Thai Journal of Mathematics
Volume 11 (2013) Number 1 : 189–198
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On a-Maximal Ideals and C-Ideals of Ordered Γ-Semigroups

Jian Tang[†] **and Xiang-Yun Xie**[‡,1]

[†]School of Mathematics and Computational Science
Fuyang Normal College, Fuyang, Anhui 236041, P.R. China
e-mail : tangjian0901@126.com

[‡]School of Mathematics and Computational Science
Wuyi University, Jiangmen, Guangdong 529020, P.R. China
e-mail : xyxie@wyu.edu.cn

Abstract : In this paper, we first introduce the concept of a-maximal ideals in ordered Γ-semigroups and give some characterizations of a-maximal ideals of ordered Γ-semigroups. Furthermore, the concept of C-ideals in ordered Γ-semigroups is introduced. Some results on semigroups containing no maximal ideals are extended to ordered Γ-semigroups.

Keywords : ordered Γ-semigroups; a-maximal ideals; C-ideals; $\mathcal{J}$-class.
2010 Mathematics Subject Classification : 20M10; 06F05.

1 Introduction and Preliminaries

As we know, the notion of Γ-semigroups was introduced by Sen in 1981 [1]. Γ-semigroups generalize semigroups. Many classical notions of semigroups have been extended to Γ-semigroups (see [2, 3]), and some properties of Γ-semigroups were studied by some mathematician (for example, see [4, 5]). In 1993, Sen and Seth [6] introduced the concept of po-Γ-semigroups (some authors called ordered Γ-semigroups). Since then, some properties of ordered Γ-semigroups were studied (see [7, 8, 9]). As a continuation of Sen and Seth's works in ordered Γ-semigroups, the aim of this paper is to study and characterize maximal ideals, a-maximal ideals

[1]Corresponding author.

and C-ideals of an ordered Γ-semigroup. Some similar results of Fabrici in [10] are obtained. As an application of the results of this paper, the corresponding results of ordinary ordered semigroups can be also obtained by moderate modification.

To present the main results we first recall some definitions and notations which will be used frequently. Let $S = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two nonempty sets. Then S is called a Γ-semigroup if it satisfies $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$ [1]. A nonempty subset A of a Γ-semigroup S is called a Γ-subsemigroup of S if $a\gamma b \in A$ for all $a, b \in A$ and $\gamma \in \Gamma$. For nonempty subsets A, B of S, let $A\Gamma B := \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$. We also write $a\Gamma B,\ A\Gamma b$ and $a\Gamma b$ for $\{a\}\Gamma B,\ A\Gamma\{b\}$ and $\{a\}\Gamma\{b\}$, respectively.

An ordered Γ-semigroup is an ordered set $(S, \leq)$ at the same time a Γ-semigroup such that

$$a \leq b \Rightarrow a\beta c \leq b\beta c \text{ and } c\gamma a \leq c\gamma b$$

for all $a, b, c \in S$ and $\beta, \gamma \in \Gamma$ [6].

Definition 1.1 ([7, 11])**.** Let S be an ordered Γ-semigroup and I a nonempty subset of S. Then I is called a *left* (resp. *right*) ideal of S if

(1) $S\Gamma I \subseteq I$ (resp. $I\Gamma S \subseteq I$), and

(2) If $a \in I$ and $S \ni b \leq a$, then $b \in I$.

If I is both a left and a right ideal of S, then it is called an *(two-sided) ideal* of S.

Definition 1.2 ([8])**.** An ideal I of an ordered Γ-semigroup S is called *weakly prime* if for all ideals A, B of S such that $A\Gamma B \subseteq I$, then $A \subseteq I$ or $B \subseteq I$.

Let S be an ordered Γ-semigroup. For $\emptyset \neq H \subseteq S$, we define

$$(H] := \{t \in S |\ (\exists h \in H)\ \ t \leq h\},\ [H) := \{t \in S |\ (\exists h \in H)\ \ h \leq t\}.$$

We write $(a]$ and $[a)$ instead of $(\{a\}]$ and $[\{a\})$ $(a \in S)$, respectively. Denote by $I(a)$ the ideal of S generated by a $(a \in S)$. For convenience we use the notation $S^1 = S \cup \{1\}$, where $a\gamma 1 = 1\gamma a = a$ and $1\gamma 1 = 1$ for all $a \in S, \gamma \in \Gamma$. We have $I(a) = (S^1\Gamma a\Gamma S^1] = (a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]$ [7].

Lemma 1.3 ([8, 11])**.** *Let S be an ordered Γ-semigroup. Then, we have:*

(1) $A \subseteq (A]$ for any subset A of S.

(2) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.

(3) $(A]\Gamma(B] \subseteq (A\Gamma B]$ for any $A, B \subseteq S$.

(4) $(A] = ((A]]$ for any $A \subseteq S$.

(5) For every left (resp. right, two-sided) ideal T of S, $(T] = T$.

(6) If A, B are ideals of S, then $(A\Gamma B]$ and $A \cup B$ are ideals of S.

(7) $(S\Gamma a\Gamma S]$ is an ideal of S, $\forall\ a \in S$.

Let S be an ordered Γ-semigroup. Then we define an equivalence relation "$\mathcal{J}$" on S as follows:

$$(a, b) \in \mathcal{J} \text{ if and only if } I(a) = I(b).$$

We denote the $\mathcal{J}$-class containing a by I^a and assign a partial order relation "$\preceq$" on the $\mathcal{J}$-classes as follows:

$$I^a \preceq I^b \text{ if and only if } I(a) \subseteq I(b).$$

Definition 1.4. Let S be an ordered Γ-semigroup. Then a proper ideal I of S is called a *maximal ideal* of S if for any ideal A of S such that $I \subset A$, we have $A = S$; I is said to be the greatest ideal of S if, for any ideal A of S, we have $A \subseteq I$.

Definition 1.5. Let S be an ordered Γ-semigroup. Then the intersection $K(S)$ of all ideals of S is called the *kernel* of S, provided it is nonempty.

Definition 1.6. Let $(\mathcal{X}, \subseteq)$ be an ordered set and $M, N \in \mathcal{X}$. Then M is said to be a *cover* of N if $N \subset M$ and there does not exists $P \in \mathcal{X}$ such that $N \subset P \subset M$.

The reader is referred to [7, 12] for notation and terminology not defined in this paper.

2 a-Maximal Ideals of Ordered Γ-Semigroups

In this section $M(x, y)$ denotes the minimum of x and y, and Z^+ the set of positive integers. From now on, unless stated otherwise S means an ordered Γ-semigroup.

Definition 2.1. Let S be an ordered Γ-semigroup and I an ideal of S. Then I is called *a-maximal ideal* of S if I is a maximal ideal of S with respect to not containing the element $a \in S$.

Clearly, the a-maximal ideal I of S, if exists, is the unique a-maximal ideal of S.

Example 2.2. *Let $S = \{0, x_1, x_2, \ldots, x_n, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ with the multiplication and the relation "$\leq$ " on S defined by*

(1) $(\forall i, j \in Z^+)$ $(\forall \gamma \in \Gamma)$ $x_i \gamma x_j = x_k$*, where* $k = min\{i, j\}$.

(2) $(\forall i \in Z^+)$ $(\forall \gamma \in \Gamma)$ $0\gamma x_i = x_i \gamma 0 = 0$.

$$\leq := \{(0, 0), (0, x_i), (x_i, x_j)\}$$

for any $i \leq j$ $(i, j \in Z^+)$. Then S is an ordered Γ-semigroup. Clearly, the following subset of S

$$I_i = \{0, x_1, x_2, \ldots, x_i\}$$

is an ideal of S for any $i \in Z^+$ and I_i is a x_{i+1}-maximal ideal of S.

Proposition 2.3. *Let S be an ordered Γ-semigroup with a kernel $K(S)$ and $a \in S$. Then we have*

(1) The a-maximal ideal of S does not exist for any $a \in K(S)$.

(2) If $a \notin K(S)$, then there exists an a-maximal ideal of S.

Proof. (1) Clearly.

(2) Let $\mathcal{X}=\{I \mid I$ is an ideal of S such that $a \notin I\}$. Since $K(S) \in \mathcal{X}$, we have $\mathcal{X} \neq \emptyset$. Thus $(\mathcal{X}, \subseteq)$ is an ordered set. Let $\mathcal{Y}$ be a chain in $\mathcal{X}$. Then the set $\bigcup_{A\in\mathcal{Y}} A$ is an ideal of S and $a \notin \bigcup_{A\in\mathcal{Y}} A$. Thus the ideal $\bigcup_{A\in\mathcal{Y}} A$ is an upper bound of $\mathcal{Y}$ in $\mathcal{X}$. By Zorn's lemma, $\mathcal{X}$ has a maximal element, say M. Then M is a maximal ideal of S such that $a \notin M$. Hence M is an a-maximal ideal of S. □

The following theorem characterizes the a-maximal ideals of S.

Theorem 2.4. *Let S be an ordered Γ-semigroup and I an ideal of S. Then there exists $a \in S$ such that I is a-maximal ideal of S if and only if $S\backslash I$ contains the least $\mathcal{J}$-class among all the $\mathcal{J}$-classes contained in $S\backslash I$.*

To prove Theorem 2.4, the following two lemmas are needed.

Lemma 2.5. *Let S be an ordered Γ-semigroup and I an ideal of S. Then I is a-maximal ideal of S if and only if there exists an ideal I^* of S which is a cover of I.*

Proof. Let $I^* = \bigcap\{A \mid A$ is an ideal of S containing I and $a \in A\}$. Clearly, I^* is the least ideal containing I and is not equal to I since $a \in I^*\backslash I$. Thus I^* is a cover of I.

Conversely, let I^* be an ideal of S which is a cover of I. Then there exists $a \in I^*$ such that $a \notin I$. It is easy to see that I is an a-maximal ideal of S. □

Lemma 2.6. *Let S be an ordered Γ-semigroup and I an a-maximal ideal of S. Then the following statements are true:*

(1) $S\backslash I = \bigcup\{I^x \mid x \in S\backslash I\}$;

(2) If $\Omega = \{I^x \mid x \in S\backslash I\}$, then $I^*\backslash I$ is the least element of Ω with respect to the ordering $\preceq$ on $\mathcal{J}$-classes, and any I^x $(x \in I)$ is not greater than any element I^x in Ω.

Proof. (1) Obviously, $S\backslash I \subseteq \bigcup\{I^x \mid x \in S\backslash I\}$ since $x \in I^x$ for all $x \in S\backslash I$. To obtain the reverse inclusion, we have to show that $I^x \subseteq S\backslash I$ holds for all $x \in S\backslash I$. In fact, if $y \in I^x$, then $y \in S\backslash I$. Indeed: If $y \notin S\backslash I$, then $y \in I$. Thus we have

$$x \in I(x) = I(y) \subseteq I,$$

which is a contradiction.

(2) We first show that $I^*\backslash I$ is a $\mathcal{J}$-class. It is clear that $a \in I^*\backslash I$. If $x \in I^*\backslash I$, then I is also x-maximal. If $x \notin I(a)$, then $x \notin I(a)\cup I \neq I$, which contradicts to I

is a x-maximal ideal. Thus, $I(x) \subseteq I(a)$. Similarly, we can show that $I(a) \subseteq I(x)$. Therefore, $I(a) = I(x)$. Moreover, if $y \in S\backslash I^*$, then $I(y) \neq I(a)$. In fact, if $y \in I^x$, then $y \in I(y) \subseteq I^*$, which is a contradiction. Consequently, $I^*\backslash I$ is a $\mathcal{J}$-class.

Now, consider $y \in S\backslash I^*$. Clearly, $I(a) \subseteq I(y)$, otherwise, we would have $a \notin I(y)$. This leads to $y \in I(y) \subseteq I$ since I is an a-maximal ideal of S. However, this is also a contradiction. Thus, $I^*\backslash I$ is the least element of Ω. Suppose that $I^* \preceq I^x$ for some $x \in I$. Then, we have

$$a \in I(a) \subseteq I(x) \subseteq I.$$

But this is clearly impossible. The lemma is proved. □

We now turn to prove Theorem 2.4:

($\Rightarrow$) This part is clear by Lemma 2.6.

($\Leftarrow$) Let I^a be the least $\mathcal{J}$-class in $\Omega = \{I^x \mid x \in S\backslash I\}$. Then $a \notin I$. If I is not an a-maximal ideal of S, then, by Zorn's lemma, there exists an a-maximal ideal I_1 such that $I \subset I_1$. Let $b \in I_1\backslash I$. Then $I(a) \neq I(b)$ and $I(a) \subseteq I(b)$, by hypothesis. Thus, $a \in I(b) \subseteq I_1$, which contradicts to I_1 is a-maximal. Therefore, I is an a-maximal ideal of S.

Theorem 2.7. *Let S be an ordered Γ-semigroup. Then the following statements are equivalent:*

(1) I is a maximal ideal of S.

(2) $S\backslash I$ is a $\mathcal{J}$-class of S.

(3) $S\backslash I$ is a maximal $\mathcal{J}$-class of S.

Proof. (1) $\Rightarrow$ (2). If I is a maximal ideal of S, then S must be the cover of I. Thus, by Lemmas 2.5, 2.6, $S\backslash I$ is a $\mathcal{J}$-class of S.

(2) $\Rightarrow$ (3). This is obvious by Lemma 2.6(2).

(3) $\Rightarrow$ (1). Let $S\backslash I$ be a maximal $\mathcal{J}$-class of S. Then I is an ideal of S. Indeed: If $a \in I$ and $x \in S$, then $a\gamma x \in I$ for any $\gamma \in \Gamma$. Indeed if $a\gamma x \in S\backslash I$, then, by hypothesis, $I(a) \subseteq I(a\gamma x)$. Also, the reverse inclusion is obvious. Thus, $I(a) = I(a\gamma x)$. This implies that $a \in S\backslash I$, which is a contradiction. In the same way, we can prove that $x\gamma a \in I$ for any $\gamma \in \Gamma$. Now suppose that $a \in I, b \in S$ such that $b \leq a$. Then $I(b) \subseteq I(a)$. This leads to $b \in I$. In fact, if $b \in S\backslash I$, then $I(a) \subseteq I(b)$ by hypothesis. Hence $I(a) = I(b)$. This shows that $a \in S\backslash I$, which is impossible. Thus, I is an ideal of S.

Now if there exists a proper ideal I_1 of S such that $I \subset I_1$, then we can pick $c \in I_1\backslash I$. Since $I(x) = I(c)$ for any $x \in S\backslash I$, we have $S\backslash I \subseteq I(c) \subseteq I_1$. Thus, $S = (S\backslash I) \cup I \subseteq I_1$, which is a contradiction. It thus follows that I is a maximal ideal of S. The proof is completed. □

Theorem 2.8. *Let S be an ordered Γ-semigroup. If $S \neq (S\Gamma S]$, then the following statements hold:*

(1) The a-maximal ideal of S is of the form $S\backslash[a)$ for all $a \in S\backslash(S\Gamma S]$.

(2) Suppose S contains no maximal ideal with $a \in S\backslash(S\Gamma S]$. Then a is not a maximal element of S.

(3) $S\backslash[a)$ is not a weakly prime ideal of S for any $a \in S\backslash(S\Gamma S]$.

Proof. (1) We first show that $S\backslash[a)$ is an ideal of S. If $x \in S\backslash[a)$ and $y \in S$, then $x\gamma y, y\gamma x \in S\backslash[a)$ for any $\gamma \in \Gamma$. In fact, if $x\gamma y \geq a$ or $y\gamma x \geq a$ for some $\gamma \in \Gamma$, then $a \in (S\Gamma S]$, this is impossible. If $x \in S\backslash[a)$ with $y \in S$ and $y \leq x$, then $y \in S\backslash[a)$. In fact, if $y \notin S\backslash[a)$, then $a \leq y \leq x$. Thus $x \in [a)$, which is a contradiction. Therefore, $S\backslash[a)$ is an ideal of S with $a \notin S\backslash[a)$. Moreover, if there exists an ideal I_1 of S such that $S\backslash[a) \subset I_1 \subset S$, then there exists $b \in I_1, b \notin S\backslash[a)$. This implies that $a \leq b$ and so $a \in I_1$. Thus, $S\backslash[a)$ is an a-maximal ideal of S.

(2) If a is a maximal element of S, then $[a) = \{a\}$. Clearly, $S\backslash\{a\}$ is an a-maximal ideal of S by (1). It is obvious to see that it is a maximal ideal of S.

(3) Since $a \notin (S\Gamma S]$, we have $S\Gamma S \subseteq (S\Gamma S] \subseteq S\backslash[a)$. Obviously, $S \not\subseteq S\backslash[a)$. Thus, $S\backslash[a)$ is not a weakly prime ideal of S. □

Corollary 2.9. *Let S be an ordered Γ-semigroup. Then $a \in I(c)$ if and only if $a \leq c$ for any $a \in S\backslash(S\Gamma S], c \in S$.*

Proof. We only show that the necessity as the sufficiency is obvious.

By Theorem 2.8, $S\backslash[a)$ is an a-maximal ideal of S. If $a \in I(c)$, then $c \notin (S\Gamma S]$. If $a \not\leq c$, then $a \in (S\Gamma c \cup c\Gamma S \cup S\Gamma c\Gamma S]$. This leads to $a \in (S\Gamma S]$, which is impossible. □

Remark 2.10. *In Corollary 2.9, we can show that $I^a = \{a\}$ for any $a \in S\backslash(S\Gamma S]$. This is because if $a \neq c \in I^a$, then $I(a) = I(c)$ and $a \leq c$. Thus $c \in I(a)$ and $c \notin (S\Gamma S]$. On the other hand, we have $c \notin I(a)$. In fact, if $c \in I(a)$, then $c \in (S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]$. This implies that $c \in (S\Gamma S]$, which is impossible. Therefore, $I^a = \{a\}$.*

3 C-Ideals of Ordered Γ-Semigroups

Definition 3.1. A proper ideal I of an ordered Γ-semigroup S is called a *C-ideal* of S if $I \subseteq (S\Gamma(S - I)\Gamma S]$.

In Example 2.2, we can see that any ideal I_i $(i \in Z^+)$ of S is a C-ideal since $(S\Gamma(S - I_i)\Gamma S] = S$.

Theorem 3.2. *Let S be an ordered Γ-semigroup. If S is not simple, then S contains at least one C-ideal of S.*

Proof. Let T be any proper ideal of S. Then $(S\Gamma(S - T)\Gamma S]$ is an ideal of S, and $T \cap (S\Gamma(S - T)\Gamma S] \neq \emptyset$. If we denote $M = T \cap (S\Gamma(S - T)\Gamma S]$, then M is an ideal of S and we get $M \subseteq (S\Gamma(S - T)\Gamma S]$. Since $S - T \subseteq S - M$, the relation $M \subseteq (S\Gamma(S - T)\Gamma S]$ implies $M \subseteq (S\Gamma(S - T)\Gamma S] \subseteq (S\Gamma(S - M)\Gamma S]$, thus M is a C-ideal of S. □

Example 3.3. *Let $S = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ with the multiplication and the relation " $\leq$ " on S defined by*

γ	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	a	a

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(b,b),(c,c),(d,d)\}.$$

Then S is an ordered Γ-semigroup. Clearly, S is not simple, since $I = \{a\}$ is an ideal of S. Since $a = bcb, c \notin I$, we have $I \subseteq (S\Gamma(S-I)\Gamma S]$. By Definition 3.1, I is a C-ideal of S.

For C-ideals of an ordered Γ-semigroup, we have following properties.

Proposition 3.4. *Let S be an ordered Γ-semigroup. If S contains two different proper ideals I_1, I_2 such that $I_1 \cup I_2 = S$, then none of them is a C-ideal of S.*

Proof. If $I_1 \cup I_2 = S$, then $S - I_2 \subseteq I_1$ and $S - I_1 \subseteq I_2$. Hence none of I_1, I_2 is a C-ideal of S. Indeed: If I_1 is a C-ideal of S, then

$$I_1 \subseteq (S\Gamma(S-I_1)\Gamma S] \subseteq (S\Gamma I_2 \Gamma S] \subseteq (I_2] = I_2.$$

Since $I_1 \cup I_2 = S$, we have $I_2 = S$, which is impossible. Thus, I_1 is not a C-ideal of S. In a similar way, we can show that I_2 is also not a C-ideal of S. □

Corollary 3.5. *If an ordered Γ-semigroup S contains at least two maximal ideals, then any maximal ideal of S is not a C-ideal of S.*

Proof. Let M_1 and M_2 be two different maximal ideals of S. Then $M_1 \cup M_2 = S$. Thus, by Proposition 3.4, M_1 and M_2 are not C-ideals of S. □

Proposition 3.6. *Let S be an ordered Γ-semigroup. If I_1, I_2 are two C-ideals of S, then $I_1 \cup I_2$ is a C-ideal of S.*

Proof. We show that if $I_1 \subseteq (S\Gamma(S-I_1)\Gamma S]$ and $I_2 \subseteq (S\Gamma(S-I_2)\Gamma S]$, then $I_1 \cup I_2 \subseteq (S\Gamma(S-(I_1 \cup I_2))\Gamma S]$.

Let $x \in I_1$, then $I_1 \subseteq (S\Gamma(S-I_1)\Gamma S]$ implies that there exists $a \in S - I_1$ such that $x \in (S\Gamma a \Gamma S]$. There are two possibilities:

(1) if $a \in S - (I_1 \cup I_2)$, then $x \in (S\Gamma(S-(I_1 \cup I_2))\Gamma S]$.

(2) if $a \in (S-I_1) \cap I_2$, then $a \in I_2 \subseteq (S\Gamma(S-I_2)\Gamma S]$. So there exists $b \in S - I_2$ such that $a \in (S\Gamma b \Gamma S]$. The element b does not belong to I_1, since otherwise we would have $a \in (S\Gamma b\Gamma S] \subseteq (S\Gamma I_1 \Gamma S] \subseteq I_1$, which is a contradiction. Therefore, $b \in S - I_1$ and $b \in S - I_2$, which imply that $b \in (S-I_1) \cap (S-I_2) = S - (I_1 \cup I_2)$. Thus we have

$$x \in (S\Gamma a \Gamma S] \subseteq (S\Gamma(S\Gamma b \Gamma S]\Gamma S] \subseteq (S\Gamma b \Gamma S] \subseteq (S\Gamma(S-(I_1 \cup I_2))\Gamma S].$$

Hence $I_1 \subseteq (S\Gamma(S-(I_1\cup I_2))\Gamma S]$. Similarly, we can prove that $I_2 \subseteq (S\Gamma(S-(I_1\cup I_2))\Gamma S]$. Thus we have $I_1\cup I_2 \subseteq (S\Gamma(S-(I_1\cup I_2))\Gamma S]$. □

Proposition 3.7. *Let S be an ordered Γ-semigroup. If I_1, I_2 are two C-ideals of S and $I_1\cap I_2 \neq \emptyset$, then $I_1\cap I_2$ is also a C-ideal of S.*

Proof. From the relation $I_1 \subseteq (S\Gamma(S-I_1)\Gamma S]$ we have

$$I_1\cap I_2 \subseteq I_1 \subseteq (S\Gamma(S-I_1)\Gamma S] \subseteq (S\Gamma(S-(I_1\cap I_2))\Gamma S].$$

Thus, $I_1\cap I_2$ is a C-ideal of S. □

If we consider the empty set $\emptyset$ as a C-ideal, then, by Propositions 3.6 and 3.7, we may state:

Corollary 3.8. *The set of all C-ideals of an ordered Γ-semigroup S is a sublattice of the lattice of all ideals of S.*

Proof. The proof is easy by Proposition 3.4, we omit it. □

The following example shows that the converse of Proposition 3.9 does not hold, in general.

Example 3.9. *Let $S=\{a,b,c,d\}$ and $\Gamma=\{\gamma\}$ with the multiplication and the relation "$\leq$" on S defined by*

γ	a	b	c	d
a	a	b	a	b
b	b	a	b	a
c	a	b	a	b
d	b	a	b	c

$$\leq := \{(a,a),(a,c),(b,b),(b,d),(c,c),(d,d)\}.$$

Then S is an ordered Γ-semigroup. Clearly, $I=\{a,b,c\}$ is the only maximal ideal of S and any proper ideal of S is contained in I. By Definition 1.4, I is the greatest ideal of S. But I is not a C-ideal of S since $I \not\subseteq (S\Gamma(S-I)\Gamma S]=\{a,b\}$.

Theorem 3.10. *Let S be an ordered Γ-semigroup which satisfies just one of the following conditions:*

(1) S contains the greatest ideal I_g and it is a C-ideal.

(2) $S=(S\Gamma S]$ and for any proper ideal I and for every principal ideal $I(a)\subseteq I$, there exists $b\in S\backslash I$ such that $I(a)\subseteq I(b)$.

Then every proper ideal of S is C-ideal.

Proof. Let I be any proper ideal of S. If (1) holds, then $I \subseteq I_g$ and $S - I_g \subseteq S - I$. Then

$$I \subseteq I_g \subseteq (S\Gamma(S - I_g)\Gamma S] \subseteq (S\Gamma(S - I)\Gamma S],$$

which implies that I is C-ideal of S.

Let (2) be satisfied. If $x \in I$, then $I(x) \subseteq I$, which implies that there exists $b \in S \backslash I$ such that $I(x) \subseteq I(b) \subseteq S$. It is obvious that $I(x) \neq I(b)$. Since $S = (S\Gamma S]$, we have $S = (S\Gamma S\Gamma S]$ and so $b \in (S\Gamma d\Gamma S]$ for some $d \in S$. We show that $d \notin I$. In fact, if $d \in I$, then $b \in (S\Gamma d\Gamma S] \subseteq (S\Gamma I\Gamma S] = I$, which is a contradiction. Thus, we have

$$x \in I(x) \subseteq I(b) \subseteq (S\Gamma d\Gamma S] \subseteq (S\Gamma(S - I)\Gamma S],$$

from which we deduce that $I \subseteq (S\Gamma(S - I)\Gamma S]$. Therefore, I is a C-ideal of S. □

Theorem 3.11. *Let any proper ideal of an ordered Γ-semigroup S be C-ideal. Then just one of the following conditions holds:*

(1) S contains the greatest ideal I_g.

(2) $S = (S\Gamma S]$ and for any proper ideal I and any principal ideal $I(a) \subseteq I$, there exists a principal ideal $I(b)$ of S such that $I(a) \subset I(b)$ and $b \in S \backslash I$.

Proof. We first show that if any proper ideal of S is C-ideal, then S cannot contain two different maximal $\mathcal{J}$-classes. Indeed:

If I^c, I^d are maximal $\mathcal{J}$-classes and $I^c \neq I^d$, then, by Theorem 2.7, $S \backslash I^c$ and $S \backslash I^d$ are two different maximal ideals of S and none of them is C-ideal of S by Corollary 3.5. There are two possibilities:

(1) If S contains just one maximal $\mathcal{J}$-class I^c, then $S \backslash I^c$ is again a maximal ideal of S and moreover it is a C-ideal. By Proposition 3.9, we have that $I_g = S \backslash I^c$.

(2) Suppose that S does not contain maximal $\mathcal{J}$-classes. First we show that $S = (S\Gamma S]$. If $S \neq (S\Gamma S]$, then for $y \in S \backslash (S\Gamma S]$, $I(y) \neq S$. In fact, if $I(y) = S$, then I_y is a maximal $\mathcal{J}$-class of S, which is a contradiction. By assumption, $I(y)$ is a C-ideal of S, i.e., $I(y) \subseteq (S\Gamma(S - I(y))\Gamma S]$. Thus, $y \in I(y) \subseteq (S\Gamma S\Gamma S] \subseteq (S\Gamma S]$, which is impossible. Therefore, $S = (S\Gamma S]$.

Let I be any proper ideal of S. Then, by assumption, $I \subseteq (S\Gamma(S - I)\Gamma S]$. Let $a \in I$. Then there exist $b \in S \backslash I$ such that $a \in (S\Gamma b\Gamma S]$. This implies $I(a) \subseteq (S\Gamma b\Gamma S] \subseteq I(b)$. Since S does not contain maximal $\mathcal{J}$-classes, we have $I(b) \neq S$. Moreover, $I(a) \neq I(b)$, because $a \in I, b \in S \backslash I$. □

Theorem 3.12. *Let S be an ordered Γ-semigroup. If every principal ideal of S is a C-ideal of S, then S does not contain maximal ideals.*

Proof. Let I^a be a $\mathcal{J}$-class of S. Then $I(a) \neq S$ since S is not a C-ideal of S. By Definition 3.1, $I(a) \subseteq (S\Gamma(S - I(a))\Gamma S]$. Thus there exists $b \in S \backslash I(a)$ such that $a \leq x\alpha b\beta y$ for some $x, y \in S$ and $\alpha, \beta \in \Gamma$. Hence $I(a) \subset I(b) \neq S$. Therefore, I_a is not a maximal $\mathcal{J}$-class of S. By Theorem 2.7, S does not contain maximal ideals. □

Acknowledgements : Authors should express their heart-felt thanks to three anonymous referees of this paper for their interest, extremely valuable remarks and suggestions to the paper. This work was supported by the National Natural Science Foundation (10961014), the Science and technology projects in Guangdong Province (2010B010600039), the Guangdong Provincial Natural Science Foundation of China (S2011010003681), the University Excellent Youth Talent Key Project of Anhui Province (2012SQRL115ZD), the University Natural Science Project of Anhui Province (KJ2012B133) and the Fuyang Normal College Natural Science Foundation (2007LZ01).

References

[1] M.K. Sen, On Γ-semigroups, Proceeding of International Symposium on Algebra and Its Applications, DeckerPublication, New York (1981), 301-308.

[2] S. Chattopadhyay, Right inverse Γ-semigroups, Bull. Cal. Math. Soc. 93 (2001) 435–442.

[3] S. Chattopadhyay, Right orthodox Γ-semigroups, SEA Bull. Math. 29 (2005) 23–30.

[4] Y.B. Jun, Roughness of Γ-subsemigroups/ideals in Γ-semigroups, Bull. Korean Math. Soc. 40 (3) (2003) 531–536.

[5] R. Chinram, Rough prime ideals and rough fuzzy prime ideals in Γ-semigroups, Commun. Korean Math. Soc. 24 (3) (2009) 341–351.

[6] M.K. Sen, A. Seth, On po-Γ-semigroups, Bull. Calcutta Math. Soc. 85 (5) (1993) 445–450.

[7] Y.I. Kwon, S.K. Lee, Some special elements in ordered Γ-semigroups, Kyungpook Math. J. 35 (3) (1996) 679–685.

[8] Y.I. Kwon, S.K. Lee, On weakly prime ideals of ordered Γ-semigroups, Commun. Korean Math. Soc. 13 (2) (1998) 251–256.

[9] R. Chinram, K.Tinpun, A note on minimal bi-ideals in ordered Γ-semigroups, International Math. Forum 4 (1) (2009) 1–5.

[10] I. Fabrici, On bases and maximal ideals in semigroups, Math. Slovaca 31 (1981) 115–120.

[11] N. Kehayopulu, M. Tsingelis, On weakly prime ideals of ordered semigroups, Math. Japonica 35 (6) (1990) 1051–1056.

[12] X.Y. Xie, An introduction to ordered semigroup theory, Kexue Press, Beijing, 2001.

(Received 18 May 2011)
(Accepted 13 February 2012)