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A Common Fixed Point for a Family of Compatible Maps Involving a Quasi-contraction on Cone Metric Spaces

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Abstract : In this paper, we establish a common fixed point result for a family of compatible maps satisfying a quasi-contraction on a complete cone metric space without assumption of normality condition of the cone. The result is an extension of a recent paper of Janković et al. [S. Janković, Z. Golubović, S. Radenović, Compatible and weakly compatible mappings in cone metric spaces, Math. Comput. Modelling 52 (2010) 1728–1738].

Keywords : cone metric space; common fixed point; weakly compatible mappings; compatible pair of mappings.
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1 Introduction

Fixed point theory is a mixture of analysis, topology and geometry. The theory of existence of fixed points of maps has been revealed as a very powerful and important tool in the study of nonlinear phenomena. Huang and Zhang [1] re-introduced the concept of a cone metric space, replacing the set of real numbers by an ordered Banach space and obtain some fixed point theorems for mappings satisfying different contraction conditions. The study of fixed point theorems in such spaces is followed by some other mathematicians, see [2–17]. In this paper, we give a common fixed point result for a family of compatible maps satisfying a quasi-contraction on a complete cone metric space.

In the sequel, the letter $\mathbb{N}^*$ will denote the set of all positive natural numbers. Let E be a real Banach space and 0_E is the zero vector of E.

Definition 1.1. A non-empty subset P of E is called a cone if the following conditions hold:

(i) P is closed and $P \neq \{0_E\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Longrightarrow ax + by \in P$;

(iii) $x \in P$, $-x \in P \Longrightarrow x = 0_E$.

Given a cone $P \subset E$, a partial ordering $\leq_E$ with respect to P is naturally defined by $x \leq_E y$ if and only if $y - x \in P$, for $x, y \in E$. We shall write $x <_E y$ to indicate that $x \leq_E y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int}P$, where $\operatorname{int}P$ denotes the interior of P. P is said solid if $\operatorname{int}P$ is non-empty.

Definition 1.2 ([1])**.** Let X be a non-empty set and $d : X \times X \to P$ satisfies

(i) $d(x, y) = 0_E$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq_E d(x, z) + d(z, y)$ for all $x, y, z \in E$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.3 ([1])**.** Let (X, d) be a cone metric space, $\{x_n\}$ is a sequence in X and $x \in X$.

(i) If for every $c \in E$ with $0_E \ll_E c$, there is $N \in \mathbb{N}$ such that $d(x_n, x) \ll_E c$ for all $n \geq N$, then $\{x_n\}$ is said to be convergent to x. This limit is denoted by $\lim_{n \to +\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$.

(ii) If for every $c \in E$ with $0_E \ll_E c$, there is $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll_E c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X.

(iii) If every Cauchy sequence in X is convergent in X, then (X, d) is called a complete cone metric space.

We start by recalling some useful definitions.

Definition 1.4. Let X be a non-empty set, N is a natural number such that $N \geq 2$ and $T_1, T_2, ..., T_N : X \to X$ are given self-mappings on X. If $w = T_1x = T_2x = \cdots = T_Nx$ for some $x \in X$, then x is called a coincidence point of $T_1, T_2, ..., T_{N-1}$ and T_N, and w is called a point of coincidence of $T_1, T_2, ..., T_{N-1}$ and T_N. If $w = x$, then x is called a common fixed point of $T_1, T_2, ..., T_{N-1}$ and T_N.

Definition 1.5 ([18])**.** Let X be a non-empty set and $T_1, T_2 : X \to X$ are given self-maps on X. The pair $\{T_1, T_2\}$ is said to be weakly compatible if $T_1T_2t = T_2T_1t$, whenever $T_1t = T_2t$ for some t in X.

The following definition extends the notion of compatibility of a pair of self-mappings on a metric space introduced by Jungck in [19] to a cone metric space.

Definition 1.6. Let (X, d) be a cone metric space and $f, g : X \to X$ are given self-mappings on X. The pair $\{f, g\}$ is said to be compatible if for any $c \gg 0_E$ we have $d(fgx_n, gfx_n) \ll c$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some t in X.

Definition 1.7. Let (X, d) be a cone metric space and $T : X \to X$ be a given mapping. We say that T is continuous on $x_0 \in X$ if for every sequence $\{x_n\}$ is X, we have

$$x_n \to x_0 \text{ as } n \to +\infty \Longrightarrow Tx_n \to Tx_0 \text{ as } n \to +\infty.$$

If T is continuous on each point $x_0 \in X$, then we say that T is continuous on X.

The following lemmas from [20] will be useful for the rest of the paper.

Lemma 1.8. *Let u, v and w be vectors from Banach space E*

(1) If $u \leq_E v$ and $v \ll w$, then $u \ll w$.

(2) If $0_E \leq_E u \ll c$ for each $c \in intP$, then $u = 0_E$.

Lemma 1.9. *If $c \in intP$ and $0_E \leq_E a_n$ with $a_n \to 0_E$, then there exists an n_0 such that for all $n > n_0$, we have $a_n \ll c$.*

Lemma 1.10. *Let x_n and x be given in X. If $0_E \leq_E d(x_n, x) \leq_E b_n$ and $b_n \longrightarrow 0_E$, then $d(x_n, x) \ll c$ for each $c \in intP$.*

Lemma 1.11. *If $0_E \leq_E a_n \leq_E b_n$ and $a_n \longrightarrow a$ and $b_n \to b$, then $a \leq_E b$ for each cone P.*

Lemma 1.12. *If $a \leq_E \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = 0_E$.*

Using the concept of weak compatibility and compatibility of pairs of self maps, the aim of this paper is to establish a common fixed point result for $(2n+2)$ mappings involving a quasi-contraction on a complete cone metric space without assumption of normality condition of the cone.

2 Main Results

We start by citing our main theorem

Theorem 2.1. *Let (X, d) be a complete cone metric space with a solid cone P. Let $n \in \mathbb{N}^*$, P_1, P_2,...,P_{2n}, Q_0 and Q_1 be self-mappings on X satisfying the following conditions:*

(c_1) $Q_0X \subseteq P_1P_3 \cdots P_{2n-1}X$ and $Q_1X \subseteq P_2P_4 \cdots P_{2n}X$,

(c_2)

$$
\begin{aligned}
P_2(P_4\cdots P_{2n}) &= (P_4\cdots P_{2n})P_2,\\
P_2(P_4\cdots P_{2n}) &= (P_4\cdots P_{2n})P_2,\\
P_2P_4(P_6\cdots P_{2n}) &= (P_6\cdots P_{2n})P_2P_4,\\
&\vdots\\
P_2\cdots P_{2n-2}(P_{2n}) &= (P_{2n})P_2\cdots P_{2n-2},\\
Q_0(P_4\cdots P_{2n}) &= (P_4\cdots P_{2n})Q_0,\\
Q_0(P_6\cdots P_{2n}) &= (P_6\cdots P_{2n})Q_0,\\
&\vdots\\
Q_0P_{2n} &= P_{2n}Q_0,\\
P_1(P_3\cdots P_{2n-1}) &= (P_3\cdots P_{2n-1})P_1,\\
P_1P_3(P_5\cdots P_{2n-1}) &= (P_5\cdots P_{2n-1})P_1P_3,\\
&\vdots\\
P_1\cdots P_{2n-3}(P_{2n-1}) &= (P_{2n-1})P_1\cdots P_{2n-3},\\
Q_1(P_3\cdots P_{2n-1}) &= (P_3\cdots P_{2n-1})Q_1,\\
Q_1(P_5\cdots P_{2n-1}) &= (P_5\cdots P_{2n-1})Q_1,\\
&\vdots\\
Q_1P_{2n-1} &= P_{2n-1}Q_1,
\end{aligned}
$$

(c_3) the pair $\{Q_0, P_2\cdots P_{2n}\}$ is compatible and the pair $\{Q_1, P_1\cdots P_{2n-1}\}$ is weakly compatible,

(c_4) $P_2\cdots P_{2n}$ or Q_0 is continuous,

(c_5) for some $\lambda \in [0,1)$ and for any $x, y \in X$ there exists

$$u(x,y) \in \left\{ \begin{array}{l} d(P_2P_4\cdots P_{2n}x, Q_0x), \;\; d(P_1P_3\cdots P_{2n-1}y, Q_1y), \\ d(P_2P_4\cdots P_{2n}x, P_1P_3\cdots P_{2n-1}y), \\ \frac{1}{2}[d(P_1P_3\cdots P_{2n-1}y, Q_0x) + d(P_2P_4\cdots P_{2n}x, Q_1y)] \end{array} \right\}$$

such that $d(Q_0x, Q_1y) \leq_E \lambda\, u(x,y)$.

Then the $(2n+2)$ maps, $P_1, P_2, ..., P_{2n}, Q_0$ and Q_1 have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. From the condition (c_1), there exist x_1, x_2 in X such that $Q_0x_0 = P_1P_3\cdots P_{2n-1}x_1 = y_0$ and $Q_1x_1 = P_2P_4\cdots P_{2n}x_2 = y_1$. Inductively, we can construct sequences $\{x_k\}$ and $\{y_k\}$ in X defined by

$$Q_0x_{2k} = P_1P_3\cdots P_{2n-1}x_{2k+1} = y_{2k} \text{ and } Q_1x_{2k+1} = P_2P_4\cdots P_{2n}x_{2k+2} = y_{2k+1} \tag{2.1}$$

for all $k \in \mathbb{N}$. Denote

$$A = P_2 P_4 \cdots P_{2n} \quad \text{and} \quad B = P_1 P_3 \cdots P_{2n-1}, \tag{2.2}$$

then, (2.1) becomes

$$Q_0 x_{2k} = B x_{2k+1} = y_{2k} \text{ and } Q_1 x_{2k+1} = A x_{2k+2} = y_{2k+1}, \text{ for all } k \in \mathbb{N}. \tag{2.3}$$

Step 1: We will show that

$$d(y_{k+1}, y_k) \leq_E \lambda\, d(y_k, y_{k-1}) \text{ for any } k \in \mathbb{N}^*. \tag{2.4}$$

Putting $x = x_{2k}$ and $y = x_{2k+1}$ in the condition (c_5), we have

$$d(y_{2k+1}, y_{2k}) = d(y_{2k}, y_{2k+1}) = d(Q_0 x_{2k}, Q_1 x_{2k+1}) \leq_E \lambda u_1,$$

where

$$u_1 \in \left\{ \begin{array}{l} d(Ax_{2k}, Q_0 x_{2k}),\ d(Bx_{2k+1}, Q_1 x_{2k+1}),\ d(Ax_{2k}, Bx_{2k+1}), \\ \frac{1}{2}[d(Bx_{2k+1}, Q_0 x_{2k}) + d(Ax_{2k}, Q_1 x_{2k})] \end{array} \right\}.$$

By (2.3), we get

$$\begin{aligned} u_1 &\in \left\{ d(y_{2k-1}, y_{2k}),\ d(y_{2k}, y_{2k+1}),\ d(y_{2k-1}, y_{2k}),\ \frac{1}{2} d(y_{2k-1}, y_{2k+1}) \right\} \\ &= \left\{ d(y_{2k-1}, y_{2k}),\ d(y_{2k}, y_{2k+1}),\ \frac{1}{2} d(y_{2k-1}, y_{2k+1}) \right\}. \end{aligned}$$

Letting $x = x_{2k+2}$ and $y = x_{2k+1}$ in the condition (c_5), we obtain

$$d(y_{2k+2}, y_{2k+1}) = d(Q_0 x_{2k+2}, Q_1 x_{2k+1}) \leq_E \lambda u_2,$$

where

$$u_2 \in \left\{ \begin{array}{l} d(Ax_{2k+2}, Q_0 x_{2k}),\ d(Bx_{2k+1}, Q_1 x_{2k+1}), \\ d(Ax_{2k+2}, Bx_{2k+1}),\ \frac{1}{2}[d(Bx_{2k+1}, Q_0 x_{2k+2}) + d(Ax_{2k+2}, Q_1 x_{2k})] \end{array} \right\}.$$

Again, from (2.3)

$$\begin{aligned} u_2 &\in \left\{ d(y_{2k+2}, y_{2k+1}),\ d(y_{2k}, y_{2k+1}),\ d(y_{2k+1}, y_{2k}),\ \frac{1}{2} d(y_{2k}, y_{2k+2}) \right\} \\ &= \left\{ d(y_{2k+2}, y_{2k+1}),\ d(y_{2k+1}, y_{2k}),\ \frac{1}{2} d(y_{2k}, y_{2k+2}) \right\}. \end{aligned}$$

Then, we have the six following cases:
$\mathbf{1}^0$ $u_1 = d(y_{2k}, y_{2k-1})$, $\mathbf{2}^0$ $u_1 = d(y_{2k+1}, y_{2k})$, $\mathbf{3}^0$ $u_1 = \frac{1}{2} d(y_{2k+1}, y_{2k-1})$,
$\mathbf{4}^0$ $u_2 = d(y_{2k+2}, y_{2k+1})$, $\mathbf{5}^0$ $u_2 = d(y_{2k+1}, y_{2k})$, $\mathbf{6}^0$ $u_2 = \frac{1}{2} d(y_{2k+2}, y_{2k})$.

In the case $\mathbf{1}^0$, we get $d(y_{2k+1}, y_{2k}) \leq_E \lambda d(y_{2k}, y_{2k-1})$, so (2.4) holds. For the case $\mathbf{2}^0$, we have $d(y_{2k+1}, y_{2k}) \leq_E \lambda d(y_{2k+1}, y_{2k})$. Since $\lambda < 1$, hence $d(y_{2k+1}, y_{2k}) = 0$, and then (2.4) holds. Now, in the case $\mathbf{3}^0$, using the triangular inequality and $\lambda < 1$, we get

$$d(y_{2k+1}, y_{2k}) \leq_E \lambda u_1 = \frac{\lambda}{2} d(y_{2k+1}, y_{2k-1}) \leq_E \frac{1}{2} d(y_{2k+1}, y_{2k}) + \frac{\lambda}{2} d(y_{2k}, y_{2k-1}).$$

We rewrite this as follows

$$d(y_{2k+1}, y_{2k}) \leq_E \lambda d(y_{2k}, y_{2k-1}).$$

For the cases $\mathbf{4}^0$ and $\mathbf{5}^0$, it is immediate that (2.4) holds. In the case $\mathbf{6}^0$, using $\lambda < 1$ and the triangular inequality, we obtain

$$\begin{aligned} d(y_{2k+2}, y_{2k+1}) \leq_E \lambda u_2 &= \frac{\lambda}{2} d(y_{2k+2}, y_{2k}) \\ &\leq_E \frac{1}{2} d(y_{2k+2}, y_{2k+1}) + \frac{\lambda}{2} d(y_{2k+1}, y_{2k}), \end{aligned}$$

which immediately yields that (2.4) holds.

Step 2: $\{y_k\}$ is a Cauchy sequence. Thanks to (2.4), one can write

$$d(y_{k+1}, y_k) \leq_E \lambda d(y_k, y_{k-1}) \leq_E \cdots \leq_E \lambda^k d(y_1, y_0) \ \ \forall\, k \in \mathbb{N}. \tag{2.5}$$

Using (2.5) and the triangular inequality, we get for $p > q$

$$\begin{aligned} d(y_p, y_q) &\leq_E d(y_p, y_{p-1}) + d(y_{p-1}, y_{p-2}) + \cdots + d(y_{q+1}, y_q) \\ &\leq_E (\lambda^{p-1} + \lambda^{p-2} + \cdots + \lambda^q) d(y_1, y_0) \\ &= \lambda^q (1 + \lambda + \cdots + \lambda^{p-q-1}) d(y_1, y_0) \\ &\leq_E \frac{\lambda^q}{1-\lambda} d(y_1, y_0) \longrightarrow 0_E \quad \text{as } \ q \to +\infty. \end{aligned}$$

Referring to Lemmas 1.8 and 1.9, it follows that $d(y_p, y_q) \ll c$ for any $c \gg 0_E$, that is, $\{y_k\}$ is a Cauchy sequence in the cone metric space (X, d), which is complete, hence there exists $z \in X$ such that $y_k \to z$. For its subsequences, we have

$$Q_0 x_{2k} \to z, \ \ A x_{2k} \to z, \ \ Q_1 x_{2k+1} \to z, \ \ B x_{2k+1} \to z. \tag{2.6}$$

Step 3: We will show that z is a coincidence point. We have two cases related to the condition (c_4). We start with:

<u>Case 1</u>. If $A = P_2 P_4 \cdots P_{2n}$ is continuous. For the rest, k is taken large enough and $c \in E$ with $c \gg 0_E$. From the condition (c_3), the pair $\{Q_0, A\}$ is compatible, and since $Q_0 x_{2k} \to z$ and $A x_{2k} \to z$, then

$$d(Q_0 A x_{2k}, A Q_0 x_{2k}) \ll \frac{c}{2}.$$

On the other hand, A is continuous, so (2.6) yields that $AQ_0x_{2k} \longrightarrow Az$. Thus,

$$d(AQ_0x_{2k}, Az) \ll \frac{c}{2}.$$

By triangular inequality, we have

$$d(Q_0Ax_{2k}, Az) \leq_E d(Q_0Ax_{2k}, AQ_0x_{2k}) + d(AQ_0x_{2k}, Az) \ll \frac{c}{2} + \frac{c}{2} = c. \quad (2.7)$$

Therefore, $Q_0Ax_{2k} \longrightarrow Az$.
(a) We first need to show that $Az = z$. The triangular inequality gives us

$$d(Az, z) \leq_E d(Az, Q_0Ax_{2k}) + d(Q_0Ax_{2k}, Q_1x_{2k+1}) + d(Q_1x_{2k+1}, z). \quad (2.8)$$

Thanks to (2.6)-(2.7), we are able to control the first and the third terms of the right-hand side of (2.8). It rests only the control of the term $d(Q_0Ax_{2k}, Q_1x_{2k+1})$. To do this, we apply condition (c_5) for $x = Ax_{2k}$ and $y = x_{2k+1}$ to get

$$d(Q_0Ax_{2k}, Q_1x_{2k+1}) \leq_E \lambda u_1,$$

where

$$u_1 \in \left\{ \begin{array}{l} d(Q_0Ax_{2k}, A^2x_{2k}),\ \ d(Bx_{2k+1}, Q_1x_{2k+1}),\ d(A^2x_{2k}, Bx_{2k+1}), \\ \frac{1}{2}[d(Bx_{2k+1}, Q_0x_{2k}) + d(A^2x_{2k}, Q_1x_{2k+1})] \end{array} \right\}.$$

We have the following four cases:
$\mathbf{1}^0$: In this case, we have

$$d(Q_0Ax_{2k}, Q_1x_{2k+1}) \leq_E \lambda d(Q_0Ax_{2k}, A^2x_{2k}) \leq_E \lambda d(Q_0Ax_{2k}, Az) + \lambda d(Az, A^2x_{2k}).$$

By the fact that A is continuous, we get from (2.6), $A^2x_{2k} \longrightarrow Az$. Also, by (2.8), it follows that

$$\begin{aligned} d(Az, z) &\leq_E (1+\lambda)d(Q_0Ax_{2k}, Az) + \lambda d(Az, A^2x_{2k}) + d(Q_1x_{2k+1}, z) \\ &\ll (1+\lambda)\frac{c}{3(1+\lambda)} + \lambda\frac{c}{3\lambda} + \frac{c}{3} = \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c \end{aligned}$$

$\mathbf{2}^0$: Here, we have

$$d(Q_0Ax_{2k}, Q_1x_{2k+1}) \leq_E \lambda d(Q_1Ax_{2k+1}, Bx_{2k+1}) \leq_E \lambda d(Q_1x_{2k+1}, z) + \lambda d(z, Bx_{2k+1}).$$

Again, (2.8) yields

$$\begin{aligned} d(Az, z) &\leq_E d(Az, Q_0Ax_{2k}) + (1+\lambda)d(Q_1x_{2k+1}, z) + \lambda d(Bx_{2k+1}, z) \\ &\ll \frac{c}{3} + (1+\lambda)\frac{c}{3(1+\lambda)} + \lambda\frac{c}{3\lambda} = \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c. \end{aligned}$$

$\mathbf{3}^0$: In this case, we find

$$\begin{aligned} d(Q_0Ax_{2k}, Q_1x_{2k+1}) &\leq_E \lambda d(A^2x_{2k}, Bx_{2k+1}) \\ &\leq_E \lambda d(A^2x_{2k}, Az) + \lambda d(Az, z) + \lambda d(z, Bx_{2k+1}). \end{aligned}$$

The inequality (2.8) leads to

$$\begin{aligned} d(Az,z) &\preceq_E \frac{1}{1-\lambda} d(Az,Q_0Ax_{2k}) + \frac{\lambda}{1-\lambda} d(A^2x_{2k},Az) + \frac{\lambda}{1-\lambda} d(Bx_{2k+1},z) \\ &\quad + \frac{1}{1-\lambda} d(Q_1x_{2k+1},z) \\ &\ll \frac{1}{1-\lambda}\frac{(1-\lambda)c}{4} + \frac{1}{1-\lambda}\frac{(1-\lambda)c}{4} + \frac{1}{1-\lambda}\frac{(1-\lambda)c}{4} + (1-\lambda)\frac{c}{4(1-\lambda)} \\ &= c \end{aligned}$$

$\mathbf{4}^0$: In this case, we get

$$\begin{aligned} d(Q_0Ax_{2k},Q_1x_{2k+1}) &\preceq_E \frac{\lambda}{2} d(Q_0x_{2k},Bx_{2k+1}) + \frac{\lambda}{2} d(A^2x_{2k},Q_1x_{2k+1}) \\ &\preceq_E \frac{\lambda}{2}\left(d(Q_0x_{2k},Az) + d(Az,z) + d(z,Bx_{2k+1})\right) \\ &\quad + \frac{\lambda}{2}\left(d(A^2x_{2k},Az) + d(Az,z) + d(z,Q_1x_{2k+1})\right) \\ &= \frac{\lambda}{2}\left(d(Q_0x_{2k},Az) + d(z,Bx_{2k+1})\right) \\ &\quad + \frac{\lambda}{2}\left(d(A^2x_{2k},Az) + d(z,Q_1x_{2k+1})\right) + \lambda d(Az,z). \end{aligned}$$

By (2.8), we have

$$\begin{aligned} d(Az,z) &\preceq_E \frac{2+\lambda}{2(1-\lambda)} d(Az,Q_0Ax_{2k}) + \frac{\lambda}{2(1-\lambda)} d(A^2x_{2k},Az) \\ &\quad + \frac{\lambda}{2(1-\lambda)} d(Bx_{2k+1},z) + \frac{\lambda+1}{2(1-\lambda)} d(Q_1x_{2k+1},z) \\ &\ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c. \end{aligned}$$

In all the four above cases, $d(Az,z) \ll c$, then referring to Lemma 1.8, we find $Az = z$.

(b) Here, we prove that $Q_0z = z$. For this, from the condition (c_5), let us write for $x = z$ and $y = x_{2k+1}$

$$d(Q_0z,z) \preceq_E d(Q_0z,Q_1x_{2k+1}) + d(z,Q_1x_{2k+1}) \preceq_E \lambda u_2 + d(z,Q_1x_{2k+1}), \quad (2.9)$$

where

$$u_2 \in \left\{ \begin{array}{l} d(z,Q_0z),\ d(Q_1x_{2k+1},Bx_{2k+1}),\ d(z,Bx_{2k+1}), \\ \frac{1}{2}[d(Q_0z,Bx_{2k+1}) + d(Q_1x_{2k+1},z)] \end{array} \right\}.$$

Here, we have used $Az = z$. Also, we have the following four cases:

$\mathbf{1}^0$: In this case, using (2.9), we have

$$d(Q_0z,z) \preceq_E \lambda d(z,Q_0z) + d(Q_1x_{2k+1},z).$$

Therefore, referring to (2.6), we find

$$d(Q_0z,z) \preceq_E \frac{1}{1-\lambda} d(Q_1x_{2k+1},z) \ll \frac{1}{1-\lambda}(1-\lambda)c = c.$$

$\mathbf{2}^0$: Here, using the triangular inequality

$$\begin{aligned} d(Q_0z,z) &\preceq_E \lambda d(Q_1x_{2k+1},Bx_{2k+1}) + d(Q_1x_{2k+1},z) \\ &\preceq_E (\lambda+1)d(Q_1x_{2k+1},z) + \lambda d(Bx_{2k+1},z) \\ &\ll (\lambda+1)\frac{c}{2(1+\lambda)} + \lambda\frac{c}{2\lambda} = c. \end{aligned}$$

$\mathbf{3}^0$: Similarly we have

$$d(Q_0z,z) \preceq_E \lambda d(Bx_{2k+1},z) + d(Q_1x_{2k+1},z) \ll \lambda\frac{c}{2\lambda} + \lambda\frac{c}{2\lambda} = c.$$

$\mathbf{4}^0$: In this case, we get

$$\begin{aligned} d(Q_0z,z) &\preceq_E \frac{\lambda}{2}[d(Q_0z,Bx_{2k+1}) + d(Q_1x_{2k+1},z)] + d(Q_1x_{2k+1},z) \\ &\preceq_E \frac{\lambda}{2}d(Q_0z,z) + \frac{\lambda}{2}d(z,Bx_{2k+1}) + \left(\frac{\lambda}{2}+1\right)d(Q_1x_{2k+1},z) \\ &\preceq_E \frac{\lambda}{2}d(Q_0z,z) + \frac{1}{2}d(z,Bx_{2k+1}) + \frac{3}{2}d(Q_1x_{2k+1},z). \end{aligned}$$

It follows that

$$d(Q_0z,z) \preceq_E d(z,Bx_{2k+1}) + 3d(Q_1x_{2k+1},z) \ll \frac{c}{2} + 3\frac{c}{6} = c.$$

(c) Here we prove that $P_{2n}z = z$. From the condition (c_2), we have $Q_0P_{2n} = P_{2n}Q_0$ and $AP_{2n} = P_{2n}A$. Letting $x = P_{2n}z$ and $y = x_{2k+1}$ in the condition (c_5), we have

$$\begin{aligned} d(P_{2n}z,z) &= d(P_{2n}Q_0z,z) = d(Q_0P_{2n}z,z) \\ &\preceq_E d(Q_0P_{2n}z,Q_1x_{2k+1}) + d(Q_1x_{2k+1},z) \\ &\preceq_E \lambda u_3 + d(Q_1x_{2k+1},z) \end{aligned}$$

where

$$u_3 \in \left\{ \begin{array}{l} d(P_{2n}z,P_{2n}z) = 0_E,\ d(Q_1x_{2k+1},Bx_{2k+1}),\ d(P_{2n}z,Bx_{2k+1}), \\ \frac{1}{2}[d(P_{2n}z,Bx_{2k+1}) + d(P_{2n}z,Q_1x_{2k+1})] \end{array} \right\},$$

using $Q_0z = Az = z$. As the two precedent steps, we have the four cases:
$\mathbf{1}^0$:

$$d(P_{2n}z,z) \preceq_E \lambda d(P_{2n}z,P_{2n}z) + d(Q_1x_{2k+1},z) = d(Q_1x_{2k+1},z) \ll c.$$

$\mathbf{2}^0$:

$$\begin{aligned} d(P_{2n}z, z) &\leq_E \lambda d(Q_1 x_{2k+1}, Bx_{2k+1}) + d(Q_1 x_{2k+1}, z) \\ &\leq_E (\lambda+1) d(Q_1 x_{2k+1}, z) + \lambda d(Bx_{2k+1}, z) \\ &\ll (\lambda+1)\frac{c}{2(1+\lambda)} + \lambda \frac{c}{2\lambda} = c. \end{aligned}$$

$\mathbf{3}^0$:

$$\begin{aligned} d(P_{2n}z, z) &\leq_E \lambda d(P_{2n}z, Bx_{2k+1}) + d(Q_1 x_{2k+1}, z) \\ &\leq_E \lambda d(P_{2n}z, z) + \lambda d(z, Bx_{2k+1}) + d(Q_1 x_{2k+1}, z). \end{aligned}$$

It is clear that $d(P_{2n}z, z) \ll c$.
$\mathbf{4}^0$:

$$\begin{aligned} d(P_{2n}z, z) &\leq_E \frac{\lambda}{2}[d(P_{2n}z, Bx_{2k+1}) + d(Q_1 x_{2k+1}, P_{2n}z)] + d(Q_1 x_{2k+1}, z) \\ &\leq_E \frac{\lambda}{2} d(P_{2n}z, z) + \frac{\lambda}{2} d(z, Bx_{2k+1}) + d(Q_1 x_{2k+1}, z) \\ &\quad + \frac{\lambda}{2} d(Q_1 x_{2k+1}, z) + \frac{\lambda}{2} d(z, P_{2n}z). \end{aligned}$$

We obtain

$$d(P_{2n}z, z) \leq_E \frac{\lambda}{2(1-\lambda)} d(z, Bx_{2k+1}) + \frac{2+\lambda}{2(1-\lambda)} d(Q_1 x_{2k+1}, z) \ll c.$$

Hence, following the four cases, we obtain $P_{2n}z = z$.
(d) From the condition (c_1), $Q_0 X \subseteq BX$, hence there exists $v \in X$ such that $z = Q_0 z = Bv$. First, we need to show that $Bv = Q_1 v$. For this we have

$$d(Bv, Q_1 v) = d(Q_0 z, Q_1 v) \leq_E \lambda u_4,$$

where thanks to condition (c_5) with $x = z$ and $y = v$,

$$u_4 \in \left\{0_E,\ d(Bv, Q_1 v),\ 0_E,\ \frac{1}{2}[0_E + d(Bv, Q_1 v)]\right\}.$$

Since $d(Bv, Q_1 v) \leq_E \lambda d(Bv, Q_1 v)$ and $d(Bv, Q_1 v) \leq_E \frac{\lambda}{2} d(Bv, Q_1 v)$, which implies $d(Bv, Q_1 v) = 0$, hence $Bv = Q_1 v$. From the condition (c_3), (Q_1, B) is weakly compatible, then $Q_1 Bv = BQ_1 v$. We deduce $Q_1 z = Bz$.
(e) Here, we write

$$d(Q_1 z, z) = d(Q_1 z, Q_0 z) = d(Q_0 z, Q_1 z) \leq_E \lambda u_5,$$

where applying $x = y = z$ in condition (c_5), we obtain

$$u_5 \in \left\{d(Az, Q_0 z) = 0_E,\ d(Bz, Q_1 z) = 0_E,\ d(Az, Bz), \frac{1}{2}[d(Bz, Q_0 z) + d(Az, Q_1 z)]\right\}.$$

Then, since $Q_0z = Az = z$, we get

$$u_5 \in \left\{0_E,\ d(z, Bz),\ \frac{1}{2}[d(Bz, z) + d(z, Q_1z)]\right\}.$$

Proceeding as the precedent steps, we easily find $Q_1z = z = Bz$.
(f) We will check that $P_2z = z$. For this,

$$d(Q_0P_4 \cdots P_{2n}z, z) = d(Q_0P_4 \cdots P_{2n}z, Q_1z) \leq_E \lambda u_6.$$

Letting $x = P_4 \cdots P_{2n}z$ and $y = z$ in the condition (c_5), we have

$$u_6 \in \left\{ \begin{array}{l} d(AP_4 \cdots P_{2n}, Q_0P_4 \cdots P_{2n}),\ d(Bz, Q_1z),\ d(AP_4 \cdots P_{2n}z, Bz), \\ \frac{1}{2}[d(Bz, Q_0P_4 \cdots P_{2n}z) + d(AP_4 \cdots P_{2n}z, Q_1z)] \end{array} \right\}.$$

Thanks to condition (c_2), we have $AP_4 \cdots P_{2n} = P_4 \cdots P_{2n}A$ and $Q_0P_4 \cdots P_{2n} = P_4 \cdots P_{2n}Q_0$. Therefore, using $Bz = Q_1z = z = Az = Q_0z$, we deduce

$$u_6 \in \left\{ \begin{array}{l} 0_E,\ d(P_4 \cdots P_{2n}z, z), \\ \frac{1}{2}[d(z, P_4 \cdots P_{2n}z) + d(P_4 \cdots P_{2n}z, z)] \end{array} \right\}.$$

It is clear that $P_4 \cdots P_{2n}z = z$. Thus, $z = Az = P_2(P_4 \cdots P_{2n})z = P_2z$. Proceeding similarly, one can find

$$z = P_2z = P_4z = \cdots = P_{2n}z.$$

(g) We will prove that $P_{2n-1}z = z$. Since $P_{2n-1}Q_1 = Q_1P_{2n-1}$, we have

$$d(P_{2n-1}z, z) = d(P_{2n-1}Q_1z, Q_0z) = d(Q_0z, Q_1P_{2n-1}z) \leq_E \lambda u_7,$$

where $u_7 \in \{0_E, d(z, P_{2n-1}z)\}$. Here we have taken $x = z$ and $y = P_{2n-1}z$. It is obvious that $P_{2n-1}z = z$. Using the same strategy, it can be shown that

$$z = P_1z = P_3z = \cdots = P_{2n-1}z.$$

As a conclusion, we conclude

$$z = Q_0z = Q_1z = P_mz\ \forall\, m \in \mathbb{N}^*,\ \ 1 \leq m \leq 2n,$$

that is z is a coincidence point.
Case 2. If Q_0 is continuous. The mapping Q_0 is continuous, then $Q_0^2x_{2k} \longrightarrow Q_0z$ and $Q_0Ax_{2k} \longrightarrow Q_0z$, as $k \longrightarrow +\infty$. From the condition (c_3), the pair $\{Q_0, A\}$ is compatible, then for each $c \in E$ with $c \gg 0_E$

$$d(Q_0Ax_{2k}, AQ_0x_{2k}) \ll \frac{c}{2}.$$

It follows that $AQ_0x_{2k} \longrightarrow Q_0z$. Indeed

$$d(AQ_0x_{2k}, Q_0z) \leq_E d(AQ_0x_{2k}, Q_0Ax_{2k}) + d(Q_0Ax_{2k}, Q_0z) \ll \frac{c}{2} + \frac{c}{2} = c. \quad (2.10)$$

(h) We need to show that $Q_0z = z$. First, the triangular inequality gives us

$$\begin{aligned} d(Q_0z, z) &\leq_E d(Q_0z, Q_0^2x_{2k}) + d(Q_0^2x_{2k}, Q_1x_{2k+1}) + d(Q_1x_{2k+1}, z) \\ &\ll \frac{c}{3} + d(Q_0^2x_{2k}, Q_1x_{2k+1}) + \frac{c}{3}. \end{aligned} \tag{2.11}$$

It rests only the control of the term $d(Q_0^2x_{2k}, Q_1x_{2k+1})$. To do this, we apply condition (c_5) for $x = Q_0x_{2k}$ and $y = x_{2k+1}$ to get

$$d(Q_0^2x_{2k}, Q_1x_{2k+1}) \leq_E \lambda u_8,$$

where

$$u_8 \in \left\{ \begin{array}{l} d(AQ_0x_{2k}, Q_0^2x_{2k}),\ d(Bx_{2k+1}, Q_1x_{2k+1}),\ d(AQ_0x_{2k}, Bx_{2k+1}), \\ \frac{1}{2}[d(Bx_{2k+1}, Q_0^2x_{2k}) + d(AQ_0x_{2k}, Q_1x_{2k+1})] \end{array} \right\}.$$

We have four cases. We need to do the first case, the others will be the same. For this,

$$\begin{aligned} d(Q_0^2x_{2k}, Q_1x_{2k+1}) &\leq_E \lambda u_8 = \lambda d(AQ_0x_{2k}, Q_0^2x_{2k}) \\ &\leq_E \lambda d(AQ_0x_{2k}, Q_0z) + \lambda d(Q_0z, Q_0^2x_{2k}) \\ &\ll \lambda\frac{c}{6\lambda} + \lambda\frac{c}{6\lambda} = \frac{c}{3}. \end{aligned}$$

Combining this to (2.11) yields that $Q_0z = z$. Now, using similarly steps (d), (e) and (g) leads to

$$Q_1z = P_1z = P_3z = \cdots = P_{2n-1}z = z.$$

(i) We know that $Q_1X \subseteq AX$, there exists $w \in X$ such that $z = Q_1z = Aw$. We shall show that $Q_0w = Aw$. For this we write

$$d(Q_0w, Aw) = d(Q_w, Q_1z) \leq_E \lambda u_9$$

where $u_9 \in \{0_E,\ \frac{1}{2}d(Q_0w, z)\}$, that is, $d(Q_0w, Aw) = d(Q_0w, z) = \frac{\lambda}{2}d(Q_0w, z)$, i.e, $Q_0w = Aw = z$. Since (Q_0, A) is weakly compatible, we have $z = Q_0z = Az$. Then, $P_{2n}z = z$ follows from step (c), and from this and step (f), $P_2z = P_4z = \cdots = P_{2n}z = z$. As a consequence, even in this second case, z is a common fixed point of Q_0, Q_1 and all the $2n$ mappings (P_k), $k = 1, 2, ..., 2n$.

Step 4: Proof of uniqueness. Let z_1 be an other common fixed point of the above maps, then

$$z_1 = Q_0z_1 = Q_1z_1 = P_mz_1\ \forall\, m \in \mathbb{N},\ 1 \leq m \leq 2n.$$

We put $x = z$ and $y = z_1$ in the condition (c_5), then

$$d(z, z_1) = d(Q_0z, Q_1z_1) \leq_E \lambda u_{10}$$

where

$$u_{10} \in \left\{d(Az, Q_0z),\ d(Bz_1, Q_1z_1),\ d(Az, Bz_1),\ \frac{1}{2}[d(Bz_1, Q_0z) + d(Az, Q_1z_1)\right\}.$$

We get $u_{10} \in \{0_E,\ d(z,z_1)\}$. If $u_{10} = 0_E$, then $d(z,z_1) = 0_E$, so $z = z_1$. While, if $u_{10} = d(z,z_1)$, then $d(z,z_1) \leq_E \lambda d(z,z_1)$. Following lemma 1.12, we obtain $z = z_1$. Hence, there is a unique common fixed point. The proof of Theorem 2.1 is completed. □

Our main theorem is an extension of a recent result of Janković et al. [20]. In fact when we let $n = 2$ in Theorem 2.1, we find Theorem 2.1 of [20], which is

Theorem 2.2. *Let A, B, S, T, L and M be self-maps in the complete cone metric space (X, d) satisfying the conditions:*

(i) *$LX \subset ST(X)$ and $M(X) \subset AB(X)$,*

(ii) *$AB = BA$, $ST = TS$, $LB = BL$ and $MT = TM$,*

(iii) *for some $\lambda \in [0,1)$ and for all $x, y \in X$, there exists*

$$u(x,y) \in \left\{ \begin{array}{l} d(Lx, ABx), d(My, STy), d(ABx, STy), \\ \frac{1}{2}(d(Lx, STy) + d(ABx, My)) \end{array} \right\}$$

such that $d(Lx, My) \leq_E \lambda u(x,y)$,

(iv) *the pair $\{L, AB\}$ is compatible and the pair $\{M, ST\}$ is weakly compatible,*

(v) *either AB or L is continuous.*

Then A, B, S, T, L and M have a unique common fixed point.

Proof. It suffice to take $Q_0 = L$, $Q_1 = M$, $P_1 = S$, $P_3 = T$, $P_2 = A$ and $P_4 = B$ in Theorem 2.1. □

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