Let $U(\alpha,\beta)$ and $U(\gamma,\delta)$ be two UDCMs and let μ be an eigenvalue of the above matrix $U(\alpha,\beta) + U(\gamma,\delta)$. Then*

$$\mathbf{w} = \begin{bmatrix} 1[2^{n-2}(b_1 + d_1) + 2^{n-3}(b_2 + d_2)\mu + \cdots + (b_{n-1} + d_{n-1})\mu^{n-2} + \mu^{n-1}] \\ \vdots \\ 2^{n-3}[2(b_{n-2} + d_{n-2}) + (b_{n-1} + d_{n-1})\mu + \mu^2] \\ 2^{n-2}[(b_{n-1} + d_{n-1}) + \mu] \\ 2^{n-1} \end{bmatrix}$$

is an eigenvector of $U(\alpha,\beta) + U(\gamma,\delta)$ corresponding to the eigenvalue μ.

Proof. Since the matrix $U(\alpha,\beta)+U(\gamma,\delta) = Z'$ is similar to the UDCM $D^{-1}Z'D =: U(\alpha+\gamma, \beta+\delta)$ where $D = Diag(1, 2, 2^2, \ldots, 2^{n-1})$, by hypothesis μ be an eigenvalue of the matrix $U(\alpha,\beta)+U(\gamma,\delta)$, then μ is also the eigenvalue of $U(\alpha+\gamma, \beta+\delta)$.

Applying Theorem 4.1 to the UDCM $U(\alpha+\gamma, \beta+\delta)$, then we have

$$\mathbf{v}' := \begin{bmatrix} 2^{n-2}(b_1+d_1) + 2^{n-3}(b_2+d_2)\mu + \cdots + (b_{n-1}+d_{n-1})\mu^{n-2} + \mu^{n-1} \\ \vdots \\ 2(b_{n-2}+d_{n-2}) + (b_{n-1}+d_{n-1})\mu + \mu^2 \\ (b_{n-1}+d_{n-1}) + \mu \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to μ of the matrix $L(\alpha+\gamma, \beta+\delta)$. But $D^{-1}Z'D = U(\alpha+\gamma, \beta+\delta)$; hence by Theorem 2.3 we have

$$\begin{aligned} \mathbf{w} &:= D\mathbf{v}' \\ &= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 2^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2^{n-1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} 2^{n-2}(b_1+d_1) + 2^{n-3}(b_2+d_2)\mu + \cdots + (b_{n-1}+d_{n-1})\mu^{n-2} + \mu^{n-1} \\ \vdots \\ 2(b_{n-2}+d_{n-2}) + (b_{n-1}+d_{n-1})\mu + \mu^2 \\ (b_{n-1}+d_{n-1}) + \mu \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1[2^{n-2}(b_1+d_1) + 2^{n-3}(b_2+d_2)\mu + \cdots + (b_{n-1}+d_{n-1})\mu^{n-2} + \mu^{n-1}] \\ \vdots \\ 2^{n-3}[2(b_{n-2}+d_{n-2}) + (b_{n-1}+d_{n-1})\mu + \mu^2] \\ 2^{n-2}[(b_{n-1}+d_{n-1}) + \mu] \\ 2^{n-1} \end{bmatrix} \end{aligned}$$

is an eigenvector corresponding to μ of the matrix $U(\alpha, \beta) + U(\gamma, \delta)$. It is easy to see that the last component in the vector $\mathbf{w}$ cannot be zero, which proves the assertion. □

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References

[1] J.C. Butcher, W.M. Wright, Applications of doubly companion matrices, Applied Numerical Mathematics 56 (2006) 358–373.

[2] W. Wanicharpichat, Nonderogatory of Sum and Product of Doubly Companion Matrices, Thai J. Math. 9 (2) (2011) 337–348.

[3] B. Malešević, D. Todorić, I. Jovović, S. Telebaković, Formulae of partial reduction for linear systems of first order operator equations, Applied Mathematics Letters 23 (2010) 1367–1371.

[4] S. Roman, *Advanced Linear Algebra*, Third Edition, Springer, New York, 2008.

[5] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1996.

[6] L. Brand, The companion matrix and its properties, The American Mathematical Monthly 71 (6) (1964) 629–634.

[7] S. Moritsugu, K. Kuriyama, A linear algebra method for solving systems of algebraic equations, J. Jap. Soc. Symb. Alg. Comp. (J. JSSAC) 7 (2000) 2–22.

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