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On the Spectrum of the Option Price of Stock Markets from the Black-scholes Equation

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Abstract : It is well known that the option price of stock can be obtained from the Black-scholes equation. Such option price is the solution of the Black-scholes equation. In this paper we studied the spectrum which contains such option price and also found the interesting properties of the kernel of such option price which is related to the spectrum. However, the results of this paper may not be useful in the real world application, but at least it may create the new knowledges in Financial Mathematics.

Keywords : option price; Black-scholes equation; stock markets.
2010 Mathematics Subject Classification : 62P05.

1 Introduction

In financial mathematics, the well known equation named the Black-scholes equation plays an important role in solving the option price of stock.

Such Black-scholes equation is given by

$$\frac{\partial u(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 u(s,t)}{\partial s^2} + rs\frac{\partial u(s,t)}{\partial s} - ru(s,t) = 0 \tag{1.1}$$

with the terminal condition

$$u(s,t) = (s_T - p)^+ \tag{1.2}$$

for $0 \le t \le T$ where $u(s,t)$ is the option price at time t, r is the interest rate, s is the price of stock at time t, s_T is the price of stock at the expiration time T, σ is the volatility of stock and p is the strike price.

They obtain the solution $u(s,t)$ of (1.1) that satisfies (1.2) of the from

$$u(s,t) = s\Phi\Big(\frac{\ln(\frac{s}{p}) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big) - pe^{-r(T-t)}\Phi\Big(\frac{\ln(\frac{s}{p}) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big) \tag{1.3}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-y^2}{2}}dy$$

see [1, p.91] the equation (1.3) is called Black-scholes formula.

In this paper, we solve the solution of (1.1) satisfies (1.2) in the other form which is different from (1.3). We let $R = \ln s$ and $\tau = T - t$ write $u(s,t) = V(R,\tau)$, by changing s to R, then (1.1) is transformed to

$$\frac{\partial V(R,\tau)}{\partial\tau} - \frac{1}{2}\sigma^2\frac{\partial^2 V(R,\tau)}{\partial\tau^2} - (r - \frac{\sigma^2}{2})\frac{\partial V(R,\tau)}{\partial\tau} + rV(R,\tau) = 0. \tag{1.4}$$

Now, for $\tau = 0$ we have $t = T$. Thus (1.2) corresponds to the initial condition $V(R,0) = (s-p)^+ = (e^R - p)^+$. Let

$$V(R,0) = (e^R - p)^+ = f(R) \tag{1.5}$$

where f is the function of R

By taking the Fourier transform with respect to R to (1.4) and (1.5), we obtain the solution $V(R,\tau)$ of (1.4) in the convolution form

$$V(R,\tau) = K(R,\tau) * f(R) \tag{1.6}$$

where $K(R,\tau)$ is the Kernel or elementary solution of (1.4) of the form

$$K(R,\tau) = \frac{1}{\sqrt{2\pi\tau\sigma^2}}e^{-r\tau}\exp\Big[\frac{-[(r-\frac{\sigma}{2})^2\tau - R]^2}{2\sigma^2\tau}\Big]. \tag{1.7}$$

Now, from (1.6) we have

$$V(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iR\omega}\widehat{V}(\omega,\tau)d\omega \tag{1.8}$$

which is the inverse Fourier transform. We define the closed interval $[a,b]$ as the spectrum of $V(R,\tau)$ that is $[a,b] = supp\widehat{V}(\omega,\tau)$ which is the support of $\widehat{V}(\omega,\tau)$, a and b are constant. By the concept of spectrum, we define

$$\widehat{V}(\omega,\tau) = \begin{cases} \widehat{f}(\omega)\exp\Big(-\big[\frac{\sigma^2}{2}\omega^2 + (r-\frac{\sigma^2}{2})(i\omega) + r\big]\tau\Big) & \text{for} \quad \omega \in [a,b] \\ 0 & \text{for} \quad \omega \notin [a,b] \end{cases} \tag{1.9}$$

and the size of $[a,b] = b-a$.

Now, from (1.8)

$$V(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iR\omega}\widehat{f}(\omega)\exp\Big(-\big[\frac{\sigma^2}{2}\omega^2 + (r-\frac{\sigma^2}{2})(i\omega)+r\big]\tau\Big)d\omega$$
$$= \frac{1}{2\pi}\int_{b}^{a} e^{iR\omega}\widehat{f}(\omega)\exp\Big(-\big[\frac{\sigma^2}{2}\omega^2 + (r-\frac{\sigma^2}{2})(i\omega)+r\big]\tau\Big)d\omega.$$

by (1.9) thus

$$|V(R,\tau)| \leq \frac{1}{2\pi}\int_{a}^{b} |e^{iR\omega}||\widehat{f}(\omega)||e^{-(r-\frac{\sigma^2}{2})(i\omega)\tau}||e^{-(\frac{\sigma^2}{2}\omega^2+r)\tau}|d\omega.$$

Let $M = \max|\widehat{f}(\omega)|$ and $N = |e^{-(\frac{\sigma^2}{2}\omega^2+r)\tau}|$ for any fixed τ. Thus

$$|V(R,\tau)| \leq \frac{1}{2\pi}NM\int_{a}^{b} d\omega = \frac{1}{2\pi}NM(b-a).$$

It follows that

$$|V(R,\tau)| \leq \frac{NM}{2\pi}(b-a) \quad \text{for fixed} \quad \tau. \tag{1.10}$$

That means the solution $V(R,\tau)$ of (1.4) is bounded by some constants times the size of the spectrum.

Now, from (1.6) we have

$$|V(R,\tau)| = |K(R,\tau) * f(R)| \leq \frac{NM}{2\pi}(b-a).$$

Since $|K(R,\tau)*f(R)| \leq ||K(R,\tau)||\ ||f(R)||$ where $||.||$ is L^2-norm and $K(R,\tau), f(R)$ are continuous. Thus choose $||K(R,\tau)||\ ||f(R)|| = \sup|K(R,\tau)*f(R)|$
thus $||K(R,\tau)||\ ||f(R)|| \leq \frac{NM}{2\pi}(b-a)$ or

$$||K(R,\tau)|| \leq \frac{NM}{2\pi||f(R)||}(b-a). \tag{1.11}$$

This gives the Kernel $K(R,\tau)$ is bounded by some constants times the size of the spectrum.

2 Preliminaries

Before reaching the main results, the following definitions and the basic concepts are needed.

Definition 2.1. *Let $f(x) \in L(\mathbb{R})$, the space of integrable function on the set of real number $\mathbb{R}$. The Fourier transform of $f(x)$ is defined by*

$$\widehat{f}(\omega) = \mathcal{F}f(x) = \int_{-\infty}^{\infty} e^{-i\omega x}f(x)dx \tag{2.1}$$

where $\omega, x \in \mathbb{R}$. Also the inverse Fourier transform is defined by

$$f(x) = \mathcal{F}^{-1}\widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\widehat{f}(\omega)d\omega \tag{2.2}$$

Lemma 2.2. *Given the Black-scholes equation*

$$\frac{\partial u(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 u(s,t)}{\partial s^2} + rs\frac{\partial u(s,t)}{\partial s} - ru(s,t) = 0 \tag{2.3}$$

with terminal condition

$$u(s,T) = (s_T - p)^+ \tag{2.4}$$

for $0 \leq t \leq T$. By changing the variable s to R with $R = \ln s$ and $\tau = T - t$ then (2.3) *is transformed to*

$$\frac{\partial V(R,\tau)}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 V(R,\tau)}{\partial \tau^2} - (r - \frac{\sigma^2}{2})\frac{\partial V(R,\tau)}{\partial \tau} + rV(R,\tau) = 0 \tag{2.5}$$

with the initial condition $V(R,0) = (s-p)^+ = (e^R - p)^+$. Let

$$V(R,0) = (e^R - p)^+ = f(R) \tag{2.6}$$

where f is the function of R.

Proof. We have $R = \ln s$ and write $u(s,t) = V(R,\tau)$ where $\tau = T - t$.
Now, $\dfrac{\partial u(s,t)}{\partial t} = \dfrac{\partial V(R,\tau)}{\partial t} = \dfrac{\partial V}{\partial \tau}\dfrac{\partial \tau}{\partial t} = -\dfrac{\partial V}{\partial \tau}$

$$\frac{\partial u(s,t)}{\partial s} = \frac{\partial V(R,\tau)}{\partial s} = \frac{\partial V(R,\tau)}{\partial R}\frac{\partial R}{\partial s} = \frac{1}{s}\frac{\partial V(R,t)}{\partial R}$$

and

$$\frac{\partial^2 u(s,t)}{\partial s^2} = \frac{1}{s^2}\frac{\partial^2 V(R,\tau)}{\partial R^2} - \frac{1}{s^2}\frac{\partial V(R,\tau)}{\partial R}.$$

Substitute into (2.3) we obtain (2.5) and (2.6) as required. □

Lemma 2.3. *The equation* (2.5) *with the initial condition* (2.6) *has a solution in convolution form*

$$V(R,\tau) = K(R,\tau) * f(R) \tag{2.7}$$

where

$$K(R,\tau) = \frac{1}{\sqrt{2\pi\tau\sigma^2}}e^{-r\tau}\exp\Big[\frac{-[(r - \frac{\sigma^2}{2})\tau + R]^2}{2\sigma^2\tau}\Big] \tag{2.8}$$

is the Kernel of (2.5).

Proof. Take the Fourier transform given by (2.1) to both sides of (2.5), we obtain

$$\frac{\partial\widehat{V}(\omega,\tau)}{\partial\tau}+\frac{1}{2}\sigma^2\omega^2\widehat{V}(\omega,\tau)+i\omega(r-\frac{1}{2}\sigma^2)\widehat{V}(\omega,\tau)+r\widehat{V}(\omega,\tau)=0. \tag{2.9}$$

Now, for any fixed ω, (2.8) is the ordinary differential equation of variable τ and we obtain

$$\widehat{V}(\omega,\tau)=C(\omega)\exp\big[-(\frac{1}{2}\sigma^2\omega^2+i\omega(r-\frac{1}{2}\sigma^2)+r)\tau\big] \tag{2.10}$$

as the solution of (2.8). We need to find $C(\omega)$. We have from (2.6), $V(R,0)=(e^R-p)^+=f(R)$ thus $\widehat{V}(\omega,0)=\widehat{f}(\omega)$. It follows that $C(\omega)=\widehat{f}(\omega)$ from (2.9). Now, by (2.2), $V(R,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ir\omega}\widehat{V}(\omega,\tau)d\omega$. Thus, by (2.9)

$$\begin{aligned}
V(R,\tau)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega R}\widehat{f}(\omega)\exp\big[-(\frac{1}{2}\sigma^2\omega^2+i\omega(r-\frac{1}{2}\sigma^2)+r)\tau\big]d\omega\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\omega R}e^{-i\omega y}\exp\big[-(\frac{1}{2}\sigma^2\omega^2+i\omega(r-\frac{1}{2}\sigma^2)+r)\tau\big]f(y)dyd\omega\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\big[-(\frac{1}{2}\sigma^2\omega^2\tau+i(\tau(r-\frac{1}{2}\sigma^2)-R+y)\omega+r\tau)\big]f(y)dyd\omega\\
&=\frac{1}{2\pi}e^{-r\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\Big[-\frac{1}{2}\sigma^2\tau\Big(\omega^2-\frac{2i((r-\frac{1}{2}\sigma^2)\tau-R+y)\omega}{\sigma^2\tau}\Big)\Big]f(y)dyd\omega\\
&=\frac{1}{2\pi}e^{-r\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\Big[-\frac{1}{2}\sigma^2\tau\Big(\omega-\frac{i((r-\frac{1}{2}\sigma^2)\tau-R+y)}{\sigma^2\tau}\Big)^2\\
&\qquad\qquad-\frac{((r-\frac{1}{2}\sigma^2)\tau-R+y)^2}{2\sigma^2\tau}\Big]f(y)dyd\omega\\
&=\frac{1}{2\pi}e^{-r\tau}\int_{-\infty}^{\infty}\exp\Big[-\frac{((r-\frac{1}{2}\sigma^2)\tau-R+y)^2}{2\sigma^2\tau}\Big]f(y)dy\times\\
&\qquad\int_{-\infty}^{\infty}\exp\Big[-\frac{1}{2}\sigma^2\tau(\omega-i(r-\frac{1}{2}\sigma^2)\tau-R+y)^2\Big]d\omega\\
&=\frac{1}{2\pi}e^{-r\tau}\int_{-\infty}^{\infty}\exp\Big[-\frac{((r-\frac{1}{2}\sigma^2)\tau-R+y)^2}{2\sigma^2\tau}\Big]f(y)dy\times\\
&\qquad\int_{-\infty}^{\infty}\exp\Big[-\frac{1}{2}\sigma^2\tau\big(\omega-\frac{i(r-\frac{1}{2}\sigma^2)\tau-R+y}{\sigma^2\tau}\big)^2\Big]d\omega.
\end{aligned}$$

Put $u=\sigma\sqrt{\frac{\tau}{2}}(\omega-\dfrac{i(r-\frac{1}{2}\sigma^2)\tau-R+y}{\sigma^2\tau})^2$

thus $du=\sigma\sqrt{\frac{\tau}{2}}d\omega$, $d\omega=\sqrt{\frac{2}{\tau}}\frac{1}{\sigma}du$

thus

$$\begin{aligned} V(R,\tau) &= \frac{1}{2\pi}e^{-r\tau}\int_{-\infty}^{\infty}\exp\Big[-\frac{((r-\frac{1}{2}\sigma^2)\tau-R+y)^2}{2\sigma^2\tau}\Big]\frac{1}{\sigma}\sqrt{\frac{2}{\tau}}\int_{-\infty}^{\infty}e^{-u^2}du \\ &= \frac{1}{2\pi\sigma}\sqrt{\frac{2}{\tau}}\sqrt{\pi}\int_{-\infty}^{\infty}\exp\Big[-\frac{((r-\frac{1}{2}\sigma^2)\tau-R+y)^2}{2\sigma^2\tau}\Big]f(y)dy \\ &= \frac{1}{\sqrt{2\pi\tau\sigma^2}}e^{-r\tau}\int_{-\infty}^{\infty}\exp\Big[-\frac{((r-\frac{1}{2}\sigma^2)\tau+R-y)^2}{2\sigma^2\tau}\Big]f(y)dy \\ &= K(R,\tau)*f(R) \end{aligned}$$

where $K(R,\tau)=\dfrac{e^{-r\tau}}{\sqrt{2\pi\tau\sigma^2}}e^{-\frac{((r-\frac{1}{2}\sigma^2)\tau+R)^2}{2\sigma^2\tau}}$ is the Kernel of (2.5)

Actually, it is the Gaussian distribution with mean $e^{-r\tau}\tau(\frac{1}{2}\sigma^2-r)$ and variance $e^{-2r\tau}\sigma^2\tau$. If $\tau=0$ then $t=T$ and it can be shown that $\lim_{\tau\to 0}K(R,\tau)=\delta(R)$ where $\delta(R)$ is the Diract-delta distribution, see [,]. Thus, from (2.7)

$$V(R,0)=\delta(R)*f(R)=f(R).$$

It follows that (2.6) holds. □

Definition 2.4. *Let Ω be closed and bounded set and is called the spectrum of function $f(x)$ if $\Omega = supp\widehat{f}(\omega)$ where $supp\widehat{f}(\omega)$ is the support of fourier transform of f.*

Let $\Omega=[a,b]$, we define the spectrum of the solution $V(R,\tau)$ as

$$[a,b]=supp\widehat{V}(\omega,\tau) \tag{2.11}$$

where

$$\widehat{V}(\omega,\tau)=\begin{cases}\widehat{f}(\omega)\exp\Big(-\big[\frac{\sigma^2}{2}\omega^2+(r-\frac{\sigma^2}{2})(i\omega)+r\big]\tau\Big) & \text{for } \omega\in[a,b]\\ 0 & \text{for } \omega\notin[a,b]\end{cases} \tag{2.12}$$

we also define the size of the spectrum $[a,b]=b-a$.

3 Main Results

Theorem 3.1. *Given the equation*

$$\frac{\partial V(R,\tau)}{\partial\tau}-\frac{1}{2}\sigma^2\frac{\partial^2 V(R,\tau)}{\partial\tau^2}-(r-\frac{\sigma^2}{2})\frac{\partial V(R,\tau)}{\partial\tau}+rV(R,\tau)=0. \tag{3.1}$$

with the initial condition

$$V(R,0)=(e^R-p)^+=f(R). \tag{3.2}$$

Then we obtain $V(R,\tau)=K(R,\tau)*f(R)$ as the solution of (3.1) and $|V(R,\tau)|\leq \dfrac{MN}{2\pi}(b-a)$ where $M=\max\widehat{f}(\omega)$, $N=|e^{-(\frac{\sigma^2}{2}\omega+r)\tau}|$ and $b-a$ is the size of spectrum given by (2.11).

Proof. By taking Fourier transform to (3.1) and (3.2) we obtain $V(R,\tau) = K(R,\tau) * f(R)$ by Lemma 2.3, now

$$\begin{aligned} V(R,\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega R}\widehat{V}(\omega,\tau)d\omega \\ &= \frac{1}{2\pi}\int_a^b e^{i\omega R}\widehat{f}(\omega)\exp\Big[-(\frac{1}{2}\sigma^2\omega^2 + i\omega(r-\frac{\sigma^2}{2})+r)\tau\Big]d\omega \end{aligned}$$

by (2.12) thus

$$\begin{aligned} |V(R,\tau)| &\leq \frac{1}{2\pi}\int_a^b |e^{i\omega R}||\exp\Big[-(\frac{1}{2}\sigma^2\omega^2 + i\omega(r-\frac{\sigma^2}{2})+r)\tau\Big]||\widehat{f}(\omega)|d\omega \\ &= \frac{1}{2\pi}\int_a^b |\widehat{f}(\omega)||\exp\Big[-(\frac{1}{2}\sigma^2\omega^2 + r)\tau\Big]|d\omega \end{aligned}$$

let $M = \max|\widehat{f}(\omega)|$ and $N = |\exp\Big[-(\frac{1}{2}\sigma^2\omega^2+r)\tau\Big]|$. Thus $|V(R,\tau)| \leq \frac{MN}{2\pi}\int_a^b d\omega = \frac{MN}{2\pi}(b-a)$. It follows that $|V(R,\tau)| \leq \frac{MN}{2\pi}(b-a)$ as required. That mean $V(R,\tau)$ is bounded by some constant times the size of spectrum.

□

Theorem 3.2. ***(The properties of Kernel*** $K(R,\tau)$***)***
The Kernel given by (2.8) *of Lemma 2.3 has the following properties*

- *(i)* $K(R,\tau)$ *satisfies equation (2.12)*
- *(ii)* $e^{r\tau}\int_{-\infty}^{\infty} K(R,\tau)dR = 1$
- *(iii)* $\lim_{\tau\to 0} K(R,\tau) = \delta(R)$ *where* $\delta(R)$ *is the Dirac-delta distribution.*
- *(iv)* $K(R,\tau)$ *is Gaussian distribution with mean* $e^{-r\tau}\tau(\frac{1}{2}\sigma^2 - r)$ *and variance* $e^{-2r\tau}\sigma^2\tau$
- *(v)* $\| K(R,\tau) \| \leq \frac{MN}{2\pi\|f\|}(b-a)$ *where* $\| \,.\, \|$ *is* L^2*-norm.*

Proof. (i) By computing directly, $K(R,\tau)$ satisfies (2.12).

(ii)

$$e^{r\tau}\int_{-\infty}^{\infty} K(R,\tau)dR = \frac{1}{\sqrt{2\pi\tau\sigma^2}}\int_{-\infty}^{\infty} e^{\frac{-((r-\frac{1}{2}\sigma^2)\tau+R)^2}{2\sigma^2\tau}}dR$$

let $u = \dfrac{(r-\frac{1}{2}\sigma^2)\tau + R}{\sqrt{2\tau}\sigma}$ then $dR = \sqrt{2\tau}\sigma du$ thus

$$\begin{aligned} e^{r\tau}\int_{-\infty}^{\infty} K(R,\tau)dR &= \frac{1}{\sqrt{2\pi\tau\sigma^2}}\sqrt{2\tau}\sigma\int_{-\infty}^{\infty} e^{-u^2}du \\ &= \frac{1}{\sqrt{2\pi\tau\sigma^2}}\sqrt{2\pi\tau\sigma^2} \\ &= 1 \end{aligned}$$

(since $\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$).

(iii)

$$\lim_{\tau\to 0} K(R,\tau) = \lim_{\tau\to 0}\Big(\frac{e^{\frac{-((r-\frac{1}{2}\sigma^2)\tau+R)^2}{2\sigma^2\tau}}}{\sqrt{2\pi\tau\sigma^2}}\Big)$$
$$= \delta(R)$$

see [1, pp. 36-37].

(iv)

$$mean = E(K(R,\tau)) = e^{-r\tau}E\Big[\frac{1}{\sqrt{2\pi\tau\sigma^2}}e^{\frac{-((r-\frac{1}{2}\sigma^2)\tau+R)^2}{2\sigma^2\tau}}\Big]$$
$$= e^{-r\tau}\tau(\frac{\sigma^2}{2} - r)$$

$$variance = V\Big[e^{-r\tau}\frac{1}{\sqrt{2\pi\tau\sigma^2}}e^{\frac{-((r-\frac{1}{2}\sigma^2)\tau+R)^2}{2\sigma^2\tau}}\Big]$$
$$= e^{-2r\tau}\sigma^2\tau.$$

(v) We have

$$V(R,\tau) = K(R,\tau) * f(R)$$
$$|V(R,\tau)| = |K(R,\tau) * f(R)| \le \frac{MN}{2\pi}(b-a)$$

by Theorem 3.1.
Now $|K(R,\tau) * f(R)| \le\| K(R,\tau) \|\| f(R) \|$ where $\| . \|$ is L^2-norm since $K(R,\tau)$ and $f(R)$ are continuous, we can define

$$\| K(R,\tau) \|\| f(R) \|= \sup|K(R,\tau) * f(R)|$$

thus $\| K(R,\tau) \|\| f(R) \|\le \frac{MN}{2\pi}(b-a)$. That means the norm of $K(R,\tau)$ is bounded by some constants times the size of spectrum.

□

4 Conclusion

The Black-scholes formula given by (1.3) which is the solution of (1.1) is used widely in the real world applications. But, when (1.1) is transformed to (1.4) and obtain (1.6) as a solution. Such solution needs more mathematical concepts than (1.3), particularly in the part of main result shows the theorem concerning the boundedness and the norms of the option price and the Kernel. Those are bounded by the size of the spectrum. Such main results may not be useful directly in the real world applications but at least it opens the door to the mathematical analysis in the area of Financial Mathematics and obtain the new knowledge in such area.

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