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# Reduced Differential Transform Method and Its Application on Kawahara Equations

**Reza Abazari**†,1 **and Babak Soltanalizadeh**‡

†Department of Mathematics, Ardabil Branch
Islamic Azad University, Ardabil, Iran
e-mail : abazari-r@uma.ac.ir; abazari.r@gmail.com

‡Department of Mathematics, Sarab Branch
Islamic Azad University, Sarab, Iran
e-mail : babak.soltanalizadeh@gmail.com

**Abstract :** In this paper, the reduced form of differential transform method (called reduced DTM), is employed to approximate the solutions of Kawahara and modified Kawahara equations. These equations, proposed first by Kawahara [T. Kawahara, Oscillatory solitary waves in dispersive media, J. Phys. Soc. Jpn. 33 (1972) 260–264.] in 1972, occurs in the theory of shallow water waves and plays an important role in the modeling of many physical phenomena such as plasma waves, magneto–acoustic wave. In the last few years, considerable efforts have been expended in formulating accurate and efficient methods to solve these equations. In this paper, we first present the two–dimensional reduced DTM then employ to approximate solutions of the Kawahara and modified Kawahara equations. This method provides remarkable accuracy for the approximate solutions when compared to the exact solutions, especially in large scale domain. Numerical results demonstrate that the methods provide efficient approaches to solving these equations.

**Keywords :** Kawahara equation; modified Kawahara equation;
reduced differential transform method; closed form solutions.

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[1]Corresponding author.

# 1 Introduction

Nonlinear partial differential equations (PDEs) and their modified forms arise in a large number of mathematical and engineering problems. The world around us is inherently nonlinear. Nonlinear problems are more difficult to solve than linear ones. When studying these nonlinear phenomena, which particularly plays a major role in many fields of physics (such as fluid mechanics, solid state physics, plasma physics, etc.), one can encounter with an equation of the form:

$$\frac{\partial u}{\partial t} + \alpha u^m \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} - \gamma \frac{\partial^5 u}{\partial x^5} = 0, \tag{1.1}$$

where $\alpha, \beta$ and $\gamma$ are nonzero positive arbitrary constants and $m = 1, 2, 3, ...$, with $u, u_x, u_{3x}, u_{5x} \to 0$, when $|x| \to \infty$. When $m = 1$ and $m = 2$, Eq. (1.1) is called Kawahara and modified Kawahara equations, respectively. Kawahara equation, proposed first by Kawahara [1] in 1972, occurs in the theory of shallow water waves and plays an important role in the modeling of many physical phenomena such as plasma waves, magneto–acoustic waves, see [2–4] and the references therein.

In the Eq. (1.1), the second term is convective part and the third term is dispersive part. Karpman and Vanden–Broeck showed numerically that the fifth order term in Eq. (1.1) is of critical importance for the soliton stability [5]. If the coefficient of the term having third–order derivative is dominant over that of the fifth order, then a monotone solitary wave solution is found. If the fifth–order derivative is dominating over the third one, oscillatory structure of the solitary waves forms, which are called as Kawahara solitons. The existence and uniqueness of solutions are obtained by Shuangping and Shuangbin [6]. In the literature, this equation is also referred as a fifth–order KdV equation [7]. The modified Kawahara equation also has wide applications in physics such as plasma waves, capillary–gravity water waves, water waves with surface tension, shallow water waves and so on [1–4]. This equation is also called the singularly perturbed KdV equation [7].

There are some valuable efforts that focus on finding the analytical and numerical methods for solving the Kawahara and modified Kawahara equations. These analytical and numerical methods include the $(\frac{G'}{G})$–expansion method [8], tanh–function method [9], extended tanh-function method [10], sine–cosine method [11], Jacobi elliptic function method [12], direct algebraic method [13], Bäcklund transformation [14], Adomian decomposition method [15], He's variational method [16], Homotopy analysis method [17], homotopy perturbation method [18] and Crank–Nicolson–Differential quadrature algorithm [19].

On the other hand, in recent years, the differential transform method (DTM) has been developed for solving ordinary and partial differential equations. The DTM was first introduced by Zhou in a study about electrical circuits [20]. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylors series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders.

With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. With this technique, the given partial differential equation and related initial conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful for obtaining exact and approximate solutions of linear and nonlinear ordinary and partial differential equations. There is no need for linearization or perturbations, large computational work and round–off errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations.

DTM has been successfully applied to solve system of differential equations [21], differential-algebraic equations [22], difference equations [23], differential difference equations [24], partial differential equations [25–29], partial differential equations with proportional delay [30], system of PDEs [31], some coupled PDEs [32] and in [33] the author extended DTM to solve the first and second kind of the Riccati matrix differential equations. And finally, authors of [34, 35], applied a similar method to obtain the numerical solution of PDEs with nonlocal boundary conditions.

Although, one of the advantage of the two–dimensional DTM over other methods, such as the Adomian's decomposition method (ADM), variational iteration method (VIM), homotopy perturbation method (HPM) and homotopy analysis method (HAM) are that the two–dimensional DTM are exact, the two–dimensional DTMs recursive equation generate exactly all the multivariate Taylor series coefficients of exact solutions. Our first interest in the present work is introducing a reduced form of two–dimensional DTM as reduced-DTM, where generate the multivariate Taylor series coefficients of exact solutions $u$ with respect to the variable weights $U_0$, and implementing the present method to stress its power in handling nonlinear equations. The next interest is to employ the reduced-DTM to solve the nonlinear Kawahara equation (1.1).

Rest of the paper is organized as follows: In Section 2, the differential transform method is produced. Section 3 is devoted to employed the method on the problems related to the Kawahara equation and their modified form. Section 4 is the brief conclusion of this paper. Finally some references are listed in the end.

# 2 Basic Definitions

With reference to the articles [20–27, 30–33], the basic definitions of differential transformation are introduced as follows:

## 2.1 One-Dimensional Differential Transform Method

The transformation of the k-th derivative of a function in one variable is as follows:

**Definition 2.1.** If $u(t) \in R$ can be expressed as a Taylor series about fixed point $t_0$, then $u(t)$ can be represented as

$$u(t) = \sum_{k=0}^{\infty} \frac{u^{(k)}(t_0)}{k!}(t-t_0)^k. \tag{2.1}$$

If $u_n(t) = \sum_{k=0}^{n} \frac{u^{(k)}(t_0)}{k!}(t-t_0)^k$, is the $n$-partial sums of a Taylor series Eq. (2.1), then

$$u(t) = \sum_{k=0}^{n} \frac{u^{(k)}(t_0)}{k!}(t-t_0)^k + R_n(t). \tag{2.2}$$

where $u_n(t)$ is called the $n$-th Taylor polynomial for $u(t)$ about $t_0$ and $R_n(t)$ is remainder term. If $U(k)$ is defined as

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}, \tag{2.3}$$

where $k = 0, 1, ..., \infty$ then Eq. (2.1) reduce to

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_0)^k \tag{2.4}$$

and the $n$-partial sums of a Taylor series Eq. (2.2) reduce to

$$u_n(t) = \sum_{k=0}^{n} U(k)(t-t_0)^k + R_n(t). \tag{2.5}$$

The $U(k)$ defined in Eq. (2.5), is called the differential transform of function $u(t)$.

For simplicity assume that $t_0 = 0$, then the Eq. (2.5) reduce to

$$u_n(t) = \sum_{k=0}^{n} U(k)t^k + R_n(t). \tag{2.6}$$

From the above definitions, it can be found that the concept of the one-dimensional differential transform is derived from the Taylor series expansion. With relationships (2.3)–(2.6), the fundamental mathematical operations performed by onedimensional differential transform can readily be obtained and listed in Table 1. (See [25–27] and their references).

## 2.2 Two–Dimensional Reduced DTM

Consider a function of two variables $w(x,t)$, and suppose that it can be represented as a product of two single-variable function, i.e., $w(x,t) = f(x)g(t)$. Based

Table 1: The fundamental operations of one–dimensional DTM.

| Original function | Transformed function |
|---|---|
| $w(t) = u(t) \pm v(t)$ | $W(k) = U(k) \pm V(k)$ |
| $w(t) = \frac{d^m u(t)}{dt^m}$ | $W(k) = \frac{(k+m)!}{k!} U(k+m)$ |
| $w(t) = u(t)v(t)$ | $W(k) = U(k) \star V(k) = \sum_{l=0}^{k} U(l)V(k-l)$ |
| $w(x) = x^m$ | $W(k) = \delta(k-m) = \begin{cases} 1 & k = m, \\ 0 & \text{otherwise} \end{cases}$ |
| $w(t) = \exp(\lambda t)$ | $W(k) = \frac{\lambda^k}{k!}$ |
| $w(t) = \sin(\alpha t + \beta)$ | $W(k) = \frac{\alpha^k}{k!} \sin(\frac{k\pi}{2} + \beta)$ |
| $w(t) = \cos(\alpha t + \beta)$ | $W(k) = \frac{\alpha^k}{k!} \cos(\frac{k\pi}{2} + \beta)$ |

on the properties of one–dimensional differential transform, the function $w(x,t)$ can be represented as

$$w(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} W(i,j)x^i t^j, \tag{2.7}$$

where $W(i,j) = F(i)G(j)$ is called the spectrum of $w(x,t)$.

**Remark 2.2.** *The poisson function series generates a multivariate Taylor series expansion of the input expression $w$, with respect to the variables $X$, to order $n$, using the variable weights $W$.*

**Remark 2.3.** *The relationship introduce in (2.7) is the poisson series form of the input expression $w(x,t)$, with respect to the variables $x$ and $t$, to order $N$, using the variable weights $W_k(x)$.*

Similar on previous section, the basic definitions of two–differential reduced differential transformation are introduced as follows:

**Definition 2.4.** If $w(x,t)$ is analytical function in the domain of interest, then the spectrum function

$$W_k(x) = \frac{1}{k!} \left[ \frac{\partial^k}{\partial t^k} w(x,t) \right]_{t=t_0} \tag{2.8}$$

is the reduced transformed function of $w(x,t)$.

Similarly on previous sections, the lowercase $w(x,t)$ respect the original function while the uppercase $W_k(x)$ stand for the reduced transformed function. The differential inverse transform of $W_k(x)$ is defined as:

$$w(x,t) = \sum_{k=0}^{\infty} W_k(x)(t-t_0)^k. \tag{2.9}$$

Combining Eq. (2.2) and Eq. (2.3), it can be obtained that

$$w(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}w(x,t)\right]_{t=t_0}(t-t_0)^k.$$

From the above proposition, it can be found that the concept of the reduced two–dimensional differential transform is derived from the two–dimensional differential transform method. With Eq. (2.2) and Eq. (2.3), the fundamental mathematical operations performed by reduced two–dimensional differential transform can readily be obtained and listed in Table 2.

Table 2: The fundamental operations of two–dimensional reduced DTM.

| Original function | Reduced transformed function |
|---|---|
| $w(x,t)=u(x,t)\pm v(x,t)$ | $W_k(x)=U_k(x)\pm V_k(x)$ |
| $w(x,t)=\frac{\partial}{\partial x}u(x,t)$ | $W_k(x)=\frac{d}{dx}U_k(x)$ |
| $w(x,t)=\frac{\partial}{\partial t}u(x,t)$ | $W_k(x)=(k+1)U_{k+1}(x)$ |
| $w(x,t)=\frac{\partial^{r+s}}{\partial x^r\partial t^s}u(x,t)$ | $W_k(x)=\frac{(k+s)!}{k!}\frac{d^r}{dx^r}U_{k+s}(x)$ |
| $w(x,t)=u(x,t)v(x,t)$ | $W_k(x)=\sum_{r=0}^{k}U_r(x)V_{k-r}(x)$ |
| $w(x,t)=x^mt^n$ | $W_k(x)=x^m\delta(k-n)=\begin{cases}x^m & k=n\\ 0 & \text{otherwise}\end{cases}$ |

## 3 Application

Consider the generalized Kawahara equation (1.1). When $m=1$, Eq. (1.1) is called Kawahara equation

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}-\gamma\frac{\partial^5 u}{\partial x^5}=0, \tag{3.1}$$

and when $m=2$, Eq. (1.1) is called modified Kawahara equation

$$\frac{\partial u}{\partial t}+\alpha u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}-\gamma\frac{\partial^5 u}{\partial x^5}=0. \tag{3.2}$$

Recently, the hyperbolic function solutions of Kawahara Eq. (3.1) and modified Kawahara Eq. (3.2) was derived by Öziş and Aslan [8] using $(\frac{G'}{G})$–expansion method and are given by

$$u(x,t)=-\frac{3}{169}\frac{12(\beta^2+\gamma)}{\alpha\gamma}+\frac{105}{169}\frac{\beta^2}{\alpha\gamma}\operatorname{sech}^4\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}\left(x+\frac{36}{169}t\right)\right), \tag{3.3}$$

and

$$u(x,t)=\frac{\sqrt{10}}{10}\frac{\beta}{\sqrt{\alpha\gamma}}+\frac{\sqrt{10}}{9}\sqrt{\frac{\gamma}{\alpha}}\left\{1-\frac{3}{2}\tanh^2\left(-\frac{1}{6}x+\frac{\beta^2+\frac{5}{27}\gamma^2}{60\gamma}t\right)\right\}. \tag{3.4}$$

respectively, where $\alpha, \beta$ and $\gamma$ are nonzero arbitrary constants.

In the followings, we demonstrate the advantages of present method on Kawahara equation (3.1) and modified Kawahara equation (3.2).

## 3.1 Kawahara Equation

Firstly, we consider the Kawahara equation (3.1) subject to initial conditions:

$$u(x,0) = -\frac{3}{169}\frac{12(\beta^2+\gamma)}{\alpha\gamma} + \frac{105}{169}\frac{\beta^2}{\alpha\gamma}\operatorname{sech}^4\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}\, x\right), \tag{3.5}$$

where $\alpha, \beta$, and $\gamma$, are nonzero arbitrary constants. Therefore, by applying the reduced differential transform method on Kawahara equation (3.1), for $k = 0, 1, 2, ...$, we get the following recursive equation

$$U_{k+1}(x) = \frac{-1}{k+1}\left\{\alpha\sum_{r=0}^{k} U_r(x)\frac{d}{dx}U_{k-r}(x) + \beta\frac{d^3}{dx^3}U_k(x) - \gamma\frac{d^5}{dx^5}U_k(x)\right\}, \tag{3.6}$$

and their initial value is obtained from initial condition (3.5) as follow

$$U_0(x) = -\frac{3}{169}\frac{12(\beta^2+\gamma)}{\alpha\gamma} + \frac{105}{169}\frac{\beta^2}{\alpha\gamma}\operatorname{sech}^4\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}\, x\right) \tag{3.7}$$

then by utilize the initial condition (3.7) in recursive equation (3.6) for $k = 0, 1, 2, 3$, the first four terms of $U_k(x)$ obtain as follow

$$\begin{aligned}
U_1(x) &= -\frac{7560\sqrt{13}}{371293}\frac{\beta^2\sqrt{\frac{\beta}{\gamma}}\sinh\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)}{\alpha\gamma\cosh^5\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)},\\
U_2(x) &= \frac{68040}{62748517}\frac{\beta^3\left(4\cosh^2\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)-5\right)}{\alpha\gamma^2\cosh^6\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)},\\
U_3(x) &= -\frac{816480\sqrt{13}}{137858491849}\frac{\beta^3\sqrt{\frac{\beta}{\gamma}}\sinh\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}\,x\right)\left(8\cosh^2\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)-15\right)}{\alpha\gamma^2\cosh^7\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)},\\
U_4(x) &= \frac{3674160}{23298085122481}\frac{\beta^4\left(105-130\cosh^2\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)+32\cosh^4\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}\,x\right)\right)}{\alpha\gamma^3\cosh^6\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)}.
\end{aligned} \tag{3.8}$$

In the same manner, the rest of components can be obtained using the recurrence relation (3.6). Substituted the obtained quantities in inverse differential transform Eq. (2.9), the approximation solution of Kawahara equation (3.1) in the Poisson series form are:

$$
\begin{aligned}
U_4(x,t) &= U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + U_4(x)t^4 \\
&= -\frac{3}{169}\frac{12(\beta^2+\gamma)}{\alpha\gamma} + \frac{105}{169}\frac{\beta^2}{\alpha\gamma}\operatorname{sech}^4\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right) \\
&\quad - \frac{7560\sqrt{13}}{371293}\frac{\beta^2\sqrt{\frac{\beta}{\gamma}}\sinh\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)}{\alpha\gamma\cosh^5\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)}t \\
&\quad + \frac{68040}{62748517}\frac{\beta^3\left(4\cosh^2\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)-5\right)}{\alpha\gamma^2\cosh^6\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)}t^2 \\
&\quad - \frac{816480\sqrt{13}}{137858491849}\frac{\beta^3\sqrt{\frac{\beta}{\gamma}}\sinh\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)\left(8\cosh^2\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)-15\right)}{\alpha\gamma^2\cosh^7\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)}t^3 \\
&\quad + \frac{3674160}{23298085122481}\frac{\beta^4\left(105-130\cosh^2\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)+32\cosh^4\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}\,x\right)\right)}{\alpha\gamma^3\cosh^6\left(\frac{\sqrt{13}}{26}\sqrt{\frac{\beta}{\gamma}}x\right)}t^4,
\end{aligned}
\tag{3.9}
$$

which exactly is the first four terms of the poisson series of the exact solution (3.3).

To examine the accuracy and reliability of the RDTM solution for the Kawahara equation, we set $\alpha = 1, \beta = \frac{1}{2}$, and $\gamma = \frac{3}{2}$. In Table 3, the error of $N-$approximation solution obtained from RDTM and exact solution, $u - U_N$, in various $N$ are given in some points of the intervals $-10 \le x \le 10$ and $-10 \le t \le 10$. The details of these errors are shown in Fig. 1 – Fig. 4.

## 3.2 Modified Kawahara Equation

Now, consider the modified Kawahara equation (3.2) subject to initial conditions:

$$u(x,0) = \frac{\sqrt{10}}{10}\frac{\beta}{\sqrt{\alpha\gamma}} + \frac{\sqrt{10}}{9}\sqrt{\frac{\gamma}{\alpha}}\left\{1 - \frac{3}{2}\tanh^2\left(-\frac{1}{6}x\right)\right\}, \tag{3.10}$$

where $\alpha, \beta$, and $\gamma$, are nonzero arbitrary constants and $\alpha > 0$. Similar on previous section, by applying the reduced differential transform method on modified Kawahara equation (3.2), for $k = 0, 1, 2, ...$, the following recursive equation is obtained

$$
\begin{aligned}
U_{k+1}(x) = \frac{-1}{k+1}\Big\{\alpha\sum_{r=0}^{k}\Big(\sum_{\ell=0}^{r}U_\ell(x)U_{r-\ell}(x)\Big)\frac{d}{dx}U_{k-r}(x) \\
+ \beta\frac{d^3}{dx^3}U_k(x) - \gamma\frac{d^5}{dx^5}U_k(x)\Big\},
\end{aligned}
\tag{3.11}
$$

with the following initial value

$$U_0(x) = \frac{\sqrt{10}}{10}\frac{\beta}{\sqrt{\alpha\gamma}} + \frac{\sqrt{10}}{9}\sqrt{\frac{\gamma}{\alpha}}\left\{1 - \frac{3}{2}\tanh^2\left(-\frac{1}{6}x\right)\right\}, \tag{3.12}$$

then by utilize the initial condition (3.12) in recursive equation (3.11) for $k = 0, 1, 2, 3$, the first four terms of $U_k(x)$ obtain as follow

$$\begin{aligned}
U_1(x) &= \frac{\sqrt{10}}{4860}\frac{(27\beta^2+5\gamma^2)\sinh\left(\frac{1}{6}x\right)}{\sqrt{\alpha}\sqrt{\gamma}\cosh^3\left(\frac{1}{6}x\right)},\\
U_2(x) &= \frac{\sqrt{10}}{15746400}\frac{(27\beta^2+5\gamma^2)^2\left(2\cosh^2\left(\frac{1}{6}x\right)-3\right)}{\sqrt{\alpha}\sqrt{\gamma^3}\cosh^4\left(\frac{1}{6}x\right)},\\
U_3(x) &= \frac{\sqrt{10}}{19131876000}\frac{(27\beta^2+5\gamma^2)^3\left(\cosh^2\left(\frac{1}{6}x\right)-3\right)\sinh\left(\frac{1}{6}x\right)}{\sqrt{\alpha}\sqrt{\gamma^5}\cosh^5\left(\frac{1}{6}x\right)},\\
U_4(x) &= \frac{\sqrt{10}}{123974556480000}\frac{(27\beta^2+5\gamma^2)^4\left(2\cosh^4\left(\frac{1}{6}x\right)-15\cosh^2\left(\frac{1}{6}x\right)+15\right)}{\sqrt{\alpha}\sqrt{\gamma^7}\cosh^6\left(\frac{1}{6}x\right)}.
\end{aligned} \tag{3.13}$$

In the same manner, the rest of components can be obtained using the recurrence relation (3.11). Substituted the obtained quantities in inverse differential transform Eq. (2.9), the approximation solution of modified Kawahara equation (3.2) in the POISSON series form are:

$$\begin{aligned}
U_4(x,t) = {}& \frac{\sqrt{10}}{10}\frac{\beta}{\sqrt{\alpha\gamma}} + \frac{\sqrt{10}}{9}\sqrt{\frac{\gamma}{\alpha}}\left\{1-\frac{3}{2}\tanh^2\left(-\frac{1}{6}x\right)\right\}\\
&+ \frac{\sqrt{10}}{4860}\,\frac{(27\beta^2+5\gamma^2)\sinh\left(\frac{1}{6}x\right)}{\sqrt{\alpha}\sqrt{\gamma}\ \cosh^3\left(\frac{1}{6}x\right)}\ t\\
&+ \frac{\sqrt{10}}{15746400}\,\frac{(27\beta^2+5\gamma^2)^2\left(2\cosh^2\left(\frac{1}{6}x\right)-3\right)}{\sqrt{\alpha}\sqrt{\gamma^3}\ \cosh^4\left(\frac{1}{6}x\right)}\ t^2\\
&+ \frac{\sqrt{10}}{19131876000}\,\frac{(27\beta^2+5\gamma^2)^3\left(\cosh^2\left(\frac{1}{6}x\right)-3\right)\sinh\left(\frac{1}{6}x\right)}{\sqrt{\alpha}\sqrt{\gamma^5}\ \cosh^5\left(\frac{1}{6}x\right)}\ t^3\\
&+ \frac{\sqrt{10}}{123974556480000}\,\frac{(27\beta^2+5\gamma^2)^4\left(2\cosh^4\left(\frac{1}{6}x\right)-15\cosh^2\left(\frac{1}{6}x\right)+15\right)}{\sqrt{\alpha}\sqrt{\gamma^7}\ \cosh^6\left(\frac{1}{6}x\right)}\ t^4,
\end{aligned} \tag{3.14}$$

which exactly is the first four terms of the poisson series of the exact solution (3.4).

Similar on previous section and to examine the accuracy and reliability of the RDTM solution for the modified Kawahara equation, we set $\alpha = \beta = \gamma = 1$. In Table 4, the error of $N$–approximation solution obtained from RDTM and exact solution, $u - U_N$, in various $N$ are given in some points of the intervals $-10 \le x \le 10$ and $-10 \le t \le 10$. Fig. 5 – Fig. 8 shown these details.

# 4 Conclusions

In this paper, we have shown that the reduced differential transform method can be used successfully for solving the a famous partial differential differential equation with physical interest namely, the Kawahara equation and their modified form. The results of the test examples show that the differential transform method results are equal to ADM, HPM and HAM results. The advantage of the reduced differential transform method over other methods, such as ADM, HPM and HAM, is that the differential transform method is exact. Nonetheless, it is rather straightforward to apply. The present method reduces the computational difficulties of the other methods and all the calculations can be made with simple manipulations. On the other hand the results are quite reliable. Therefore, this method can be applied to many complicated linear and nonlinear PDEs and systems of PDEs and does not require linearization, discretization or perturbation. Another considerable advantage of the method is that Poisson coefficients of the solution are found very easily by using the computer programs. Also, the accuracy of the series solution increases when the number of terms in the series solution is increased.

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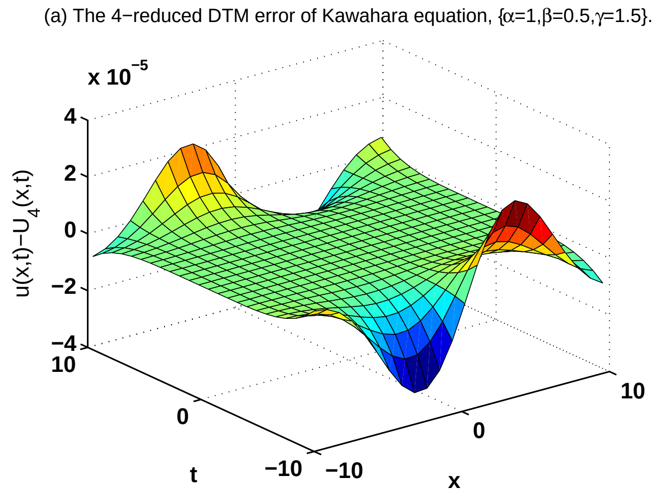


Figure 1: The error between exact and 4-RDTM solution of Kawahara equation (3.1), when $\alpha = 1, \beta = \frac{1}{2}$, and $\gamma = \frac{3}{2}$.

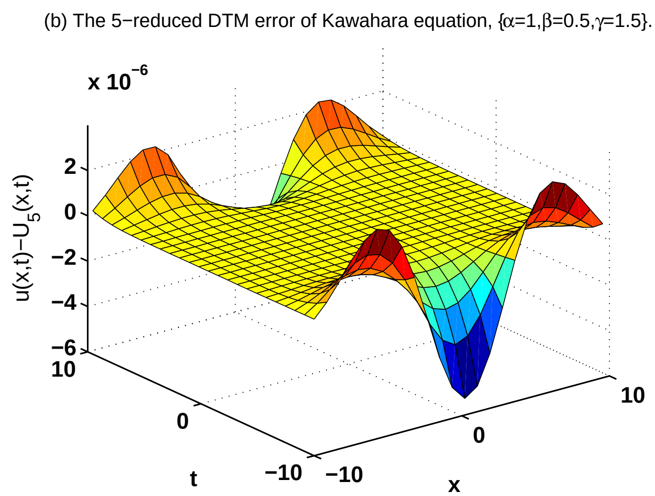


Figure 2: The error between exact and 5-RDTM solution of Kawahara equation (3.1), when $\alpha = 1, \beta = \frac{1}{2}$, and $\gamma = \frac{3}{2}$.

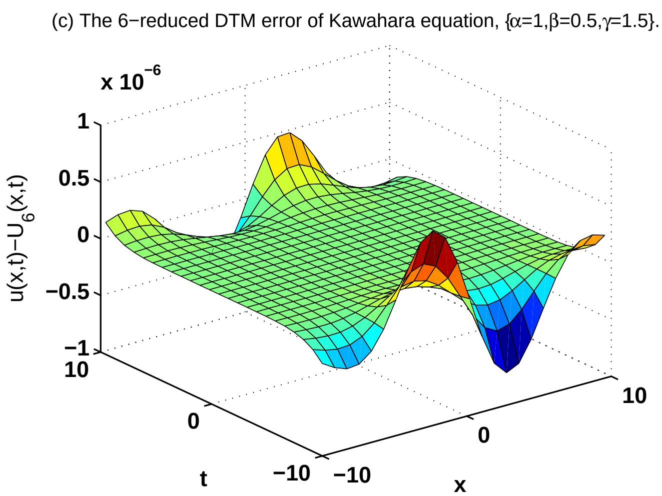


Figure 3: The error between exact and 6-RDTM solution of Kawahara equation (3.1), when $\alpha = 1, \beta = \frac{1}{2}$, and $\gamma = \frac{3}{2}$.

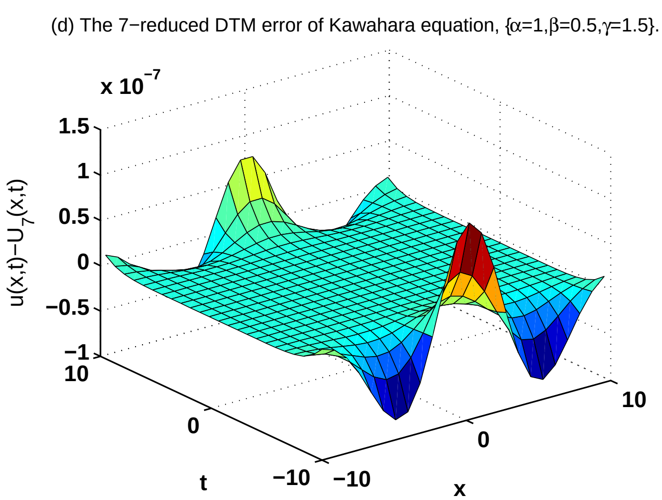


Figure 4: The error between exact and 7-RDTM solution of Kawahara equation (3.1), when $\alpha = 1, \beta = \frac{1}{2}$, and $\gamma = \frac{3}{2}$.

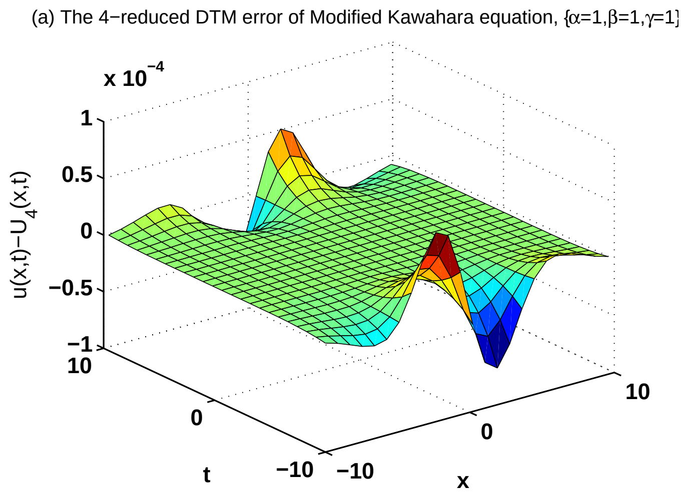


Figure 5: The error between exact and 4-RDTM solution of Modified Kawahara equation (3.1), when $\alpha = 1, \beta = 1$, and $\gamma = 1$.

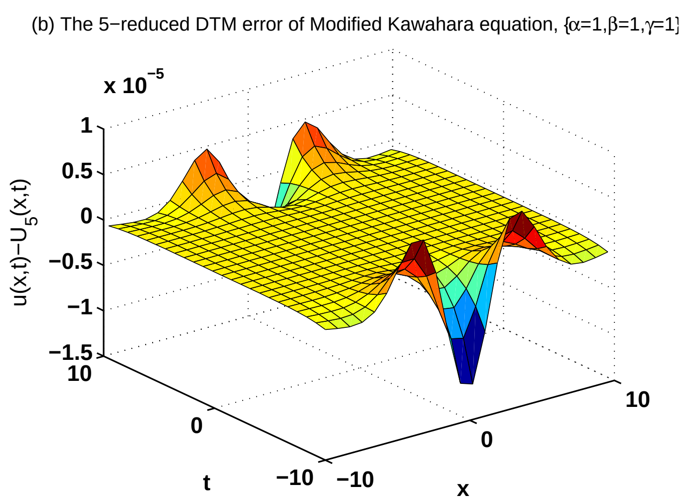


Figure 6: The error between exact and 5-RDTM solution of Modified Kawahara equation (3.1), when $\alpha = 1, \beta = 1$, and $\gamma = 1$.

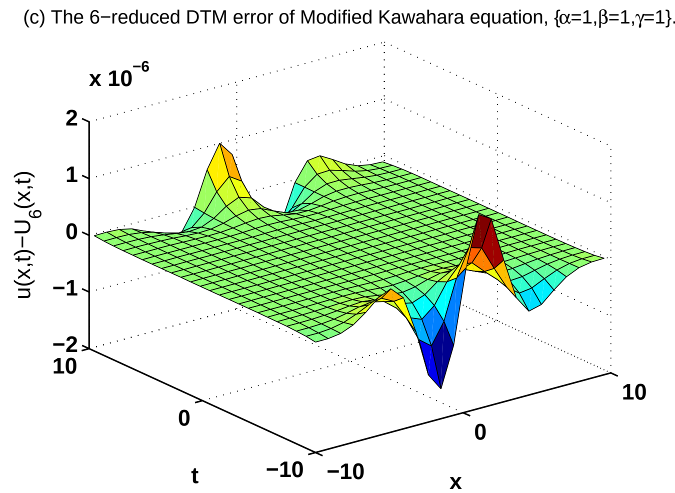


Figure 7: The error between exact and 6-RDTM solution of Modified Kawahara equation (3.1), when $\alpha = 1, \beta = 1$, and $\gamma = 1$.

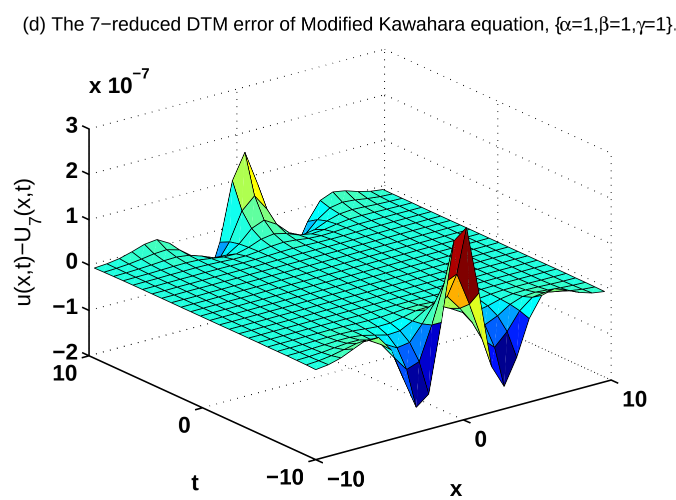


Figure 8: The error between exact and 7-RDTM solution of Modified Kawahara equation (3.1), when $\alpha = 1, \beta = 1$, and $\gamma = 1$.

Table 3: The numerical results for the $N$-term approximate solutions (3.9) obtained by RDTM in comparison with the exact solution (3.3), $u(x,t) - U_N(x,t)$, of Kawahara equation (3.1) at some points and in various $N$, when $\alpha = 1, \beta = \frac{1}{2}$ and $\gamma = \frac{3}{2}$.

| $x$ | $t$ | $u(x,t) - U_4(x,t)$ | $u(x,t) - U_5(x,t)$ | $u(x,t) - U_6(x,t)$ | $u(x,t) - U_7(x,t)$ |
|---|---|---|---|---|---|
| 10 | 10 | +9.43666706E-006 | +1.96237243E-008 | -1.84217766E-007 | +2.23968836E-008 |
| | 7.5 | +2.24564681E-006 | +1.09382804E-008 | -2.53411255E-008 | +2.23860105E-009 |
| | 5 | +2.95940482E-007 | +1.65787792E-009 | -1.52714538E-009 | +8.70315764E-011 |
| | 2.5 | +9.23382462E-009 | +3.74932307E-011 | -1.22727661E-011 | +3.37979644E-013 |
| | -2.5 | -9.13362302E-009 | +6.27083663E-011 | +1.29423694E-011 | +3.31623617E-013 |
| | -5 | -2.89399667E-007 | +4.88293692E-009 | +1.69791362E-009 | +8.37366565E-011 |
| | -7.5 | -2.16873920E-006 | +6.59693233E-008 | +2.96899173E-008 | +2.11019080E-009 |
| | -10 | -8.98593251E-006 | +4.31110825E-007 | +2.27269334E-007 | +2.06546844E-008 |
| 5 | 10 | -1.78075529E-005 | +3.10265273E-006 | +1.68172323E-007 | -7.71271327E-008 |
| | 7.5 | -4.41486737E-006 | +5.47222429E-007 | +2.49479629E-008 | -7.79556186E-009 |
| | 5 | -6.05982773E-007 | +4.74611517E-008 | +1.60989541E-009 | -3.06506598E-010 |
| | 2.5 | -1.96899289E-008 | +7.30193794E-010 | +1.37679035E-011 | -1.20398136E-012 |
| | -2.5 | +2.11203670E-008 | +7.00244307E-010 | -1.61815839E-011 | -1.20969901E-012 |
| | -5 | +6.97069467E-007 | +4.36255425E-008 | -2.22571381E-009 | -3.09311798E-010 |
| | -7.5 | +5.44372235E-006 | +4.81632551E-007 | -4.06419159E-008 | -7.89839111E-009 |
| | -10 | +2.35209814E-005 | +2.61077585E-006 | -3.23704554E-007 | -7.84050979E-008 |
| 0 | 10 | -5.20756163E-006 | -5.20756163E-006 | +1.15820790E-007 | +1.15820790E-007 |
| | 7.5 | -9.35752298E-007 | -9.35752298E-007 | +1.16954022E-008 | +1.16954022E-008 |
| | 5 | -8.27186814E-008 | -8.27186814E-008 | +4.59168814E-010 | +4.59168814E-010 |
| | 2.5 | -1.29785360E-009 | -1.29785360E-009 | +1.80030990E-012 | +1.80030990E-012 |
| | -2.5 | -1.29785360E-009 | -1.29785360E-009 | +1.80030990E-012 | +1.80030990E-012 |
| | -5 | -8.27186814E-008 | -8.27186814E-008 | +4.59168814E-010 | +4.59168814E-010 |
| | -7.5 | -9.35752298E-007 | -9.35752298E-007 | +1.16954022E-008 | +1.16954022E-008 |
| | -10 | -5.20756163E-006 | -5.20756163E-006 | +1.15820790E-007 | +1.15820790E-007 |
| -5 | 10 | +2.35209814E-005 | +2.61077585E-006 | -3.23704554E-007 | -7.84050979E-008 |
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| | 5 | +6.97069467E-007 | +4.36255425E-008 | -2.22571381E-009 | -3.09311798E-010 |
| | 2.5 | +2.11203670E-008 | +7.00244307E-010 | -1.61815839E-011 | -1.20969901E-012 |
| | -2.5 | -1.96899289E-008 | +7.30193794E-010 | +1.37679035E-011 | -1.20398136E-012 |
| | -5 | -6.05982773E-007 | +4.74611517E-008 | +1.60989541E-009 | -3.06506598E-010 |
| | -7.5 | -4.41486737E-006 | +5.47222429E-007 | +2.49479629E-008 | -7.79556186E-009 |
| | -10 0 | -1.78075529E-005 | +3.10265273E-006 | +1.68172323E-007 | -7.71271327E-008 |
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| | 7.5 | -2.16873920E-006 | +6.59693233E-008 | +2.96899173E-008 | +2.11019080E-009 |
| | 5 | -2.89399667E-007 | +4.88293692E-009 | +1.69791362E-009 | +8.37366565E-011 |
| | 2.5 | -9.13362302E-009 | +6.27083663E-011 | +1.29423694E-011 | +3.31623617E-013 |
| | -2.5 | +9.23382462E-009 | +3.74932307E-011 | -1.22727661E-011 | +3.37979644E-013 |
| | -5 | +2.95940482E-007 | +1.65787792E-009 | -1.52714538E-009 | +8.70315764E-011 |
| | -7.5 | +2.24564681E-006 | +1.09382804E-008 | -2.53411255E-008 | +2.23860105E-009 |
| | -10 | +9.43666706E-006 | +1.96237243E-008 | -1.84217766E-007 | +2.23968836E-008 |

Table 4: The error of fourth, fifth, sixth and seventh approximation solutions, $u(x,t)-U_N(x,t)$, of modified Kawahara equation by RDTM at some points and in various $N$, when $\alpha=\beta=\gamma=1$.

| $x$ | $t$ | $u(x,t)-U_4(x,t)$ | $u(x,t)-U_5(x,t)$ | $u(x,t)-U_6(x,t)$ | $u(x,t)-U_7(x,t)$ |
|---|---|---|---|---|---|
| 10 | 10 | -3.74002863E-006 | -6.12797553E-007 | -5.09011784E-008 | -1.33079830E-009 |
| | 7.5 | -8.48870378E-007 | -1.06763784E-007 | -6.75830178E-009 | -1.41454792E-010 |
| | 5 | -1.06898683E-007 | -9.17271162E-009 | -3.93080762E-010 | -5.81215631E-012 |
| | 2.5 | -3.19416765E-009 | -1.40231049E-010 | -3.04931080E-012 | -2.37865283E-014 |
| | -2.5 | +2.91975477E-009 | -1.34181832E-010 | +2.99990588E-012 | -2.56183963E-014 |
| | -5 | +8.93268727E-008 | -8.39909850E-009 | +3.80532356E-010 | -6.73625045E-012 |
| | -7.5 | +6.48540776E-007 | -9.35658177E-008 | +6.43966491E-009 | -1.77182075E-010 |
| | -10 | +2.61309466E-006 | -5.14136419E-007 | +4.77599551E-008 | -1.81042492E-009 |
| 5 | 10 | -5.37459857E-006 | +3.70866427E-006 | +5.31666182E-007 | +3.67594633E-009 |
| | 7.5 | -1.51900532E-006 | +6.36495525E-007 | +7.10581216E-008 | +5.79932768E-010 |
| | 5 | -2.30055997E-007 | +5.37959670E-008 | +4.15537194E-009 | +3.04481995E-011 |
| | 2.5 | -8.06236589E-009 | +8.08007994E-010 | +3.23737148E-011 | +1.47770685E-013 |
| | -2.5 | +9.61398189E-009 | +7.43608009E-010 | -3.20262705E-011 | +1.99673611E-013 |
| | -5 | +3.29424737E-007 | +4.55727730E-008 | -4.06782208E-009 | +5.71016567E-011 |
| | -7.5 | +2.65206492E-006 | +4.96564071E-007 | -6.88733323E-008 | +1.60485653E-009 |
| | -10 | +1.17496012E-005 | +2.66633834E-006 | -5.10659747E-007 | +1.73304892E-008 |
| 0 | 10 | -1.15916742E-005 | -1.15916742E-005 | +2.35879770E-007 | +2.35879770E-007 |
| | 7.5 | -2.08123817E-006 | -2.08123817E-006 | +2.38123365E-008 | +2.38123365E-008 |
| | 5 | -1.83870823E-007 | -1.83870823E-007 | +9.34707978E-010 | +9.34707978E-010 |
| | 2.5 | -2.88392188E-009 | -2.88392188E-009 | +3.66451314E-012 | +3.66451314E-012 |
| | -2.5 | -2.88392188E-009 | -2.88392188E-009 | +3.66451314E-012 | +3.66451314E-012 |
| | -5 | -1.83870823E-007 | -1.83870823E-007 | +9.34707978E-010 | +9.34707978E-010 |
| | -7.5 | -2.08123817E-006 | -2.08123817E-006 | +2.38123365E-008 | +2.38123365E-008 |
| | -10 | -1.15916742E-005 | -1.15916742E-005 | +2.35879770E-007 | +2.35879770E-007 |
| -5 | 10 | +1.17496012E-005 | +2.66633834E-006 | -5.10659747E-007 | +1.73304892E-008 |
| | 7.5 | +2.65206492E-006 | +4.96564071E-007 | -6.88733323E-008 | +1.60485653E-009 |
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