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Between Preopen and Open Sets in Topological Spaces

Halgwrd M. Darwesh

Department of Mathematics, College of Science
University of Sulaimani, Sulaimani, Kurdistan Region, Iraq
e-mail : darweshymath@yahoo.com

Abstract : The aim of the present paper is to introduce the class of ω_p-open sets which is strictly placed between the class of all preopen and the class of all open subsets of X. Relationships with some other type of sets are given. Furthermore, we introduce a new type of continuity termed the class of ω-almost continuous functions which is lies strictly between the class of all continuous and the class of all almost-continuous functions. New decompositions of continuity are obtained.

Keywords : ω-open set; preopen set; ω_p-open set; almost continuous functions; ω-almost continuous functions.
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1 Introduction

In 1966, Husain [1] introduced the notion of almost continuous function. The notion of preopen sets and precontinuity introduced by Mashhour et al. [2] in 1982. They showed where that the notion of almost continuity and precontinuity are equivalent. In 1968, Veličko [3] introduced the class of δ-open sets to give more characterization of H-closed spaces. In 1982, Hdeib [4] introduced the notion of ω-open sets. In 1993, Raychaudhuri et al. [5] used the notion of δ-open sets to introduce the notion of δ-preopen sets and δ-almost continuous functions as a strictly weaker forms of preopen sets and almost continuous functions in the sense of Husain [1]. As continuation of this work, in Section 3, we introduce and study a new class of sets namely ω_p-open sets which is properly placed between the class of preopen sets and the class of open sets. In Section 4, the class of ω-almost continuous function are introduced and investigated. This class of almost continuity is strictly stronger than almost continuity according to Husain and strictly weaker than continuity. Finally, in section 5, some decompositions of continuity are obtained.

2 Preliminaries

Throughout this paper $(X,\Im)$ (simply X) represent non-empty topological spaces (briefly, spaces) on which no separation axioms are assumed unless explicitly stated. If A is a subset of X, then the closure and interior of A in X are denoted by ClA and $IntA$, respectively. A subset U of a space X is called δ-open [3] (resp. ω-open [4]) if for each $x \in U$, there exists an open set G containing x such that $IntClG \subseteq U$ (resp. $G - U$ is countable). In a space $(X,\Im)$ the family of all δ-open and ω-open sets are denoted by $\Im_\delta$ and $\Im^\omega$, respectively and it is well known that $\Im_\delta \subset \Im \subset \Im^\omega$. The complement of a δ-open (resp. ω-open) set is called δ-closed (resp. ω-closed). For a subset A of a space X, the δ-closure (resp. δ-interior, ω-closure and ω-interior) is denoted by $Cl_\delta A$ (resp. $Int_\delta A$, $Cl_\omega A$ and $Int_\omega A$). A subset A of a space X is said to be preopen [2] (resp. δ-preopen [5], pre-ω-open [6]) if $A \subseteq IntClA$ (resp. $A \subseteq IntCl_\delta A$, $A \subseteq Int_\omega ClA$). The family of all preopen (resp. δ-preopen, pre-ω-open) subsets of a space X will be denoted by $PO(X)$ (resp. $\delta PO(X)$, PωO(X)).

Definition 2.1 ([7])**.** A space X is said to *locally countable* if for each $x \in X$, there exists a countable open subset G of X containing x.

Definition 2.2 ([7])**.** A space X is said to *anti-locally countable* if each non-empty open subset of X is uncountable.

Lemma 2.3 ([7])**.** *If a space X is an anti-locally countable space, then:*

1. $Cl_\omega A = ClA$ *for each ω-open subset of X.*
2. $Int_\omega A = IntA$ *for each ω-closed subset of X.*

Lemma 2.4 ([8])**.** *If $(A,\Im_A)$ is an anti-locally countable subspace of a space X, then $Cl_\omega A = ClA$.*

Lemma 2.5 ([6])**.** *For a subset A, if G is open then $Cl(G \cap A) = Cl(G \cap ClA)$, and hence $G \cap ClA \subseteq Cl(G \cap A)$.*

As an immediate consequence of the above lemma we have:

Corollary 2.6. *For a subset A, if G is ω-open then $Cl_\omega(G \cap A) = Cl_\omega(G \cap Cl_\omega A)$, and hence $G \cap Cl_\omega A \subseteq Cl_\omega(G \cap A)$.*

Lemma 2.7 ([7, 8])**.** *If A is any subset of a space $(X,\Im)$, then $(\Im_A)^\omega = (\Im^\omega)_A$.*

Definition 2.8 ([9])**.** A subset A of a space X is said to be *ω-dense* if $Cl_\omega A = X$.

Definition 2.9 ([1])**.** A function $f : X \to Y$ is said to be *almost continuous in the sense of Husain* if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $U \subseteq Cl f^{-1}(V)$.

Definition 2.10. A function $f : X \to Y$ is said to be *precontinuous* [2] (resp. pre-ω-continuous [6]) if $f^{-1}(V)$ is preopen (resp. pre-ω-open) in X for each open subset V of Y.

Mashhor et al. [2] proved that the almost continuity in the sense of Husain and precontinuity are equivalent.

3 ω_p-Open Sets

Definition 3.1. A subset A of a space X is said to be *ω_p-open* if $A \subseteq IntCl_\omega A$.

The complement of such set is called ω_p-closed. The collection of all ω_p-open (resp. ω_p-closed) subsets of X will be denoted by ω_pO(X) (resp. ω_pC(X)).

Remark 3.2. *It is obvious that:*

1. *Every open set is ω_p-open.*
2. *Every ω_p-open is preopen, δ-preopen and pre-ω-open.*

The following example shows that the converses of neither parts of the above remark are true.

Example 3.3. *Consider the set of all real numbers* $\mathbf{R}$ *with the indiscrete topology τ, then the set $A = \{0\}$ is preopen, δ-preopen and pre-ω-open but not ω_p-open. However, the set $B = (0,1)$ is ω_p-open but it is neither ω-open nor open.*

The following example with the above example show that the ω-openness and ω_p-openness are independent topological concepts.

Example 3.4. *Let the set of all natural numbers* $\mathbf{N}$ *be equipped with the topology* $\Im = \{\mathbf{N}\} \bigcup \{A \subseteq \mathbf{N} : 0 \notin A\}$. *Then the set $A = \{0\}$ is ω-open and pre-ω-open but it is neither ω_p-open, preopen, δ-preopen nor open.*

However, the following example with Example 3.3 show that the δ-preopenness and pre-ω-openness are independent topological concepts.

Example 3.5. *Consider the uncountable point included topology* $\Im = \{\phi\} \bigcup \{G \subseteq \mathbf{R} : (0,1) \subseteq G\}$ *on* $\mathbf{R}$ [10, Example 10 , p.44]. *Then $\Im_\delta = \{\phi, \mathbf{R}\}$, and hence every subset of* $\mathbf{R}$ *is δ-preopen. While, the set $\{0\}$ is not pre-ω-open.*

We say a subset A of a space $(X, \Im)$ is a preω-open subset of X, if it is a preopen subset of $(X, \Im^\omega)$. The family of all preω-open subsets of a space X denotes by $PO(X^\omega)$.

Remark 3.6. *We notice that:*

1. *Every ω_p-open set is preω-open, but not conversely. Since the set $A = \{0\}$ in Example 3.4 is preω-open but it is not ω_p-open.*
2. *Every preω-open is pre-ω-open, but not conversely. Since the set $\{\frac{1}{2}\}$ in Example 3.5 is pre-ω-open, but it is not preω-open.*
3. *The concepts of preopenness and preω-openness are independent as well as the δ-preopenness and preω-openness. Since in Example 3.3, the set $\{0\}$ is preopen and δ-preopen but it is not preω-open. However, in Example 3.4, the set $\{0\}$ is preω-open but it is neither preopen nor δ-preopen.*

The following diagram indicates the relationships among the above type of sets:

$$\begin{array}{ccccccc} & & \omega-open & \Rightarrow & pre\omega-open & \Rightarrow & pre-\omega-open \\ & \nearrow & \not\updownarrow & \nearrow & \not\updownarrow & \nearrow & \not\updownarrow \\ open & \Rightarrow & \omega_p-open & \Rightarrow & preopen & \Rightarrow & \delta-preopen \end{array}$$

where A $\Rightarrow$ B (resp. A $\nLeftrightarrow$ B) means A implies B (resp. A and B are independent).

Proposition 3.7. *A subset A of a space X is ω_p-closed if and only if $ClInt_\omega A \subseteq A$.*

Proof. A is ω_p-closed if and only if $X - A$ is ω_p-open if and only if $X - A \subseteq IntCl_\omega(X - A)$ if and only if $ClInt_\omega A \subseteq A$. □

Proposition 3.8. *Arbitrary union (intersection) of ω_p-open (ω_p-closed) sets is ω_p-open (ω_p-closed).*

Proof. Straightforward. □

Remark 3.9. *The intersection of even two ω_p-open sets may not be ω_p-open. Thus the family of ω_p-open sets in a space X does not always form a topology on X. For example, the sets $A = [-1, 0]$ and $B = [0, 1]$ in Example 3.3 are ω_p-open sets but $A \bigcap B = \{0\}$ is not ω_p-open.*

Proposition 3.10. *If X is an anti-locally countable space, then $\omega_pO(X) = PO(X^\omega)$.*

Proof. $A \in \omega_pO(X)$ if and only if $A \subseteq IntCl_\omega A$ {by Definition 3.1} if and only if $A \subseteq Int_\omega Cl_\omega A$ {by Lemma 2.3} if and only if $A \in PO(X^\omega)$. □

Proposition 3.11. *If X is an anti-locally countable space, then $PO(X) = P\omega O(X)$.*

Proof. The proof is analogous to the proof of Proposition 3.10. □

In view of Example 3.3, we notice that it is not necessary that $\omega_pO(X) = PO(X)$ or $PO(X) = PO(X^\omega)$ even in an anti-locally countable space.

Proposition 3.12. *Let $(A, \Im_A)$ be an anti-locally countable subspace of a space $(X, \Im)$. Then*

1. *A is ω_p-open if and only if it is preopen.*
2. *A is preω-open if and only if it is pre-ω-open.*

Proof. (1) A is ω_p-open if and only if $A \subseteq IntCl_\omega A$ if and only if $A \subseteq IntClA$ {by Lemma 2.4} if and only if A is preopen.

(2) The proof is analogous to the first part. □

Theorem 3.13. *A singleton subset of a space X is ω_p-open if and only if it is open.*

Proof. Let $\{x\}$ be an ω_p-open subset of X. Then $\{x\} \subseteq IntCl_\omega \{x\}$. Since each singleton subset of any space is ω-closed, then $Cl_\omega \{x\} = \{x\}$. Thus $\{x\} \subseteq Int \{x\}$. Hence $\{x\}$ is open. The proof of the converse part is followed by part (1) of Remark 3.2. □

Corollary 3.14. *A space X is discrete if and only if every subset of X is ω_p-open.*

Proof. It is a direct consequence of Theorem 3.13. □

Analogous to the above results we can obtain the following result:

Proposition 3.15. *A singleton subset of a space X is preω-open if and only if it is ω-open. Hence, a space X is locally countable if and only if every subset of X is preω-open.*

Proof. Similar to the proof of Theorem 3.13 and Corollary 3.14. □

Theorem 3.16. *A subset A of a space X is ω_p-open if and only if it is the intersection of an open and an ω-dense subset of X.*

Proof. Let A be an ω_p-open subset of X. Then $A \subseteq IntCl_\omega A$. Put $G = IntCl_\omega A$ and $D = (X - IntCl_\omega A) \bigcup A$, then G is an open subset of X and $A = G \bigcap D$, so we have only to show D is an ω-dense subset of X. Since $Cl_\omega D = Cl_\omega[(X - IntCl_\omega A) \bigcup A] = Cl_\omega(X - IntCl_\omega A) \bigcup Cl_\omega A = Cl_\omega ClInt_\omega(X-A) \bigcup Cl_\omega A \supseteq Int_\omega(X-A) \bigcup Cl_\omega A = X$. Thus D is an ω-dense subset of X. Conversely, let $A = G \bigcap D$, where G and D are open and ω-dense subsets of X, respectively. We have to show $G \subseteq Cl_\omega(G \bigcap D)$. Suppose that $G \nsubseteq Cl_\omega(G \bigcap D)$, then there exists a point $y \in G$ but $y \notin Cl_\omega(G \bigcap D)$. This means that there exists an ω-open subset V of X containing y such that $V \bigcap (G \bigcap D) = \phi$. Hence $(V \bigcap G) \bigcap D = \phi$. This implies that $y \notin Cl_\omega D = X$ which is impossible. Therefore, $G \subseteq Cl_\omega(G \bigcap D)$, and hence $A \subseteq G \subseteq IntCl_\omega G \subseteq IntCl_\omega(G \bigcap D) = IntCl_\omega A$. Thus A is ω_p-open. □

Theorem 3.17. *A subset A of a space X is ω_p-open if and only if there exists an open subset G of X such that $A \subseteq G \subseteq Cl_\omega A$.*

Proof. It is very simple, so we omitted. □

Theorem 3.18. *A subset A of a space X is ω_p-open if and only if $G \bigcap A$ is ω_p-open for each open subset G of X.*

Proof. Let A be an ω_p-open subset of X. Let G be any open subset of X. Then $G \bigcap A \subseteq G \bigcap IntCl_\omega A$ {since A is ω_p-open} $= Int(G \bigcap Cl_\omega A) \subseteq IntCl_\omega(G \bigcap A)$ {by Corollary 2.6}. Hence $G \bigcap A$ is ω_p-open. The other part is followed from $A = X \bigcap A$. □

Corollary 3.19. *The intersection of an open set and an ω_p-open set is ω_p-open.*

Theorem 3.20. *If A is ω_p-open and $U \subseteq A \subseteq Cl_\omega U$, then U is ω_p-open.*

Proof. Let A be ω_p-open and $U \subseteq A \subseteq Cl_\omega U$. Then by Theorem 3.17, there exists an open subset V of X such that $A \subseteq V \subseteq Cl_\omega A$. This implies that $U \subseteq V \subseteq Cl_\omega U$. Hence by Theorem 3.17, U is ω_p-open. □

If A is a subspace of a space X and $B \subseteq A$, then the ω-closure and ω-interior of B in A are denoted by $Cl^A_\omega B$ and $Int^A_\omega B$, respectively.

Theorem 3.21. *Let $A \in \omega_p O(X)$ and $V \in \omega_p O(A)$. Then $V \in \omega_p O(X)$.*

Proof. Let $A \in \omega_p O(X)$ and $V \in \omega_p O(A)$. Then by Theorem 3.17, there are two open subsets G and O of X such that $A \subseteq G \subseteq Cl_\omega A$ and $V \subseteq O \bigcap A \subseteq Cl^A_\omega V$. It follows from Lemma 2.7, $Cl^A_\omega V = Cl_\omega V \bigcap A$. Since A is ω_p-open and O is open, then by Corollary 3.19, $O \bigcap A$ is an ω_p-open subset of X. Hence by Theorem 3.17, there exists an open subset U of X such that $O \bigcap A \subseteq U \subseteq Cl_\omega(O \bigcap A)$. Thus $V \subseteq U \subseteq Cl_\omega(O \bigcap A) \subseteq Cl_\omega(Cl_\omega V \bigcap A) \subseteq Cl_\omega V$. So by Theorem 3.17, $V \in \omega_p O(X)$. □

Theorem 3.22. *If $B \in \omega_p O(X)$ and $B \subseteq A \subseteq X$, then $B \in \omega_p O(A)$.*

Proof. Straightforward. □

Definition 3.23. The intersection (union) of all ω_p-closed (ω_p-open) subsets of X containing (contained in) A is called *the ω_p-closure (ω_p-interior) of A* and is denoted by $\omega_p ClA$ ($\omega_p IntA$).

Proposition 3.24. *For subsets A and B of a space X, we have:*

1. *A is ω_p-closed (ω_p-open) if and only if $\omega_p ClA = A$ ($\omega_p IntA = A$).*
2. *If $A \subseteq B$, then $\omega_p ClA \subseteq \omega_p ClB$ and $\omega_p IntA \subseteq \omega_p IntA$.*
3. *$\omega_p ClA$ ($\omega_p IntA$) is $\omega_p -$ closed ($\omega_p -$ open).*
4. *$\omega_p Cl(\omega_p ClA) = \omega_p ClA$ and $\omega_p Int(\omega_p IntA) = \omega_p IntA$.*
5. *$\omega_p Cl(X - A) = X - \omega_p IntA$ and $\omega_p Int(X - A) = X - \omega_p ClA$.*
6. *$x \in \omega_p ClA$ ($x \in \omega_p IntA$) if and only if for each (there exists an) $\omega_p -$ open set V containing x, $V \bigcap A \neq \phi$ ($V \subseteq A$).*

7. $\omega_pClA = A\bigcup ClInt_\omega A$ *and* $\omega_pIntA = A\bigcap IntCl_\omega A$.

Proof. The proof of (1)-(6) are followed by using [11, Lemma 3.1 and Lemma 3.3]. We only write the proof of part (7). Since $ClInt_\omega(A\bigcup ClInt_\omega\ A) \subseteq ClInt_\omega A\bigcup ClInt_\omega ClInt_\omega A = ClInt_\omega A \subseteq A\bigcup ClInt_\omega A$, then by Proposition 3.7, we have $A\bigcup ClInt_\omega A$ is an ω_p-closed set containing A, Therefore, by definition of ω_pClA, we obtain $\omega_pClA \subseteq A\bigcup ClInt_\omega A$. Conversely, let x be any point of $A\bigcup ClInt_\omega A$. Then $x \in A$ or $x \in ClInt_\omega A$. If $x \in A$, then $x \in \omega_pClA\{$ Since $A \subseteq \omega_pClA\}$. Now, If $x \in ClInt_\omega A$, let V be any ω_p-open subset of X containing x. Then by Theorem 3.17, there exists an open set G contains x such that $V \subseteq G \subseteq Cl_\omega V$. Since $x \in ClInt_\omega A$, then $G\bigcap Int_\omega A \neq \phi$, so $V\bigcap Int_\omega A \neq \phi$, therefore $V\bigcap A \neq \phi$. Thus $x \in \omega_pClA$, and hence $A\bigcup ClInt_\omega A \subseteq \omega_pClA$. So $\omega_pClA = A\bigcup ClInt_\omega A$. Then by using part (5), we obtain $\omega_pIntA = A\bigcap IntCl_\omega A$. □

Theorem 3.25. *If A is an ω-open (ω-closed) subset of a space X, then $\omega_pClA = ClA$ ($\omega_pIntA = IntA$).*

Proof. If A is ω-open, then by part (7) of Proposition 3.24, $\omega_pClA = A\bigcup ClInt_\omega A = A\bigcup ClA = ClA$. Also, if A is ω-closed, then by part (7) of Proposition 3.24, $\omega_pIntA = A\bigcap IntCl_\omega A = A\bigcap IntA = IntA$. □

4 ω-Almost Continuous Functions

For a function $f : X \to Y$, it is well known that the continuity of f can be characterized by the condition $f(ClA) \subseteq Clf(A)$ for every subset A of X, whereas almost continuity is equivalent to the same condition when A is restricted to lie in the class of open subsets of X (see [12]). However, the δ-almost continuity is equivalent to the same condition when A is more restricted to lie in the class of δ-open subsets of X. This motivates us to define ω-almost continuous function which is equivalent to the same condition whereas A is taken through the class of all ω-open subsets of X.

Definition 4.1. A function $f : X \to Y$ is said to be *ω-almost continuous*, if $f(ClA) \subseteq Clf(A)$ for every ω-open subset A of X.

Theorem 4.2. *For a function $f : X \to Y$, the following are equivalent:*

1. *f is ω-almost continuous.*
2. *$f^{-1}(G)$ is ω_p-open, for each open set G of Y.*
3. *For each $x \in X$ and each neighborhood G of $f(x)$ in Y, $Cl_\omega f^{-1}(G)$ is a neighborhood of x in X.*
4. *For each $x \in X$ and each neighborhood G of $f(x)$ in Y, there exists an ω_p-open set S such that $x \in S \subseteq Cl_\omega f^{-1}(G)$.*

Proof. (1) ⇒ (2) Let G be any open subset of Y. Then $X - Cl_\omega f^{-1}(G)$ is an ω-open subset of X. So, by ω-almost continuity of f, we have $f(Cl(X - Cl_\omega f^{-1}(G))) \subseteq Cl(f(X - Cl_\omega f^{-1}(G))) \subseteq Y - Int(f(Cl_\omega f^{-1}(G)))$. This implies that $X - IntCl_\omega f^{-1}(G) \subseteq f^{-1}(f(X - IntCl_\omega f^{-1}(G))) \subseteq f^{-1}(Y - Int(f(Cl_\omega f^{-1}(G)))) = X - f^{-1}(Int(Cl_\omega f^{-1}(G))) \subseteq X - f^{-1}(G)$. Therefore, $f^{-1}(G) \subseteq IntCl_\omega f^{-1}(G)$. Thus $f^{-1}(G)$ is ω_p-open.

(2) ⇒ (3) It followes by using Theorem 3.17.

(3) ⇒ (1) Let V be an ω-open subset of X. Let $x \in ClV$ but $f(x) \notin Clf(V)$. Then there exists an open subset G of Y containing $f(x)$ such that $G \bigcap f(V) = \phi$. Hence $f^{-1}(G) \bigcap V = \phi$. Since V is ω-open, then $Cl_\omega f^{-1}(G) \bigcap V = \phi$. Since $Cl_\omega f^{-1}(G)$ is a neighborhood of x. This implies that, $x \notin ClV$ which is against. Thus $f(ClV) \subseteq Clf(V)$. Hence f is ω-almost continuous.

(3) ⇔ (4) It is clear. □

We say a function $f : (X, \Im) \to (Y, \tau)$ is ω^*-almost continuous, if $f : (X, \Im^\omega) \to (Y, \tau)$ is almost continuous function. That is, a function $f : X \to Y$ is ω^*-almost continuous if and only if $f(Cl_\omega A) \subseteq Clf(A)$ for each ω-open subset A of X.

Remark 4.3. *It is obvious that:*

1. *Every continuous function is ω-almost continuous.*
2. *Every ω-almost continuous function is almost continuous, δ-almost continuous, ω^*-almost continuous and pre-ω-continuous.*
3. *Every ω-continuous function is ω^*-almost continuous function.*

The following examples show that the converse of neither part of the above remark is true.

Example 4.4. *Let τ and $\mathscr{U}$ be the indiscrete and usual topologies on R. Then identity function $f : (R, \tau) \to (R, \mathscr{U})$ is ω-almost continuous, and hence it is ω^*-continuous but it is neither continuous nor ω-continuous.*

Example 4.5. *Let τ and $\Im$ be the indiscrete and discrete topologies on R. Then identity function $f : (R, \tau) \to (R, \Im)$ is δ-almost-continuous, almost-continuous and pre-ω-continuous, but it is neither ω-almost continuous nor ω^*-almost continuous.*

Example 4.6. *Let (N, τ) be the topological space which given in example 3.4, and let $\Im$ be the discrete topologies on N, then the identity function $f : (N, \tau) \to (N, \Im)$ is ω^*-almost continuous but it is not ω-almost continuous. However, it is also ω-continuous.*

The following example with example 4.4 show that the ω-almost continuity and ω-continuity are independent topological concepts.

Example 4.7. *Consider the identity function $f : (\mathbf{N}, \tau) \to (\mathbf{N}, \Im)$, where τ and $\Im$ are the indiscrete and discrete topologies on $\mathbf{N}$, then f is ω-continuous, but it is not ω-almost continuous.*

Theorem 4.8. *For a function $f : X \to Y$, the following statements are equivalent:*

1. *f is ω-almost continuous.*
2. *for each $x \in X$ and each open set G of Y containing $f(x)$, there exists an ω_p-open set U of X containing x such that $f(U) \subseteq G$.*
3. *$f^{-1}(F)$ is ω_p-closed, for each closed set F of Y.*
4. *$f(\omega_p ClA) \subseteq Clf(A)$, for each subset A of X.*
5. *$\omega_p Clf^{-1}(B) \subseteq f^{-1}(ClB)$, for each subset B of Y.*
6. *$f^{-1}(intB) \subseteq \omega_p Intf^{-1}(B)$, for each subset B of Y.*

Proof. Straightforward. □

Theorem 4.9. *Let $f : X \to Y$ be a function from an anti-locally countable space X into a space Y. Then*

1. *f is ω-almost continuous if and only if it is ω^*-almost continuous.*
2. *f is almost continuous if and only if it is pre-ω-continuous.*

Proof. (1) It follows from Proposition 3.10 and Theorem 4.2.
(2) It follows from Proposition 3.11. □

Proposition 4.10. *Let f be an injective function from a space X into a discrete space Y. Then f is ω-almost continuous if and only if X is a discrete space.*

Proof. It is follows from Theorem 3.13 and Theorem 4.2. □

Theorem 4.11. *If $f : X \to Y$ is an ω-almost continuous function and U an open subset of X, then $f_{/U} : U \to Y$ is ω-almost continuous.*

Proof. Let G be any open subset of Y. Then by ω-almost continuity of f and Theorem 4.2, we have $f^{-1}(G)$ is an ω_p-open subset of X. Since U is open, then by Corollary 3.19, $f_{/U}^{-1}(G) = f^{-1}(G) \bigcap U$ is an ω_p-open subset of X, and hence by Theorem 3.22, $f_{/U}^{-1}(G)$ is an ω_p-open subset of U. That is, by Theorem 4.2, $f_{/U}$ is ω-almost continuous. □

Theorem 4.12. *Let $f : X \to Y$ and let $\{U_\lambda : \lambda \in \Lambda\}$ be any ω_p-open cover X. Then f is ω-almost continuous, if $f_{/U_\lambda} : U_\lambda \to Y$ is ω-almost continuous for each $\lambda \in \Lambda$.*

Proof. Let G be any open subset of Y. Then $f^{-1}_{/U_\lambda}(G)$ is an ω_p-open subset of U_λ, for each $\lambda \in \Lambda$. Since U_λ is ω_p-open, for each $\lambda \in \Lambda$, so by Theorem 3.21 $f^{-1}_{/U_\lambda}(G)$ is an ω_p-open subset of X, for each $\lambda \in \Lambda$. Since $f^{-1}(G) = \bigcup_{\lambda \in \lambda} f^{-1}_{/U_\lambda}(G)$, then by Proposition 3.8, $f^{-1}(G)$ is an ω_p-open subset of X. Hence f is ω-almost continuous. □

Theorem 4.13. *Let Y be any regular space and let X be any space. Then a function $f : X \to Y$ is continuous if and only if it is ω-almost continuous.*

Proof. Let x be any point of X and let G be any open subset of Y containing $f(x)$. Since Y is regular, then there exists an open subset U of Y such that $f(x) \in U \subseteq ClU \subseteq G$. Since f is ω-almost continuous, then by Theorem 4.2 and Theorem 4.8, $Cl_\omega f(U)$ is a neighborhood of x in X and $Cl_\omega f^{-1}(U) \subseteq f^{-1}(ClU)$. Therefore, $x \in IntCl_\omega f^{-1}(U) \subseteq f^{-1}(G)$. Thus $f^{-1}(G)$ is a neighborhood of x in X. This implies that f is continuous. The converse part is followed by Remark 4.3. □

5 Some Decompositions of Continuity

In this section, we shall obtain some decomposition of some types of continuity. For this we recall some definitions:

Definition 5.1. A subset A of a topological space X is said to be *regular closed* [13] (resp. semi-closed [14] or t-set [15], semi-regular [16], ω-semi-open [17]) if $A = ClIntA$ (resp. $IntClA \subseteq A$, $IntClA \subseteq A \subseteq ClIntA$, $A \subseteq ClInt_\omega A$).

Remark 5.2. *The complement of ω-semi-open sets are called ω-semi-closed. It is easy to see that, a subset A of a space X is ω-semi-closed if and only if $IntCl_\omega A \subseteq A$ if and only if $IntCl_\omega A = IntA$.*

Definition 5.3. A subset A of a space X is said to be *locally closed* [18] (resp. $\mathcal{A}$-set[19], $\mathcal{B}$-set [15], $\mathcal{AB}$-set [12], $\mathcal{B}^\omega$-set) if $A = G \bigcap F$, where G is open and F is closed (resp. regular closed, semi-closed, semi-regular, ω-semi-closed).

Remark 5.4. *It is easy to see that every semi-closed set is ω-semi-closed. Hence every $\mathcal{B}$-set is $\mathcal{B}^\omega$-set.*

Definition 5.5. A function $f : X \to Y$ is said to be *contra-continuous* [20] (resp. contra-ω-continuous [9] or co-ω-continuous [21], LC-continuous [18], $\mathcal{A}$-continuous [19], $\mathcal{B}$-continuous [15], $\mathcal{AB}$-continuous [16], $\mathcal{B}^\omega$-continuous, perfectly continuous [22] or totally continuous [14]) if the inverse image of any open subset of Y is a closed (resp. ω-closed, locally closed, $\mathcal{A}$-set, $\mathcal{B}$-set, $\mathcal{AB}$-set, $\mathcal{B}^\omega$-set, clopen) subset of X.

Theorem 5.6. *A subset A of a space X is open if and only if it is ω_p-open and $\mathcal{B}^\omega$-set.*

Proof. If A is an open set, then by part (1) of Remark 3.2, the set A is ω_p-open. Now, Since $IntCl_\omega X \subseteq X$, so X is ω-semi-closed, and hence $A = A \bigcap X$ is $\mathcal{B}^\omega$-set. Conversely, let A be ω_p-open and $\mathcal{B}^\omega$-set. $A \subseteq IntCl_\omega A$ and $A = G \bigcap H$, for some open set G and ω-semi-closed set H. Then $A = A \bigcap IntCl_\omega A \subseteq G \bigcap IntCl_\omega(G \bigcap H) \subseteq G \bigcap IntCl_\omega G \bigcap IntCl_\omega H = G \bigcap IntH$ {by Remark 5.2} $= Int(G \bigcap H) = IntA$. Hence A is open. □

Corollary 5.7. *For a subset A of a space X, the following statements are equivalent:*

1. *A is open.*
2. *A is ω_p-open and locally closed.*
3. *A is ω_p-open and $\mathcal{A}$-set.*
4. *A is ω_p-open and $\mathcal{AB}$-set.*
5. *A is ω_p-open and $\mathcal{B}$-set.*

Proof. It is followed from the implications (locally closed $\Rightarrow$ B-set $\Rightarrow$ $\mathcal{B}^\omega$) and ($\mathcal{A}$-set $\Rightarrow$ $\mathcal{AB}$-set $\Rightarrow$ $\mathcal{B}$-set $\Rightarrow$ $\mathcal{B}^\omega$) and Theorem 5.6. □

Hence, we obtain the following decompositions of continuity.

Theorem 5.8. *For a function $f : X \to Y$, the following are equivalent:*

1. *f is continuous.*
2. *f is ω-almost continuous and LC-continuous.*
3. *f is ω-almost continuous and $\mathcal{A}$-continuous.*
4. *f is ω-almost continuous and $\mathcal{AB}$-continuous.*
5. *f is ω-almost continuous and $\mathcal{B}$-continuous.*
6. *f is ω-almost continuous and $\mathcal{B}^\omega$-continuous.*

Proof. It follows from Theorem 5.6 and Corollary 5.7. □

Theorem 5.9. *For a subset A of a space X, the following statements are equivalent:*

1. *A is clopen.*
2. *A is ω-closed and ω_p-open.*
3. *A is closed and ω_p-open.*

Proof. Straightforward. □

Our final result is the following decompositions of perfectly continuity.

Theorem 5.10. *For a function $f : X \to Y$, the following are equivalent:*

1. *f is perfectly continuous.*
2. *f is ω-almost continuous and contra continuous.*
3. *f is ω-almost continuous and contra ω-continuous.*

Proof. It follows from Theorem 5.9. □

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