



Transformation Tolerance Hypergroups

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Abstract : Tolerance spaces and algebraic structures with compatible tolerances play an important role in contemporary algebra and their applications. In this contribution we are presenting transformation hyperstructures, namely semihypergroups and hypergroups acting on tolerance spaces. Some basic concepts concerning the mentioned structures are introduced and their fundamental properties are examined on suitable constructions.

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1 Introduction

One of the motivating factors in developing the theory of hyperstructures was the generalization of concepts of classical mathematical structures, namely groups. According to the opinion of specialists in this field the development of the join spaces, which form a special class of hypergroup, is a very important moment in the investigation of concrete mathematical (especially geometrical) structures.

So in this contribution we generalize in a certain sense the classical concept of action of a group on a given phase space.

Transformation groups which represent the classical and developing discipline are situated in the intersection of several parts of mathematical structures.

This idea is adopted from the functorial assignment of a commutative hypergroup to an arbitrary transformation (discrete) group. We will describe this construction in more details.

Let $G = (X, T, \pi)$ be a transformation group, (i.e. X -phase set, T -phase group, π -action (projection): $X \times T \rightarrow X$). For any pair $x, y \in X$ we define

$$x *_G y = \pi(x, T) \cup \pi(y, T) = \{\pi(x, t); t \in T\} \cup \{\pi(y, t); t \in T\}.$$

It is easy to show that $(X, *_G)$ is an extensive commutative hypergroup (definition follows).

Recall first the basic terms and definitions. A *hypergroupoid* is a pair (H, \cdot) where H is a (nonempty) set and $\cdot : H \times H \rightarrow \mathcal{P}^*(H)$ ($= \mathcal{P}(H) \setminus \{\emptyset\}$) is a binary hyperoperation on the set H . If $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in H$, (associativity),

then (H, \cdot) is called a *semihypergroup*. A semihypergroup (H, \cdot) is said to be a *hypergroup* if the following axiom $a \cdot H = H = H \cdot a$ for all $a \in H$, (the reproduction axiom), is satisfied. Here, for $A, B \subseteq H$, $A \neq \emptyset \neq B$ we define as usual $A \cdot B = \bigcup \{a \cdot b; a \in A, b \in B\}$, (see e.g. [3]).

Let $(H, *)$ and (H', \star) be hypergroupoids. Then a mapping $f: H \rightarrow H'$ is called *inclusion homomorphism* if it satisfies the condition:

$$f(x * y) \subseteq f(x) \star f(y) \quad \text{for all pairs } x, y \in H.$$

Let X be a set and τ be a tolerance relation (i.e., reflexive and symmetric binary relation), see [6]. Then the pair (X, τ) is a *tolerance space*.

An *ordered semigroup* is a triple (S, \cdot, \leq) , where (S, \cdot) is a semigroup and " \leq " is an ordering on the set S with *substitution property* on (S, \cdot) (i.e., for an arbitrary quadruple of elements $a, b, c, d \in S$ for which $a \leq b$, $c \leq d$ the relation $a \cdot c \leq b \cdot d$ holds). An ordered monoid (i.e., a semigroup with unit element) such that any element has its inverse is called an *ordered group*.

It is to be noticed that the substitution property is equivalent to a simpler condition: for an arbitrary triple of elements $a, b, c \in S$ such that $a \leq b$ the relations $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$ hold.

2 Main Results

Below we will need the following result in which we denote for m from an ordered set H : $[m]_{\leq} = \{x \in H; m \leq x\}$ (principal upper end determined by m).

Lemma 2.1 *Let (H, \cdot, \leq) be an ordered semigroup and define a binary hyperoperation " $*$ " on H in this way:*

$$a * b = [a \cdot b]_{\leq} \quad \text{for any } a, b \in H.$$

*Then $(H, *)$ is a semihypergroup. It is commutative if and only if (H, \cdot) is commutative. If, moreover, (H, \cdot) is a group, then $(H, *)$ is a hypergroup.*

For the proof see [8, p. 146–147].

Definition 2.2 Let (X, τ) be a tolerance space (so called phase tolerance space), (G, \bullet) be a semihypergroup (so called phase semihypergroup) and $\pi: X \times G \rightarrow X$ a mapping such that

- (i) $\pi(\pi(x, t), s) \in \pi(x, t \bullet s)$, where $\pi(x, t \bullet s) = \{\pi(x, u); u \in t \bullet s\}$ for each $x \in X$, $s, t \in G$;
- (ii) if $x, y \in X$ are such that $x \tau y$, then $\pi(x, g) \tau \pi(y, g)$ holds for any $g \in G$.

Then (X, G, π) is a *transformation semihypergroup with phase tolerance space*.

If, moreover, the pair (G, \bullet) is a hypergroup (phase hypergroup), then the triple (X, G, π) is a *transformation hypergroup with phase tolerance space*.

Definition 2.3 Let (X_i, τ_i) be a tolerance space, (G_i, \bullet_i) be a hypergroup and a triple $\mathcal{T}_i = (X_i, G_i, \pi_i)$, where $i = 1, 2$ be a transformation hypergroup with phase tolerance space.

A pair of mappings (h_X, h_G) will be called *homomorphism of the transformation hypergroup \mathcal{T}_1 into the transformation hypergroup \mathcal{T}_2* if:

- (i) the mapping $h_X: X_1 \rightarrow X_2$ is the homomorphism of tolerance spaces (i.e., for any $x, y \in X_1$ such that $x \tau_1 y$ we have $h_X(x) \tau_2 h_X(y)$);
- (ii) the mapping $h_G: G_1 \rightarrow G_2$ is the inclusion homomorphism (i.e., $h_G(u \bullet_1 v) \subseteq h_G(u) \bullet_2 h_G(v)$);
- (iii) $h_X(\pi_1(x, u)) = \pi_2(h_X(x), h_G(u))$ for any $x \in X_1, u \in G_1$.

The class of all transformation hypergroups with all above defined homomorphism forms a category of transformation hypergroups with tolerance spaces.

If all phase hypergroups are identical, we speak about a category over the given phase hypergroup.

Example 2.4 Let $J \subset \mathbb{R}$ be an open interval and denote $C^\infty(J)$ the ring of all infinitely differentiable functions on J . Let us consider the set $\mathbb{L}\mathbb{A}_n(J)$, $n \in \mathbb{N}$, of all linear differential operators of the n^{th} order in the form

$$D(p_0, \dots, p_{n-1}) = \frac{d^n}{dx^n} + \sum_{k=0}^{n-1} p_k(x) \frac{d^k}{dx^k},$$

where $p_k \in C^\infty(J)$, $k = 0, 1, \dots, n-1$; $D(p_0, \dots, p_{n-1}): C^\infty(J) \rightarrow C^\infty(J)$, thus

$$D(p_0, \dots, p_{n-1})(f) = f^{(n)}(x) + p_{n-1}(x)f^{(n-1)}(x) + \dots + p_0(x)f(x), \quad f \in C^\infty(J).$$

Let δ_{ij} stands for the Kronecker symbol δ . For any $m \in \{0, 1, \dots, n-1\}$ we denote by

$$\mathbb{L}\mathbb{A}_n(J)_m = \{D(p_0, \dots, p_{n-1}); p_k \in C^\infty(J), p_m > 0\}.$$

Shortly we put $\mathbf{p} = (p_0(x), \dots, p_{n-1}(x))$, $x \in J$.

On the set $\mathbb{L}\mathbb{A}_n(J)_m$ we define a binary operation “ \circ_m ” and a binary relation “ \leq_m ” in this way:

For an arbitrary pair of operators $D(\mathbf{p}), D(\mathbf{q}) \in \mathbb{L}\mathbb{A}_n(J)_m$ we put

$$D(\mathbf{p}) \circ_m D(\mathbf{q}) = D(\mathbf{u}), \tag{2.1}$$

where $u_k(x) = p_m(x)q_k(x) + (1 - \delta_{km})p_k(x)$, $x \in J, 0 \leq k \leq n-1$ and

$$D(\mathbf{p}) \leq_m D(\mathbf{q}) \tag{2.2}$$

whenever $p_k(x) \leq q_k(x)$, $k \neq m$, $k \in \{0, 1, \dots, n-1\}$, $p_m(x) = q_m(x)$, $x \in J$.

It is easy to verify that $(\mathbb{L}\mathbb{A}_n(J)_m, \circ_m, \leq_m)$ is an ordered non-commutative group with the neutral element $D(\boldsymbol{\omega})$, where $\omega_k(x) = \delta_{km}$. An inverse to any $D(\mathbf{q})$ is

$$D^{-1}(\mathbf{q}) = \left(-\frac{q_0}{q_m}, \dots, \frac{1}{q_m}, \dots, -\frac{q_{n-1}}{q_m} \right).$$

Let $(\mathbb{Z}, +, \leq)$ be an additive group of all integers with an usual ordering “ \leq ”. Then by Lemma 2.1 the structure $G = (\mathbb{Z}, \star)$, where $\star: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathcal{P}^*(\mathbb{Z})$

$$k \star l = [k + l]_{\leq} = \{u \in \mathbb{Z}; k + l \leq u\},$$

is a hypergroup.

For a fixed $D(\mathbf{q}) \in \mathbb{L}\mathbb{A}_n(J)_m$ we define an action

$$\pi_q: \mathbb{L}\mathbb{A}_n(J)_m \times \mathbb{Z} \rightarrow \mathbb{L}\mathbb{A}_n(J)_m$$

as follows:

$$\pi_q(D(\mathbf{p}), k) = D^k(\mathbf{q}) \circ_m D(\mathbf{p}),$$

where

$$\begin{aligned} D^0(\mathbf{q}) &= D(\boldsymbol{\omega}), \\ D^k(\mathbf{q}) &= \underbrace{D(\mathbf{q}) \circ_m D(\mathbf{q}) \circ_m \cdots \circ_m D(\mathbf{q})}_{k\text{-times}} \quad \text{for } k > 0, \\ D^k(\mathbf{q}) &= (D^{-1}(\mathbf{q}))^{|k|} \quad \text{for } k < 0. \end{aligned}$$

Evidently $D^k(\mathbf{q}) \circ_m D^l(\mathbf{q}) = D^{k+l}(\mathbf{q})$, $k, l \in \mathbb{Z}$. Thus

$$\begin{aligned} \pi_q(\pi_q(D(\mathbf{p}), k), l) &= \pi_q(D^k(\mathbf{q}) \circ_m D(\mathbf{p}), l) \\ &= D^l(\mathbf{q}) \circ_m D^k(\mathbf{q}) \circ_m D(\mathbf{p}) \\ &= D^{k+l}(\mathbf{q}) \circ_m D(\mathbf{p}) = \pi_q(D(\mathbf{p}), k + l). \end{aligned}$$

On the other hand

$$\pi_q(D(\mathbf{p}), k \star l) = \pi_q(D(\mathbf{p}), [k + l]_{\leq}) \text{ so } \pi_q(D(\mathbf{p}), k + l) \in \pi_q(D(\mathbf{p}), k \star l).$$

Therefore $(\mathbb{L}\mathbb{A}_n(J)_m, G, \pi_q)$ is a transformation hypergroup with discrete phase tolerance (i.e., $D(\mathbf{p}) \tau D(\mathbf{q})$ if and only if $D(\mathbf{p}) = D(\mathbf{q})$).

Remark 2.5 Analogously it is possible to proceed in the case of any dynamical system (X, G, π) , where $G = (\mathbb{Z}, +)$ or $(\mathbb{R}, +)$.

Example 2.6 Similarly as in Example 2.4 let $J \subset \mathbb{R}$ be an open interval. Without loss of generality we will suppose e.g. $J = (0, \infty)$. Let $C^n(J)$ be the ring of all functions having continuous derivatives up to the order n , $n \in \mathbb{N}_0$. Let the set $\mathbb{L}\mathbb{A}_n(J)_m$ has an analogous meaning as in the mentioned Example 2.4, i.e., if $D(\mathbf{p}) \in \mathbb{L}\mathbb{A}_n(J)_m$, then

$$D(\mathbf{p}) = \frac{d^n}{dx^n} + \sum_{k=0}^{n-1} p_k(x) \frac{d^k}{dx^k},$$

where $p_k \in C^0(J)$. Let us denote $M(J) \subset C^0(J)$ the subset of all functions bounded at infinity, i.e., $p \in M(J)$ if and only if $\limsup_{x \rightarrow \infty} |p(x)| < \infty$.

We will consider a set of linear differential operators $\mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m \subset \mathbb{L}\mathbb{A}_n(J)_m$, where $p_k \in M(J)$, $k \in 0, 1, \dots, n-1$, $\liminf_{x \rightarrow \infty} p_m(x) > 0$. With respect to the form of an inverse element

$$D^{-1}(\mathbf{q}) = \left(-\frac{q_0}{q_m}, \dots, \frac{1}{q_m}, \dots, -\frac{q_{n-1}}{q_m} \right),$$

it is easy to see that $(\mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m, \circ_m, \leq_m)$ is a subgroup of the ordered group $(\mathbb{L}\mathbb{A}_n(J)_m, \circ_m, \leq_m)$, where the operation " \circ_m " and the ordering " \leq_m " are given by (2.1) and (2.2), respectively.

Let us define a binary hyperoperation

$$\bullet: \mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m \times \mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m \rightarrow \mathcal{P}^*(\mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m)$$

in this way: For an arbitrary pair $D(\mathbf{p}) \in \mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m$, $D(\mathbf{q}) \in \mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m$, we set

$$D(\mathbf{p}) \bullet D(\mathbf{q}) = [D(\mathbf{q}) \circ_m D(\mathbf{p})]_{\leq}.$$

Then by Lemma 2.1 the pair $(\mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m, \bullet)$ is a non-commutative hypergroup.

Let $P(J) = \mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m \times C^n(J)$ be a phase space. We will define an action $\pi: P(J) \times \mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m \rightarrow P(J)$ by

$$\pi((D(\mathbf{u}), f), D(\mathbf{p})) = (D(\mathbf{u}) \circ_m D(\mathbf{p}), f).$$

We will show that the condition 1 from Definition 2.2 is satisfied :

Suppose $f \in C^n(J)$ be an arbitrary function and $D(\mathbf{p}), D(\mathbf{q}) \in \mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m$ be an arbitrary linear operators. Then we have

$$\begin{aligned} \pi\left(\pi((D(\mathbf{u}), f), D(\mathbf{p})), D(\mathbf{q})\right) &= \pi\left((D(\mathbf{u}) \circ_m D(\mathbf{p}), f), D(\mathbf{q})\right) \\ &= (D(\mathbf{u}) \circ_m D(\mathbf{p}) \circ_m D(\mathbf{q}), f) \in P(J). \end{aligned}$$

On the other hand

$$\begin{aligned} D(\mathbf{p}) \bullet D(\mathbf{q}) &= [D(\mathbf{q}) \circ_m D(\mathbf{p})]_{\leq} = \{D(\mathbf{r}); D(\mathbf{q}) \circ_m D(\mathbf{p}) \leq_m D(\mathbf{r})\}, \\ \pi((D(\mathbf{u}), f), D(\mathbf{p}) \bullet D(\mathbf{q})) &= \{\pi((D(\mathbf{u}), f), D(\mathbf{r})); D(\mathbf{p}) \circ_m D(\mathbf{q}) \leq_m D(\mathbf{r})\} \\ &= \{(D(\mathbf{u}) \circ_m D(\mathbf{r}), f); D(\mathbf{p}) \circ_m D(\mathbf{q}) \leq_m D(\mathbf{r})\}, \end{aligned}$$

hence

$$\pi\left(\pi((D(\mathbf{u}), f), D(\mathbf{p})), D(\mathbf{q})\right) \in \pi((D(\mathbf{u}), f), D(\mathbf{p}) \bullet D(\mathbf{q})).$$

Let us define a tolerance τ_1 on $\mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m$ as follows :

For $D(\mathbf{p}), D(\mathbf{q}) \in \mathbb{B}\mathbb{L}\mathbb{A}_n(J)_m$ we set

$$D(\mathbf{p}) \tau_1 D(\mathbf{q}) \text{ if and only if } \lim_{x \rightarrow \infty} |p_k(x) - q_k(x)| = 0$$

for $k = 0, 1, \dots, n - 1$.

Let τ_2 be an arbitrary tolerance on $C^n(J)$. For $(D(\mathbf{p}), f), (D(\mathbf{q}), g) \in P(J)$ define

$$(D(\mathbf{p}), f) \tau (D(\mathbf{q}), g) \text{ if and only if } D(\mathbf{p}) \tau_1 D(\mathbf{q}) \text{ and } f \tau_2 g.$$

Evidently, τ is a tolerance on $P(J)$.

If $D(\mathbf{p}) \tau_1 D(\mathbf{q})$, then for $D(\mathbf{r}) \in \mathbb{B}L\mathbb{A}_n(J)_m$ the following relation holds: $D(\mathbf{p}) \circ_m D(\mathbf{r}) \tau_1 D(\mathbf{q}) \circ_m D(\mathbf{r})$. Indeed, suppose first that $k = m$. Then

$$\begin{aligned} & \lim_{x \rightarrow \infty} |p_m(x)r_k(x) + (1 - \delta_{km})p_k(x) - q_m(x)r_k(x) - (1 - \delta_{km})q_k(x)| \\ &= \lim_{x \rightarrow \infty} |p_m(x)r_k(x) - q_m(x)r_k(x)| \\ &= \lim_{x \rightarrow \infty} |p_m(x) - q_m(x)||r_k(x)| = 0, \end{aligned}$$

since there is a constant $\alpha \in \mathbb{R}^+$ such that $|r_k(x)| \leq \alpha$ for large $x \in J$. Now suppose $k \neq m$. Then similarly

$$\begin{aligned} & \lim_{x \rightarrow \infty} |p_m(x)r_k(x) + (1 - \delta_{km})p_k(x) - q_m(x)r_k(x) - (1 - \delta_{km})q_k(x)| \\ &= \lim_{x \rightarrow \infty} |p_m(x)r_k(x) + p_k(x) - q_m(x)r_k(x) - q_k(x)| \\ &\leq \lim_{x \rightarrow \infty} |p_m(x) - q_m(x)||r_k(x)| + \lim_{x \rightarrow \infty} |p_k(x) - q_k(x)| = 0. \end{aligned}$$

Thus, if $(D(\mathbf{p}), f) \tau (D(\mathbf{q}), g)$, then $(D(\mathbf{p}) \circ_m D(\mathbf{r}), f) \tau (D(\mathbf{q}) \circ_m D(\mathbf{r}), g)$ and the condition 2 from Definition (2.2) is satisfied, which shows that $(P(J), \mathbb{B}L\mathbb{A}_n(J)_m, \tau)$ is a transformation hypergroup with tolerance.

In the following example we will use the relation of proximity and for our purposes the most appropriate version is that one by E. Čech [2].

Definition 2.7 A binary relation \mathbf{p} on the family of all subsets of the set H is called a *proximity* on the set H if \mathbf{p} satisfies the following conditions:

- (i) $\emptyset \text{ non } \mathbf{p} H$
- (ii) The relation \mathbf{p} is symmetric, i.e., $A, B \subset H$, $A \mathbf{p} B$ implies $B \mathbf{p} A$.
- (iii) For any pair of subset $A, B \subset H$, $A \cap B \neq \emptyset$ implies $A \mathbf{p} B$.
- (iv) If A, B, C are subsets of H , then $(A \cup B) \mathbf{p} C$ if and only if either $A \mathbf{p} C$ or $B \mathbf{p} C$.

A *proximity space* is a pair (H, \mathbf{p}_H) consisting of a set H and a proximity \mathbf{p}_H on the set H .

Example 2.8 Let $(P, \mathcal{T}_1, \mathbf{p})$ be a topological space with proximity relation $\mathbf{p} \subset P(P) \times P(P)$. Let us denote by S_P (as sheaf) the set of all \mathcal{T}_1 -continuous functions

into another topological space, say (E, \mathcal{T}_2) , and for all $\emptyset \neq U \subset P$, U is \mathcal{T}_1 -open, put $S_U = \{f|U: U \rightarrow E; f \in S_P\}$. Now denote $X = \{S_U; \emptyset \neq U \subset P, U \text{ is } \mathcal{T}_1\text{-open}\}$ and for $S_U, S_V \in X$ we set

$$S_U \tau_{\mathbf{p}} S_V, \quad \text{whenever } U \mathbf{p} V.$$

Then evidently $(X, \tau_{\mathbf{p}})$ is a tolerance space. Further, let $G = \mathcal{T}_1 \setminus \{\emptyset\}$ and for $U, V \in G$ define $U \star V = \mathcal{P}^*(U \cup V) \cap \mathcal{T}_1$. The pair (G, \star) is a commutative hypergroup. In fact, it is possible to check that for $U, V, W \in G$ there is $(U \star V) \star W = \mathcal{P}^*(U \cup V \cup W) \cap \mathcal{T}_1 = U \star (V \star W)$, which gives associativity law. Further, for any $U \in G$ there is $U \star G \supset U \star P = \mathcal{P}^*(U \cup P) \cap \mathcal{T}_1 = G$. As trivially $U \star G \subset G$ we obtain reproduction axiom.

Now, define a mapping $\pi: X \times G \rightarrow X$ by $\pi(S_U, V) = S_{U \cup V}$. We will verify that the conditions of Definition 2.2 hold.

(i) For all $U, V, W \in G$ there is

$$\begin{aligned} \pi(\pi(S_U, V), W) &= \pi(S_{U \cup V}, W) = S_{U \cup V \cup W}, \\ \pi(S_U, V \star W) &= \{\pi(S_U, T); \emptyset \neq T \subset V \cup W, T \text{ is } \mathcal{T}_1\text{-open}\} \\ &= \{S_{U \cup T}; \emptyset \neq T \subset V \cup W, T \text{ is } \mathcal{T}_1\text{-open}\}. \end{aligned}$$

Since $S_{U \cup V \cup W} \in \{S_{U \cup T}; \emptyset \neq T \subset V \cup W, T \text{ is } \mathcal{T}_1\text{-open}\}$ for $T = V \cup W$, we have $\pi(\pi(S_U, V), W) \in \pi(S_U, V \star W)$.

(ii) Suppose $S_U, S_V \in X$, $S_U \tau_{\mathbf{p}} S_V$, i.e., $U \mathbf{p} V$. Then for an arbitrary $W \in G$ we get $\pi(S_U, W) = S_{U \cup W}$, $\pi(S_V, W) = S_{V \cup W}$, $U \subset U \cup W$, $V \subset V \cup W$, thus $(U \cup W) \mathbf{p} (V \cup W)$, so $S_{U \cup W} \tau_{\mathbf{p}} S_{V \cup W}$, consequently $\pi(S_U, W) \tau_{\mathbf{p}} \pi(S_V, W)$.

Therefore the triple (X, G, π) is a transformation tolerance hypergroup.

Example 2.9 Let $\Omega \subseteq \mathbb{R}^n$ be a domain, i.e., an open connected subset of the n -dimensional Euclidean space of n -tuples of reals. As usual, $C^1(\Omega)$ stands for the ring of all continuous functions of n -variables $u: \Omega \rightarrow \mathbb{R}$ with continuous first partial derivatives $\frac{\partial u}{\partial x_k}$, $k = 1, 2, \dots, n$. We will consider linear first-order partial differential operators of the form

$$D(a_1, \dots, a_n, p) = \sum_{k=1}^n a_k(x_1, \dots, x_n) \frac{\partial}{\partial x_k} + p(x_1, \dots, x_n) \text{Id},$$

where $a_k \in C^1(\Omega)$ for $k = 1, 2, \dots, n$ and $p \in C^1(\Omega)$, $p(x_1, \dots, x_n) > 0$ for any point $[x_1, \dots, x_n] \in \Omega$. Denote by $\mathbb{L}^1\mathbb{D}(\Omega)$ the set of all such operators which are associated to linear first-order homogeneous partial differential equations

$$\sum_{k=1}^n a_k(x_1, \dots, x_n) \frac{\partial u(x_1, \dots, x_n)}{\partial x_k} + p(x_1, \dots, x_n) u(x_1, \dots, x_n) = 0,$$

with $a_k, p \in C^1(\Omega)$. The above operator $D(a_1, \dots, a_n, p)$ can be also denoted in the following vector form: $D(\mathbf{a}, p)$, where $\mathbf{a} = (a_1, \dots, a_n)$. Define a binary operation

“.” and a binary relation “ \leq ” on the set $\mathbb{L}^1\mathbb{D}(\Omega)$ by the rule

$$D(\mathbf{a}, p) \cdot D(\mathbf{b}, q) = D(c_1, \dots, c_n, pq),$$

where

$$c_k(x_1, \dots, x_n) = a_k(x_1, \dots, x_n) + p(x_1, \dots, x_n)b_k(x_1, \dots, x_n)$$

and

$$D(\mathbf{a}, p) \leq D(\mathbf{b}, q) \text{ whenever } p \equiv q \text{ and } a_k(x_1, \dots, x_n) \leq b_k(x_1, \dots, x_n)$$

for any $[x_1, \dots, x_n] \in \Omega$ and $k = 1, 2, \dots, n$.

The pair $(\mathbb{L}^1\mathbb{D}(\Omega), \cdot)$ is a non-commutative group, see [12]—compare also Example 2.2.

Evidently, the relation “ \leq ” on $\mathbb{L}^1\mathbb{D}(\Omega)$ is reflexive, antisymmetric and transitive, hence $(\mathbb{L}^1\mathbb{D}(\Omega), \cdot, \leq)$ is an ordered group.

Let us define a binary hyperoperation “ \star ” on $\mathbb{L}^1\mathbb{D}(\Omega)$ by

$$\begin{aligned} D(\mathbf{a}, p) \star D(\mathbf{b}, q) &= \{D(\mathbf{c}, s); D(\mathbf{a}, p) \cdot D(\mathbf{b}, q) \leq D(\mathbf{c}, s), c_k, s \in C^1(\Omega)\} \\ &= \{D(\mathbf{c}, pq); a_k + pb_k \leq c_k, c_k \in C^1(\Omega)\}, \end{aligned}$$

where $k = 1, 2, \dots, n$.

Let $M \subseteq \Omega$ be a finite subset. Denote

$$\mathbb{L}_M^1\mathbb{D}(\Omega) = \{D(\mathbf{a}, p) \in \mathbb{L}^1\mathbb{D}(\Omega); \text{grad } p|_\xi = 0 \text{ for any } \xi \in M\}.$$

Evidently $(\mathbb{L}_M^1\mathbb{D}(\Omega), \cdot)$ is a subgroup of the group $(\mathbb{L}^1\mathbb{D}(\Omega), \cdot)$. We define a binary relation R_M on the set of operators $\mathbb{L}_M^1\mathbb{D}(\Omega)$ by the condition

$$D(\mathbf{a}, p) R_M D(\mathbf{b}, q) \text{ whenever } p = q \text{ and } \text{grad } a_k|_\xi = \text{grad } b_k|_\xi$$

for any $\xi \in M$ and $k = 1, 2, \dots, n$. Clearly, R_M is an equivalence relation (even a congruence, see [12]) on the set $\mathbb{L}_M^1\mathbb{D}(\Omega)$.

Now, set for any pair of subsets $A, B \subseteq \mathbb{L}_M^1\mathbb{D}(\Omega)$ that

$$A \mathbf{p}_{R_M} B, \text{ whenever } D(\mathbf{a}, p) R_M D(\mathbf{b}, q)$$

for some pair $[D(\mathbf{a}, p), D(\mathbf{b}, q)] \in A \times B$. Evidently, \mathbf{p}_{R_M} is reflexive and symmetric, thus it is a tolerance (it is a proximity as well, see [12]).

Denote $T = \mathcal{P}^*(\mathbb{L}_M^1\mathbb{D}(\Omega))$. Then (T, \mathbf{p}_{R_M}) is a tolerance space. (G, \star) , where $G = \mathbb{L}_M^1\mathbb{D}(\Omega)$, be an acting hypergroup and let us define an action $\pi: T \times G \rightarrow T$ as follows:

$$\begin{aligned} \pi(U, D(a_1, \dots, a_n, p)) &= \{D(b_1, \dots, b_n, q) \cdot D(a_1, \dots, a_n, p); D(b_1, \dots, b_n, q) \in U\} \\ &= \{D(\mathbf{b}, q) \cdot D(\mathbf{a}, p); D(\mathbf{b}, q) \in U\}. \end{aligned}$$

We will verify that the conditions of Definition 2.2 are fulfilled.

(i) For any $U \in T$ and $D(\mathbf{a}, p), D(\mathbf{b}, q) \in G$ there is

$$\begin{aligned}\pi(\pi(U, D(\mathbf{a}, p)), D(\mathbf{b}, q)) &= \pi(U \cdot D(\mathbf{a}, p), D(\mathbf{b}, q)) = U \cdot D(\mathbf{a}, p) \cdot D(\mathbf{b}, q) \\ &= \{D(\mathbf{c}, r) \cdot D(\mathbf{a}, p) \cdot D(\mathbf{b}, q); D(\mathbf{c}, r) \in U\}, \\ \pi(U, D(\mathbf{a}, p) \star D(\mathbf{b}, q)) &= \{U \cdot D(\mathbf{s}, t); D(\mathbf{a}, p) \cdot D(\mathbf{b}, q) \leq D(\mathbf{s}, t)\}.\end{aligned}$$

Evidently

$$U \cdot D(\mathbf{a}, p) \cdot D(\mathbf{b}, q) \in \{U \cdot D(\mathbf{s}, t); D(\mathbf{a}, p) \cdot D(\mathbf{b}, q) \leq D(\mathbf{s}, t)\}$$

i.e.,

$$\pi(\pi(U, D(\mathbf{a}, p)), D(\mathbf{b}, q)) \in \pi(U, D(\mathbf{a}, p) \star D(\mathbf{b}, q)).$$

(ii) For $U, V \in T$, $U \mathbf{p}_{R_M} V$ there exist $D(\mathbf{a}, p) \in U$ and $D(\mathbf{b}, q) \in V$ such that $D(\mathbf{a}, p) R_M D(\mathbf{b}, q)$. For an arbitrary $D(\mathbf{c}, r) \in G$ we have

$$\begin{aligned}D(\mathbf{a}, p) \cdot D(\mathbf{c}, r) R_M D(\mathbf{b}, q) \cdot D(\mathbf{c}, r), \\ D(\mathbf{a}, p) \cdot D(\mathbf{c}, r) \in \pi(U, D(\mathbf{c}, r)), \\ D(\mathbf{b}, q) \cdot D(\mathbf{c}, r) \in \pi(V, D(\mathbf{c}, r)),\end{aligned}$$

therefore $\pi(U, D(\mathbf{c}, r)) \mathbf{p}_{R_M} \pi(V, D(\mathbf{c}, r))$.

We have proved that the (T, G, π) is a transformation hypergroup with phase tolerance space.

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