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On k-Quasiposinormal Weighted Composition Operators

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Abstract : For a positive integer k, an operator A on a Hilbert space $\mathcal{H}$ is called k-quasiposinormal operator if $A^{*k}(AA^*)A^k \leq c^2 A^{*(k+1)}A^{(k+1)}$ for some $c > 0$. In this paper, we describe the conditions for the composition and weighted composition operators to be k-quasiposinormal operators.

Keywords : quasiposinormal operator; k-quasiposinormal operator; posinormal operator; weighted composition operator.
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1 Introduction

Throughout the paper, by an operator we mean a bounded linear operator on a Hilbert space. If $\mathcal{H}$ denotes a separable complex Hilbert space, denote the algebra of all operators on $\mathcal{H}$ by $\mathfrak{B}(\mathcal{H})$ and the kernel and range of an operator A on $\mathcal{H}$ by Ker(A) and Ran(A) respectively. An operator $A \in \mathfrak{B}(\mathcal{H})$ is called

- *hyponormal* if $AA^* \leq A^*A$;
- *p-hyponormal* if $(AA^*)^p \leq (A^*A)^p$, where $0 < p \leq 1$;
- *quasihyponormal* if $A^*(AA^*)A \leq A^{*2}A^2$ equivalently $(A^*A)^2 \leq A^{*2}A^2$;
- *k-quasihyponormal* if $A^{*k}(AA^*)A^k \leq A^{*(k+1)}A^{(k+1)}$, where k is a positive integer;

- *(p, k)-quasihyponormal* if $A^{*k}(AA^*)^p A^k \leq A^{*k}(A^*A)^p A^k$, where k is a positive integer and $0 < p \leq 1$;
- *posinormal* if $AA^* \leq c^2 A^*A$ for some $c > 0$;
- *p-posinormal* if $(AA^*)^p \leq c^2(A^*A)^p$ for some $c > 0$, where $0 < p \leq 1$;
- *(p, k)-quasiposinormal* if $A^{*k}(AA^*)^p A^k \leq c^2 A^{*k}(A^*A)^p A^k$ for some $c > 0$.

It is clear that for $p = 1$, p-hyponormal, p-posinormal and (p, k)-quasihyponormal are hyponormal, posinormal and k-quasihyponormal respectively. Also for $k = 1$, (p, k)-quasihyponormal, (p, k)-quasiposinormal and k-quasihyponormal are p-quasihyponormal, p-quasiposinormal and quasihyponormal respectively.

Definition 1.1. An operator $A \in \mathfrak{B}(\mathcal{H})$ is called

- *quasiposinormal* if $A^*(AA^*)A \leq c^2 A^{*2} A^2$ for some $c > 0$;
- *k-quasiposinormal* if $A^{*k}(AA^*)A^k \leq c^2 A^{*(k+1)} A^{(k+1)}$ for some $c > 0$, where k is a positive integer.

One can see from the definitions, as expected, for $p = 1$,

$$(p, k) - \text{quasiposinormal} = k - \text{quasiposinormal}$$

and for $k = 1$,

$$k - \text{quasiposinormal} = \text{quasiposinormal}.$$

Also one can easily verify that

$$\text{hyponormal} \subseteq \text{quasihyponormal} \subseteq \text{quasiposinormal} \subseteq k\text{-quasiposinormal};$$

$$k - \text{quasihyponormal} \subseteq k - \text{quasiposinormal};$$

$$k - \text{quasiposinormal} \subseteq k' - \text{quasiposinormal}$$

for positive integers $k < k'$.

The readers are referred to [1–6] and the references therein for more details and applications of hyponormal, p-hyponormal, k-quasihyponormal and (p, k)-quasihyponormal operators.

Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma-$ finite measure space. A measurable transformation $T : \Omega \to \Omega$ satisfying

$$\mu(T^{-1}(B)) = 0 \text{ whenever } \mu(B) = 0 \text{ for } B \in \mathcal{A}$$

is said to be *a non-singular measurable transformation.* If T is non-singular, then the measure μT^{-1} given by

$$(\mu T^{-1})(B) = \mu(T^{-1}(B)) \text{ for } B \in \mathcal{A},$$

is absolutely continuous with respect to the measure μ and we denote it by writing $\mu T^{-1} << \mu$. Hence by the Radon Nikodym theorem, there exists a non-negative measurable function h such that

$$(\mu T^{-1})(B) = \int_B h d\mu,$$

for every $B \in \mathcal{A}$. The function h is called the Radon Nikodym derivative of the measure μT^{-1} with respect to the measure μ. It is denoted by $h = d\mu T^{-1}/d\mu$.

A weighted composition operator $W(= W_{(u,T)})$ acting on the Hilbert space L^2, induced by a complex-valued measurable function u and a measurable transformation T is given by

$$Wf = u \cdot f \circ T \quad \text{for each } f \in L^2.$$

In case $u = 1$ *a.e.*, W becomes a composition operator denoted by C_T.

We use a symbol E very frequently in the paper, which denotes the conditional expectation operator $E(./T^{-1}(\mathcal{A})) = E(f)$. $E(f)$ is defined for each non-negative function f or for each $f \in L^p(\Omega, \mathcal{A}, \mu), 1 \leq p < \infty$, and is uniquely determined by the conditions

(i) $E(f)$ is $T^{-1}(\mathcal{A})$-measurable, and

(ii) if B is any $T^{-1}(\mathcal{A})$-measurable set for which $\int_B f\ d\mu$ exists, we have

$$\int_B f\ d\mu \ = \ \int_B E(f)\ d\mu.$$

The conditional expectation operator E has the following properties:

$E1$. $E(f \cdot g \circ T) \ = \ E(f) \cdot (g \circ T)$.

$E2$. If $f \geq g$ almost everywhere, then $E(f) \geq E(g)$ almost everywhere.

$E3$. $E(1) = I$.

$E4$. $|E(fg)|^2 \leq E(|f|^2)E(|g|^2)$.

$E5$. For $f > o$ almost everywhere, $E(f) > 0$ almost everywhere.

For each measurable function f, there exists a measurable function g such that $E(f) = g \circ T$. If we assume that the support of g lies in the support of h, then $E(f) = g \circ T$ for exactly one measurable function.

In particular, $g = E(f) \circ T^{-1}$ is a well defined measurable function.

As an operator on L^p, E is the projection operator onto the closure of the range of the composition operator C_T. This operator plays a vital role in the study of composition and weighted composition operators on various Banach function spaces (see [1] and [7]) and in this paper we present few more applications of this operator. For a deeper study of the properties of expectation operator we refer the paper of Lambert [8].

In the present paper various examples are given to show the proper inclusion amongst the classes described in the beginning. Some properties of the k-quasiposinormal operators acting on the Hilbert space $\mathcal{H}$ are discussed. Paper also provides applications of conditonal expectation operator E to characterize k-quasiposinormal composition and k-quasiposinormal weighted composition operator acting on L^2.

2 On Hilbert Space

Motivated by the result [2, Theorem 1] of Douglas, Lee and Lee [5, Theorem 2.2] obtained some characterization for the (p, k)-quasiposinormal operators, $0 < p \leq 1$ and k any natural number, introduced by them. This led us immediately to the following results about a k-quasiposinormal operator A acting on the Hilbert space $\mathcal{H}$.

Theorem 2.1. *For an operator $A \in \mathfrak{B}(\mathcal{H})$, the following are equivalent:*

1. *A is k-quasiposinormal.*
2. *$Ran(A^{*k}A) \subset Ran(A^{*(k+1)})$.*
3. *There exists $C \in \mathfrak{B}(\mathcal{H})$ satisfying $A^{*k}A = A^{*(k+1)}C$.*
4. *There exists a positive operator $P \in \mathfrak{B}(\mathcal{H})$ satisfying*
$$A^{*k}(AA^*)A^k = A^{*(k+1)}PA^{(k+1)}.$$
5. *$Ran(A^{*k}\sqrt{AA^*}) \subset Ran(A^{*k}\sqrt{A^*A})$.*
6. *There exists $\hat{C} \in \mathfrak{B}(\mathcal{H})$ satisfying $A^{*k}\sqrt{AA^*} = A^{*k}\sqrt{A^*A}\ \hat{C}$.*
7. *There exists a positive operator $\hat{P} \in \mathfrak{B}(\mathcal{H})$ satisfying*
$$A^{*k}(AA^*)A^k = A^{*k}\sqrt{A^*A}\ \hat{P}\sqrt{A^*A}\ A^{*k}.$$

Proof. Equivalence of the conditions 1, 5, 6 and 7 follow from [5, Theorem 2.2] on setting $p = 1$. The equivalence of the conditions 1, 2, 3 and 4 follow along the lines of proof of [2, Theorem 1]. □

The conditions 5, 6 and 7 are less useful being more difficult than the conditions 2, 3 and 4 to check whether an operator is k-quasiposinormal or not.

Now, it is evident that every invertible operator is k-quasiposinormal for each positive integer k and if $A \in \mathfrak{B}(\mathcal{H})$ is k-quasiposinormal then αA is k-quasiposinormal, for each $\alpha \in \mathbb{C}$. It is also apparent that if $A \in \mathfrak{B}(\mathcal{H})$ is k-quasiposinormal and $V \in \mathfrak{B}(\mathcal{H})$ is an isometry then VAV^* is k-quasiposinormal. The next result can be obtained along the computations made in [5, Theorem 2.6] with $p = 1$.

Theorem 2.2. *If $A \in \mathfrak{B}(\mathcal{H})$ is k-quasiposinormal then there exists $c > 0$ such that*

$$\|A^{k-1}x\|\|A^{k+1}x\| \geq c\|A^k x\|^2$$

for all $x \in H$.

Corollary 2.3. *If $A \in \mathfrak{B}(\mathcal{H})$ is k-quasiposinormal then $Ker(A^n) = Ker(A^k)$ for all $n \geq k$.*

Corollary 2.4. *If $A \in \mathfrak{B}(\mathcal{H})$ is k-quasiposinormal and $A^n = 0$ for some $n \geq k$, then $A^k = 0$.*

Now we discuss few examples which make the relevance of the study.

Example 2.5. *Consider the Hilbert space l^2 with orthonormal basis $\{e_n | n \geq 0\}$. Let A be the unilateral weighted shift with weight sequence $\langle \alpha_n \rangle_{n\geq 0}$, where*

$$\alpha_n = \begin{cases} 3 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 1 & \text{if } n \geq 2 \end{cases}.$$

Then $\|A^*Ae_0\|^2 = 81$, $\|A^2e_0\|^2 = 36$ and $Ran(A^*A) \subset Ran(A^{*2})$. Hence A is not quasihyponormal but is quasiposinormal. This justifies that the inclusion quasihyponormal $\subseteq$ quasiposinormal is strict.

Example 2.6. *Through this example, we show that the inclusion in k-quasihyponormal $\subseteq$ k-quasiposinormal is also strict. For, let A be the unilateral weighted shift with a positive weight sequence $\langle \alpha_n \rangle_{n\geq 0}$ with $\alpha_{k-1} > \alpha_k$ and $\alpha_k \leq \alpha_{k+1} \leq \alpha_{k+2} \leq \cdots$. Then one can see that, $\|A^*A^k e_0\| > \|A^{k+1}e_0\|$ and if $\alpha_{k-1} \leq c\alpha_k$ for some $c > 1$ then*

$$\|A^*A^k x\| \leq c\|A^{k+1}x\|$$

for all $x \in H$. Hence, A is not k-quasihyponormal but is k-quasiposinormal.

Next, we show that the inclusion k-quasiposinormal $\subseteq$ k'-quasiposinormal, where $k < k'$, is also strict.

Example 2.7. *Let A be the unilateral weighted shift with a weight sequence $\langle \alpha_n \rangle_{n\geq 0}$, where $\alpha_k = 0$ and $\alpha_n = 1$ for each $n \neq k$. Then A is (k+1)-quasiposinormal but not k-quasiposinormal.*

It is easy to verify that unilateral shift operator U on the Hilbert space l^2 satisfies the condition (3) of the Theorem 2.1 with $C = U^2$ and is k-quasiposinormal, whereas U^* is not k-quasiposinormal.

The class of k-quasiposinormal operators is not translation invariant i.e. if A is k-quasiposinormal then $(A + \alpha I)$ may not be k-quasiposinormal for $\alpha \in \mathbb{C}$. It can be verified by the fact that $A = (U^* - 2I)$ is k-quasiposinormal being invertible but $A + 2I = U^*$ is not k-quasiposinormal. This also ensures that the sum of two k-quasiposinormal operators need not be k-quasiposinormal.

In general, the product of two k-quasiposinormal operators need not be k-quasiposinormal. This can be checked by considering the unilateral shift operator A and the diagonal operator B with diagonal entries $\alpha_0 = 1, \alpha_1 = 0$ and $\alpha_n = 1$ for $n \geq 2$. Then A and B both are quasiposinormal and AB is a unilateral weighted shift operator with weight sequence $\langle \beta_n \rangle_{n\geq 0}$, where $\beta_k = 0, \beta_1 = 0$ and $\beta_n = 1$ for $n \geq 2$. Hence AB is not quasiposinormal. It is worth noticing that these operators A and B do not commute. It is yet not known, whether the product of two commuting k-quasiposinormal operators is a k-quasiposinormal operator or not. However, the next result present some affirmative answer under certain situations.

Theorem 2.8. *If A and B are k-quasiposinormal operators such that A commutes with B and B^* both then AB is k-quasiposinormal.*

Proof. We can assume that

$$A^{*k}(AA^*)A^k \leq cA^{*(k+1)}A^{(k+1)}$$

and

$$B^{*k}(BB^*)B^k \leq cB^{*(k+1)}B^{(k+1)}$$

for some $c > 0$. As the positive operators $\big(cA^{*(k+1)}A^{(k+1)} - A^{*k}(AA^*)A^k\big)$ and $\big(cB^{*(k+1)}B^{(k+1)} - B^{*k}(BB^*)B^k\big)$ commute, hence

$$\big(cA^{*(k+1)}A^{(k+1)} - A^{*k}(AA^*)A^k\big)\big(cB^{*(k+1)}B^{(k+1)} + B^{*k}(BB^*)B^k\big) \geq 0 \quad (2.1)$$

By the similar argument, we have

$$\big(cA^{*(k+1)}A^{(k+1)} + A^{*k}(AA^*)A^k\big)\big(cB^{*(k+1)}B^{(k+1)} - B^{*k}(BB^*)B^k\big) \geq 0 \quad (2.2)$$

Using (2.1) and (2.2), we find that

$$\begin{aligned}\big((AB)^{*k}(AB)(AB)^*(AB)^k\big) &= \big(A^{*k}(AA^*)A^k\big)\big(B^{*k}(BB^*)B^k\big)\\ &\leq c^2\big(A^{*(k+1)}A^{(k+1)}\big)\big(B^{*(k+1)}B^{(k+1)}\big)\\ &= c^2\big(AB)^{*(k+1)}(AB)^{(k+1)}\big).\end{aligned}$$

Hence AB is k-quasiposinormal. □

Corollary 2.9. *If A is a k-quasiposinormal and B is a normal operator such that A commutes with B then AB is k-quasiposinormal.*

Proof. As B is a normal operator, by Fuglede-Putnam theorem, A commutes with B^*. Hence the result. □

3 Weighted Composition Operators

Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma-$ finite measure space and $T : \Omega \to \Omega$ be a measurable transformation inducing composition operator $C_T(f \mapsto f \circ T)$ on L^2. For the non-singular transformation $T : \Omega \mapsto \Omega$ and a positive integer k, we mean by T^k the composition $\underbrace{T \circ T \circ \cdots \circ T}(k\ times)$. Let $T^0 = I$ and for $k \geq 1$, define the measure μT^{-k} on the measure space $(\Omega, \mathcal{A}, \mu)$ as

$$\mu T^{-k}(B) = \mu T^{-(k-1)}(T^{-1}(B)) \ \text{ for } B \in \mathcal{A}.$$

Then

$$\mu T^{-k} << \mu T^{-(k-1)} << \cdots << \mu T^{-2} << \mu T^{-1} << \mu.$$

We denote the Radon Nikodym derivative of μT^{-k} with respect to μ by h_k and the Radon Nikodym derivative of $\mu T^{-(k+1)}$ with respect to μT^{-k} by $\widetilde{h}_k$. We assume that $h_0 = 1$ and $h_1 = h$. It can be seen that $h_k = h \cdot h \circ T^{-1} \cdot h \circ T^{-2} \cdot \cdots \cdot h \circ T^{-(k-1)}$. These notations help us to present the following facts, which are either known or obtained by simple computations. For $f \in L^2$,

1. $C_T^* f = h \cdot E(f) \circ T^{-1}$.
2. For any positive integer k, $C_T^k f = f \circ T^k$ and $C_T^{*k} f = h_k \cdot E(f) \circ T^{-k}$, where $h_k = d\mu T^{-k}/d\mu$.
3. $C_T^* C_T f = h \cdot f$.
4. $C_T C_T^* f = (h \circ T) \cdot E(f)$.
5. E is the identity operator on L^2 if and only if $T^{-1}(\mathcal{A}) = \mathcal{A}$.

Theorem 3.1. *Let $C_T \in \mathfrak{B}(L^2)$. Then the following are equivalent:*

1. *C_T is quasiposinormal.*
2. *$\|h \cdot f\| \leq c\|\sqrt{h_2} \cdot f\|$ for each $f \in L^2$.*
3. *$\|h \cdot f\| \leq c\|\sqrt{h \cdot E(h) \circ T^{-1}} \cdot f\|$ for each $f \in L^2$.*
4. *$h \leq c^2 \widetilde{h}_1$, where $\widetilde{h}_1 = d\mu T^{-2}/d\mu T^{-1}$.*
5. *$h \leq c^2 E(h) \circ T^{-1}$.*

Proof. $(1) \equiv (2)$:

$$\begin{aligned}
(1) &\Longleftrightarrow C_T^*(C_T C_T^*)C_T \leq c^2 C_T^{*2} C_T^2 \\
&\Longleftrightarrow (C_T^*\ C_T)^2 \leq c^2 C_T^{*2}\ C_T^2 \\
&\Longleftrightarrow \langle h^2 f, f \rangle \leq c^2 \int |f \circ T^2|^2 d\mu \ = \ c^2 \int |f|^2 h_2 d\mu \ for\ each\ f \in L^2 \\
&\Longleftrightarrow \langle h^2 f, f \rangle \leq \langle c^2 h_2 f, f \rangle \ for\ each\ f \in L^2 \\
&\Longleftrightarrow \|h \cdot f\| \leq c\|\sqrt{h_2} \cdot f\| \ for\ each\ f \in L^2.
\end{aligned}$$

(2) ≡ (3): This part of the Theorem follows by using the observation that for each $f \in L^2$,

$$\begin{aligned}\langle h_2 f, f\rangle &= \langle C_T^{*2} C_T^2 f, f\rangle \\ &= \langle C_T^*(C_T^* C_T) C_T f, f\rangle \\ &= \langle h \cdot E(h \cdot f \circ T) \circ T^{-1}, f\rangle \\ &= \langle h \cdot E(h) \circ T^{-1} \cdot f, f\rangle,\end{aligned}$$

which yields that $h_2 = h \cdot E(h) \circ T^{-1}$.

(1) ≡ (4): (1) $\Longleftrightarrow \langle h^2 f, f\rangle \le \langle c^2 h_2 f, f\rangle$ for each $f \in L^2$. As a consequence of this, (1) $\Longleftrightarrow h^2 \le c^2 h_2$. The result follows from here by using the fact that $h_2 = d\mu T^{-2}/d\mu = (d\mu T^{-2}/d\mu T^{-1}) \cdot (d\mu T^{-1}/d\mu) = h \cdot \widetilde{h}_1$.

(1) ≡ (5): This follows by replacing the value of h_2 by $h \cdot E(h) \circ T^{-1}$ in the above arguments. Hence the Theorem. □

It is interesting to see that $h_2 = h \cdot E(h) \circ T^{-1} = h \cdot \widetilde{h}_1$, so that $\widetilde{h}_1 = E(h) \circ T^{-1}$ a.e. on the support of h.

Corollary 3.2. *If $T^{-1}(\mathcal{A}) = \mathcal{A}$ then C_T is quasiposinormal if and only if $h \le c^2 h \circ T^{-1}$.*

Theorem 3.3. *Let $C_T \in \mathfrak{B}(L^2)$. Then the following are equivalent:*

1. *C_T is k-quasiposinormal.*
2. *$\|\sqrt{h_k \cdot h \circ T^{-(k-1)}} f\| \le c\|\sqrt{h_k \cdot E(h) \circ T^{-k}} f\|$ for each $f \in L^2$.*
3. *$h_k \cdot h \circ T^{-(k-1)} \le c^2 h_k \cdot E(h) \circ T^{-k}$, where $h_k = d\mu T^{-k}/d\mu$.*
4. *$h_{k-1} \cdot (h \circ T^{-(k-1)})^2 \le c^2 h_{k-1} \cdot h \circ T^{-(k-1)} \cdot E(h) \circ T^{-k}$.*
5. *$h_{k-1} \circ T^{-1} \cdot h \circ T^{-(k-1)} \le c^2 h_{k-1} \circ T^{-1} \cdot E(h) \circ T^{-k}$.*

Proof. Using the observations,

$$\begin{aligned}(C_T^{*k}(C_T C_T^*) C_T^k) f &= C_T^{*k}(C_T C_T^*)(f \circ T^k) \\ &= C_T^{*k}(h \circ T \cdot f \circ T^k) \\ &= h_k \cdot E(h \circ T \cdot f \circ T^k) \circ T^{-k} \\ &= h_k \cdot h \circ T^{-(k-1)} \cdot f\end{aligned}$$

and

$$\begin{aligned}C_T^{*(k+1)} C_T^{(k+1)} f &= (C_T^{*k}(h \cdot f \circ T^k) \\ &= h_k \cdot E(h \cdot f \circ T^k) \circ T^{-k} \\ &= h_k \cdot E(h) \circ T^{-k} \cdot f,\end{aligned}$$

we conclude that C_T is k-quasiposinormal if and only if for each $f \in L^2$,

$$\left\langle C_T^{*(k+1)} C_T^{(k+1)} f, f \right\rangle \le c^2 \left\langle (C_T^{*k}(C_T C_T^*) C_T^k) f, f \right\rangle$$

equivalently for each $f \in L^2$,

$$\left\| \sqrt{h_k \cdot h \circ T^{-(k-1)}} f \right\| \le c \left\| \sqrt{h_k \cdot E(h) \circ T^{-k}} f \right\|$$

or

$$h_k \cdot h \circ T^{-(k-1)} \le c^2 h_k \cdot E(h) \circ T^{-k}.$$

Hence, we have (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3). Now (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) follows by using the observations $h_k = h_{k-1} \cdot h \circ T^{-(k-1)}$ being

$$\begin{aligned} \mu T^{-k}(B) &= \mu T^{-1}(T^{-(k-1)}(B)) \\ &= \int_{T^{-(k-1)}(B)} h d\mu \\ &= \int_B h_{k-1} \cdot h \circ T^{-(k-1)} d\mu \end{aligned}$$

and $h_k = h_{(k-1)} \circ T^{-1} \cdot h$ being

$$\begin{aligned} \mu T^{-k}(B) &= \mu T^{-(k-1)}(T^{-1}(B)) \\ &= \int_{T^{-(1}(B)} h_{k-1} d\mu \\ &= \int_B h \cdot h_{k-1} \circ T^{-1} d\mu \end{aligned}$$

for each $B \in \mathcal{A}$. □

Corollary 3.4. *If $T^{-1}(\mathcal{A}) = \mathcal{A}$ then C_T is k-quasiposinormal if and only if $h_k \cdot h \circ T^{-(k-1)} \le c^2 h_k \cdot h \circ T^{-k}$.*

Corollary 3.5. *If $T^{-1}(\mathcal{A}) = \mathcal{A}$ then C_T is quasiposinormal if and only if $h \le c^2 h \circ T^{-1}$.*

Theorem 3.6. *Let $C_T \in \mathfrak{B}(L^2)$. A necessary and sufficient condition for C_T^* to be k-quasiposinormal is that for each $f \in L^2$*

$$\left\langle h \circ T^k \cdot h_k \circ T^k \cdot E(f), f \right\rangle \le c^2 \left\langle h_{k+1} \circ T^{(k+1)} \cdot E(f), f \right\rangle.$$

Proof. As $C_T^{*k} f = h_k \cdot E(f) \circ T^{-k}$, we see that, for each $f \in L^2$,

$$(C_T^k (C_T^* C_T) C_T^{*k}) f = h \circ T^k \cdot h_k \circ T^k \cdot E(f)$$

and

$$(C_T^k (C_T C_T^*) C_T^{*k}) f = C_T^{(k+1)} (C_T^{*(k+1)} f) = h_{k+1} \circ T^{(k+1)} \cdot E(f).$$

Hence the result. □

Corollary 3.7. *If $T^{-1}(\mathcal{A}) = \mathcal{A}$ then C_T^* is k-quasiposinormal if and only if*

$$\left\|\sqrt{h \circ T^k \cdot h_k \circ T^k} \cdot f\right\| \leq c\left\|\sqrt{h_{k+1} \circ T^{(k+1)}} \cdot f\right\|$$

for each $f \in L^2$.

Corollary 3.8. *C_T^* is quasiposinormal if and only if for each $f \in L^2$*

$$\langle (h \circ T)^2 \cdot E(f), f\rangle \leq c^2 \langle h_2 \circ T^2 \cdot E(f), f\rangle.$$

Corollary 3.9. *If $T^{-1}(\mathcal{A}) = \mathcal{A}$, then the following are equivalent:*

1. *C_T^* is quasiposinormal.*
2. *$\|h \circ T \cdot f\| \leq c\|\sqrt{h_2 \circ T^2} \cdot f\|$, for each $f \in L^2$.*
3. *$(h \circ T)^2 \leq c^2 h_2 \circ T^2 = c^2 h \circ T^2 \cdot h \circ T$.*

Now we deal with the weighted composition operator $W = W_{(u,T)} \in \mathfrak{B}(L^2)$, $(f \mapsto u \cdot f \circ T)$ induced by the complex-valued measurable mapping u on Ω and the measurable transformation $T : \Omega \mapsto \Omega$. It is known that W^* is given by

$$W^* f = h \cdot E(u \cdot f) \circ T^{-1}$$

for each $f \in L^2$.

For a positive integer k, we put $u_k = u \cdot (u \circ T) \cdot (u \circ T^2) \cdot \cdots \cdot (u \circ T^{(k-1)})$ and $\hat{u}_k = (u \circ T^{-1}) \cdot (u \circ T^{-2}) \cdot \cdots \cdot (u \circ T^{-k})$. Then, $u_k \circ T^{-k} = \hat{u}_k$. For $k = 0$, we denote $u_k = \hat{u}_k = 1$ *and* $W^k = I$. However, h_k is used to denote the Radon Nikodym derivative of μT^{-k} with respect to μ and $h_1 = h$. For $f \in L^2$, $W^k f = u_k \cdot f \circ T^k$ so that $W^{*k} f = h_k \cdot E(u_k \cdot f) \circ T^{-k}$. The following simple computations,

$$W^* W^k f = h \cdot E(u^2) \cdot T^{-1} \cdot W^{(k-1)} f;$$
$$W^{*(k+1)} f = h_{k+1} \cdot E(u_{(k+1)} \cdot f) \circ T^{-(k+1)} = h_{k+1} \cdot E(u \cdot f) \circ T^{-(k+1)} \cdot \hat{u}_k;$$
$$W^{*k}(WW^*)W^k f = h_k \cdot h \circ T^{-(k-1)} \cdot (E(u^2) \circ T^{-k})^2 \cdot \hat{u}_{k-1}^2 \cdot f;$$
$$W^{*(k+1)} W^{(k+1)} f = h_{k+1} \cdot E(u_{k+1}^2) \circ T^{-(k+1)} \cdot f;$$

help us to conclude the following:

Theorem 3.10. *Let $W \in \mathfrak{B}(L^2)$. Then W^* is k-quasiposinormal if and only if*

$$\left\|u \cdot h_k \circ T \cdot E(u_k \cdot f) \circ T^{-(k-1)}\right\| \leq c\left\|h_{k+1} \cdot E(u_{k+1} \cdot f) \circ T^{-(k+1)}\right\|$$

for each $f \in L^2$.

Corollary 3.11. *W^* is quasiposinormal if and only if*

$$\|u \cdot h \circ T \cdot E(u \cdot f)\| \leq c\left\|h_2 \cdot E(u_2 \cdot f) \circ T^{-2}\right\|$$

for each $f \in L^2$.

Corollary 3.12. *If $T^{-1}(\mathcal{A}) = \mathcal{A}$ then W^* is quasiposinormal if and only if*

$$\left\|u^2 \cdot h \circ T \cdot f\right\| \leq c \left\|h_2 \cdot \hat{u}_2 \cdot f \circ T^{-2}\right\|$$

for each $f \in L^2$.

Theorem 3.13. *Let $W \in \mathfrak{B}(L^2)$. Then the following are equivalent:*

1. *W is k-quasiposinormal.*
2. *$\|h \cdot E(u^2) \circ T^{-1} \cdot W^{(k-1)} f\| \leq c \|u_{k+1} \cdot f \circ T^{(k+1)}\|$ for each $f \in L^2$.*
3. *$\|\sqrt{h_{k-1}} \cdot h \circ T^{-(k-1)} \cdot \hat{u}_{k-1} \cdot E(u^2) \circ T^{-k} f\| \leq c \|\sqrt{h_{k+1}} \cdot \hat{u}_{k+1} \cdot f\|$ for each $f \in L^2$.*
4. *$h_k \cdot h \circ T^{-(k-1)} \cdot (E(u^2) \circ T^{-k})^2 \cdot \hat{u}_{k-1}^2 \leq c^2 h_{k+1} \cdot E(u^2) \circ T^{-(k+1)} \cdot \hat{u}_k^2 = c^2 h_k \cdot \widetilde{h}_k \cdot \hat{u}_k^2 \cdot E(u^2) \circ T^{-(k+1)}$, where $\widetilde{h}_k = d\mu T^{-(k+1)} / d\mu T^{-k}$.*

Corollary 3.14. *Let $W \in \mathfrak{B}(L^2)$. Then the following are equivalent:*

1. *W is quasiposinormal.*
2. *$\|h \cdot E(u^2) \circ T^{-1} \cdot f\| \leq c \|u_2 \cdot f \circ T^2\|$ for each $f \in L^2$.*
3. *$\|h \cdot E(u^2) \circ T^{-1} f\| \leq c \|\sqrt{h_2} \cdot \hat{u}_2 \cdot f\|$ for each $f \in L^2$.*
4. *$h^2 \cdot (E(u^2) \circ T^{-1})^2 \leq c^2 h_2 \cdot E(u^2) \circ T^{-2} \cdot \hat{u}_1^2$*
5. *$h \cdot (E(u^2) \circ T^{-1})^2 \leq c^2 \widetilde{h}_1 \cdot E(u^2) \circ T^{-2} \cdot \hat{u}_1^2$, where $\widetilde{h}_1 = d\mu T^{-2} / d\mu T^{-1}$.*

Corollary 3.15. *If $T^{-1}(\mathcal{A}) = \mathcal{A}$ then W is quasiposinormal if and only if $h \cdot \hat{u}_1^4 \leq c^2 \widetilde{h}_1 \cdot \hat{u}_2^2$.*

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References

[1] J.T. Campbell, W.E. Hornor, Seminormal composition operators, J. Operator Theory 29 (1993) 323–343.

[2] R.G. Douglas, On Majoriation, factorization and range inclusion of operators on Hilbert Spaces, Proc. Amer. Math. Soc. 173 (1966) 413–415.

[3] S.M. Patel, On classes of non-hyponormal operators, Math. Nachr. 73 (1975) 147–150.

[4] H.C. Rhaly, Posinormal operators, Jr. Math. Soc. Japan 46 (1994) 587–605.

[5] M.Y. Lee, S.H. Lee, On (p, k)-quasiposinormal operators, Jr. Appl. Math and Computing 19 (1) (2005) 573–578.

[6] S.K. Berberian, *Introduction to Hilbert space*, Chelsea Publishing Company, New York, (1961).

[7] S.C. Arora, G. Datt, Compact weighted composition operators on Orlicz-Lorentz spaces, Acta Scientiarum Mathematicarum (Szeged) 77 (4) (2011) 567–578.

[8] A. Lambert, Localising sets for sigma-algebras and related point transformations, Proc. Royal Soc. Edinburgh (Series A) 118 (1991) 111–118.

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