



# Some I-Convergent Double Classes of Sequences of Fuzzy Numbers Defined by Orlicz Functions

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**Abstract :** In this paper, using the concept of ideal and Orlicz function, we introduce some new classes of double sequence spaces of fuzzy numbers. We study different topological properties of these sequence spaces like completeness, solidity, symmetricity etc. Also we obtain some inclusion relation involving these sequence spaces.

**Keywords :** Orlicz function;  $I$ -convergent; solid space; symmetric space; sequence algebra; convergence free.

**2010 Mathematics Subject Classification :** 40A05; 40A30; 40D25.

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## 1 Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [1] in 1965. Later on fuzzy logic became an important area of research in various branches of mathematics such as metric and topological spaces [2, 3], theory of functions [4], approximation theory [5] etc. Fuzzy set theory also finds its applications for modeling, uncertainty and vagueness in various fields of Science and Engineering, e.g. computer programming [6], nonlinear dynamical systems [7], population dynamics [8], control of chaos [9], quantum physics [10] etc. It attracted

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workers on sequence spaces to introduce different type of classes of sequences of fuzzy numbers.

The initial works on double sequences may be found in Bromwich [11]. The notion of regular convergence of double sequences of real or complex terms is introduced by Hardy [12]. Tripathy and Dutta [13, 14] introduced and investigated different types of fuzzy real valued double sequence spaces. Generalizing the concept of ordinary convergence for real sequences Kastyrko et al. [15] introduced the concept of ideal convergence which is a generalization of statistical convergence, by using the ideal  $I$  of the subsets of the set of natural numbers. Some works in this direction can be found in [16–22].

An Orlicz function  $M$  is a function  $M : [0, \infty) \rightarrow [0, \infty)$  such that it is continuous, non-decreasing and convex with  $M(0) = 0, M(x) > 0$  for  $x > 0$  and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . An Orlicz function may be bounded or unbounded. An Orlicz function also satisfies the inequality  $M(\lambda x) \leq \lambda M(x)$  for all  $\lambda$  with  $0 < \lambda < 1$ . Lindenstrauss and Tzafriri [23] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which becomes a Banach space, with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

The space  $\ell_M$  is closely related to the space  $\ell^p$ , which is an Orlicz sequence space with  $M(x) = |x|^p$ , for  $1 \leq p < \infty$ .

Subsequently the notion of Orlicz function was used to define sequence spaces by many authors such as [20, 24–32]. A fuzzy real number  $X$  is a fuzzy set on  $R$ , ie. a mapping  $X : R \rightarrow L (= [0, 1])$  associating each real number  $t$  with its grade of membership  $X(t)$ . Every real number  $r$  can be expressed as a fuzzy real number  $\bar{r}$  as follows:

$$\bar{r}(t) = \begin{cases} 1 & \text{if } t=r \\ 0 & \text{otherwise.} \end{cases}$$

The  $\alpha$ -level set of a fuzzy real number  $X$ ,  $0 < \alpha \leq 1$  denoted by  $[X]^\alpha$  is defined as  $[X]^\alpha = \{t \in R : X(t) \geq \alpha\}$ . A fuzzy real number  $X$  is called convex if  $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$ , where  $s < t < r$ . If there exists  $t_0 \in R$  such that  $X(t_0) = 1$ , then the fuzzy real number  $X$  is called normal. A fuzzy real number  $X$  is said to be upper semi-continuous if for each  $\epsilon > 0, X^{-1}([0, a + \epsilon])$ , for all  $a \in L$  is open in the usual topology of  $R$ . The set of all upper semi continuous, normal, convex fuzzy number is denoted by  $R(L)$ .

Arithmetic operations on  $R(L)$  are defined as follows:

$$\begin{aligned} (X \oplus Y)(t) &= \sup_{s \in R} \{X(s) \wedge Y(t-s)\}, t \in R, \\ (X \ominus Y)(t) &= \sup_{s \in R} \{X(s) \wedge Y(s-t)\}, t \in R, \\ (X \otimes Y)(t) &= \sup_{s \in R} \{X(s) \wedge Y(t/s)\}, t \in R, \\ (X/Y)(t) &= \sup_{s \in R} \{X(st) \wedge Y(s)\}, t \in R. \end{aligned}$$

The absolute value of  $X \in R(L)$  is defined as (one may refer to Kaleva and Seikkala [33])

$$|X(t)| = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let  $D$  be the set of all closed bounded intervals  $X=[X^L, X^R]$ . Then  $X \leq Y$  if and only if  $X^L \leq Y^L$  and  $X^R \leq Y^R$ . Also  $d(X, Y) = \max(|X^L - Y^L|, |X^R - Y^R|)$ . Then  $(D, d)$  is a complete metric space. Let  $\bar{d} : R(L) \times R(L) \rightarrow R$  be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha), \text{ for } X, Y \in R(L).$$

Then  $\bar{d}$  defines a metric on  $R(L)$ .

In this paper, we study some new double sequence spaces of fuzzy numbers defined by an Orlicz function. The spaces defined here are much more general than the existing ones and gives many of them as special cases.

## 2 Definition and Preliminaries

Let  $X$  be a non empty set. A non-void class  $I \subseteq 2^X$  (power set of  $X$ ) is called an *ideal* if  $I$  is additive (i.e.  $A, B \in I \Rightarrow A \cup B \in I$ ) and hereditary (i.e.  $A \in I$  and  $B \subseteq A \Rightarrow B \in I$ ). A non-empty family of sets  $F \subseteq 2^X$  is said to be a *filter* on  $X$  if  $\emptyset \notin F$ ;  $A, B \in F \Rightarrow A \cap B \in F$  and  $A \in F, A \subseteq B \Rightarrow B \in F$ . For each ideal  $I$  there is a filter  $F(I)$  given by  $F(I) = \{K \subseteq N : N \setminus K \in I\}$ .

A subset  $E$  of  $N$  is said to have *density*  $\delta(E)$  if

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_E(k)$$

exists, where  $\chi_E(k)$  is the characteristic function of  $E$ . Throughout the ideals of  $2^N$  and  $2^{N \times N}$  will be denoted by  $I$  and  $I_2$  respectively. Let us consider  $I_2(\rho) \subset 2^{N \times N}$  i.e. the class of all subsets of  $N \times N$  of zero natural density. Then  $I_2(\rho)$  is an ideal of  $2^{N \times N}$ . For a detailed account of different types of ideals of  $2^{N \times N}$ , one may refer to Tripathy and Tripathy [21].

Throughout a fuzzy real valued double sequence is denoted by  $(X_{nk})$  i.e. a double infinite array of fuzzy real numbers  $X_{nk}$  for all  $n, k \in N$ .

A double sequence  $(X_{nk})$  is said to be *convergent* in Pringsheim's sense to the fuzzy real number  $X_0$ , if for every  $\epsilon > 0$ , there exists  $n_0 = n_0(\epsilon)$ ,  $k_0 = k_0(\epsilon) \in N$  such that  $\bar{d}(X_{nk}, X_0) < \epsilon$  for all  $n \geq n_0$ ,  $k \geq k_0$ . A double sequence  $(X_{nk})$  is said to be *I-convergent* to the fuzzy number  $X_0$ , if for all  $\epsilon > 0$ , the set  $\{(n, k) \in N^2 : \bar{d}(X_{nk}, X_0) \geq \epsilon\} \in I_2$ . We write  $I_2 - \lim X_{nk} = X_0$ .

Throughout the article  ${}_2w^F$ ,  ${}_2\ell_\infty^F$ ,  ${}_2c^F$ ,  ${}_2c^{I(F)}$  and  ${}_2c_0^{I(F)}$  denote the classes of all, bounded, convergent, I-convergent and I-null fuzzy real number double sequences respectively.

A double sequence space  $E^F$  is said to be *solid* if  $(Y_{nk}) \in E^F$ , whenever  $\bar{d}(Y_{nk}, \bar{0}) \leq \bar{d}(X_{nk}, \bar{0})$  for all  $n, k \in N$  and  $(X_{nk}) \in E^F$ . A double sequence  $E^F$  is said to be *monotone* if  $E^F$  contains the canonical pre-image of all its step spaces. A double sequence  $E^F$  is said to be *symmetric* if  $(X_{\pi(n,k)}) \in E^F$ , whenever  $(X_{nk}) \in E^F$ , where  $\pi$  is a permutation of  $N \times N$ . A double sequence  $E^F$  is said to be *sequence algebra* if  $(X_{nk} \otimes Y_{nk}) \in E^F$ , whenever  $(X_{nk}), (Y_{nk}) \in E^F$ . A double sequence  $E^F$  is said to be *convergence free* if  $(Y_{nk}) \in E^F$ , whenever  $(X_{nk}) \in E^F$  and  $X_{nk} = \bar{0}$  implies  $Y_{nk} = \bar{0}$ .

Let  $M$  be an Orlicz function and  $p = (p_{nk})$  be a double sequence of bounded strictly positive real numbers. We introduce the following double sequence spaces:

$${}_2c^{I(F)}(M, p) = \left\{ X = (X_{nk}) : I_2 - \lim \left[ M \left( \frac{\bar{d}(X_{nk}, X_0)}{\rho} \right) \right]^{p_{nk}} = 0, \right. \\ \left. \text{for some } \rho > 0 \text{ and } X_0 \in R(L) \right\},$$

$${}_2c_0^{I(F)}(M, p) = \left\{ X = (X_{nk}) : I_2 - \lim \left[ M \left( \frac{\bar{d}(X_{nk}, \bar{0})}{\rho} \right) \right]^{p_{nk}} = 0, \text{ for some } \rho > 0 \right\},$$

$${}_2\ell_\infty^{(F)}(M, p) = \left\{ X = (X_{nk}) : \sup_{nk} \left[ M \left( \frac{\bar{d}(X_{nk}, \bar{0})}{\rho} \right) \right]^{p_{nk}} < \infty, \text{ for some } \rho > 0 \right\},$$

$${}_2\ell_\infty^{I(F)}(M, p) = \left\{ X = (X_{nk}) : \text{there exists a real number } \mu > 0 \text{ such that the set} \right. \\ \left. \left\{ (n, k) \in N \times N : \left[ M \left( \frac{\bar{d}(X_{nk}, \bar{0})}{\rho} \right) \right]^{p_{nk}} > \mu \right\} \in I_2, \text{ for some } \rho > 0 \right\}.$$

Also we write

$${}_2m^{I(F)}(M, p) = {}_2c^{I(F)}(M, p) \cap {}_2\ell_\infty^{(F)}(M, p),$$

$${}_2m_0^{I(F)}(M, p) = {}_2c_0^{I(F)}(M, p) \cap {}_2\ell_\infty^{(F)}(M, p).$$

Let  $Z$  denote any one of  ${}_2c^{I(F)}$ ,  ${}_2c_0^{I(F)}$  and  ${}_2\ell_\infty^{I(F)}$ . On giving particular values to  $M$  and  $p$ , we get the following sequence spaces from the above sequence spaces:

- (i) If  $p_{nk} = 1$ , for all  $n, k \in N$ , then we obtain  $Z(M)$  instead of  $Z(M, p)$ .

- (ii) If  $M(x) = x$ , then  $Z(M, p)$  becomes  $Z(p)$ .
- (iii) If  $M(x) = x, p_{nk} = 1$  for all  $n, k \in N$ , then we obtain  $Z$  instead of  $Z(M, p)$ .

**Lemma 2.1.** *If a sequence space  $E^F$  is solid, then it is monotone.*

For the crisp set case, one may refer to Kamthan and Gupta [34], p. 53.

**Lemma 2.2** ([17]). *For two sequences  $p = (p_{nk})$  and  $q = (q_{nk})$  we have  $(c_0^{I(F)})_2^{BP}(p) \supseteq (c_0^{I(F)})_2^{BP}(q)$  if and only if*

$$\liminf_{n,k \in K} \left( \frac{p_{nk}}{q_{nk}} \right) > 0, \text{ where } K \in F(I_2).$$

### 3 Main Results

The proof of the following result is easy, so omitted.

**Theorem 3.1.** *Let  $M$  be an Orlicz function and  $p = (p_{nk})$  be a double sequence of bounded strictly positive numbers. Then the class of sequences  ${}_2m^{I(F)}(M, p)$ ,  ${}_2m_0^{I(F)}(M, p)$  and  ${}_2\ell_\infty^{I(F)}(M, p)$  are closed under the operations of addition and scalar multiplication.*

**Theorem 3.2.** *Let the double sequence  $(p_{nk})$  be bounded. Then  ${}_2c_0^{I(F)}(M, p) \subset {}_2c^{I(F)}(M, p) \subset {}_2\ell_\infty^{I(F)}(M, p)$  and the inclusions are strict.*

*Proof.* The inclusion  ${}_2c_0^{I(F)}(M, p) \subset {}_2c^{I(F)}(M, p) \subset {}_2\ell_\infty^{I(F)}(M, p)$  is obvious. Only to show that the inclusion  ${}_2c^{I(F)}(M, p) \subset {}_2\ell_\infty^{I(F)}(M, p)$  is strict, we consider the following example.

**Example 3.3.** *Let  $I_2(P)$  denote the class of all subsets of  $N \times N$  such that  $A \in I_2(P)$  implies that there exists  $n_0, k_0 \in N$  such that  $A \subseteq N \times N - \{(n, k) \in N \times N : n \geq n_0, k \geq k_0\}$ . Let  $M(x) = x^2$  and  $n_0, k_0 \in N$  be fixed such that*

$$p_{nk} = \begin{cases} \frac{1}{2} & \text{if } 1 \leq n \leq n_0, 1 \leq k \leq k_0 \\ 2 & \text{otherwise.} \end{cases}$$

*Consider the sequence  $(X_{nk})$  defined by:  $X_{nk} = \bar{1}$ , for  $1 \leq n \leq n_0, 1 \leq k \leq k_0$ . For  $n > n_0, k > k_0$  and  $(n + k)$  even,*

$$X_{nk}(t) = \begin{cases} \frac{nt-2n+1}{n+1} & \text{for } 2 - n^{-1} \leq t \leq 3 \\ 4 - t & \text{for } 3 < t \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

*Otherwise*

$$X_{nk}(t) = \begin{cases} \frac{nt-1}{2n-1} & \text{for } n^{-1} \leq t \leq 2 \\ 3 - t & \text{for } 2 < t \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

*Then  $(X_{nk}) \in {}_2\ell_\infty^{I(F)}(M, p)$ , but  $(X_{nk}) \notin {}_2c^{I(F)}(M, p)$ .*

□

The following two results can be proved easily using simple technique.

**Theorem 3.4.** *The class of sequences  ${}_2m^{I(F)}(M, p)$  and  ${}_2m_0^{I(F)}(M, p)$  are complete metric spaces with respect to the metric  $\tau$  defined by*

$$\tau(X, Y) = \inf \left\{ \rho^{\frac{p_{nk}}{J}} > 0 : \sup_{nk} M \left( \frac{\bar{d}(X_{nk}, Y_{nk})}{\rho} \right) \leq 1, \rho > 0 \right\}$$

where  $J = \max(1, H)$ ,  $H = \sup_{nk} p_{nk}$ .

**Theorem 3.5.** *Let  $M_1$  and  $M_2$  be two Orlicz functions, then*

- (i)  $Z(M_1, p) \cap Z(M_2, p) \subseteq Z(M_1 + M_2, p)$ ;
- (ii)  $Z(M_2, p) \subseteq Z(M_1 \circ M_2, p)$ , for  $Z = {}_2m_0^{I(F)}, {}_2m^{I(F)}, {}_2\ell_\infty^{I(F)}$ .

We state the following result in view of Lemma 2.2.

**Theorem 3.6.** *For two sequences  $p = (p_{nk})$  and  $t = (t_{nk})$  we have  ${}_2m_0^{I(F)}(M, p) \supseteq {}_2m_0^{I(F)}(M, t)$  if*

$$\liminf_{n, k \in K} \left( \frac{p_{nk}}{t_{nk}} \right) > 0, \text{ where } K \in F(I_2).$$

**Theorem 3.7.** *The class of sequences  ${}_2m_0^{I(F)}(M, p)$  is solid as well as monotone.*

*Proof.* Let  $(X_{nk}) \in {}_2m_0^{I(F)}(M, p)$  and  $(Y_{nk})$  be such that  $\bar{d}(Y_{nk}, \bar{0}) \leq \bar{d}(X_{nk}, \bar{0})$  for each  $n, k \in N$ . Let  $\epsilon > 0$  be given. Then the solidness of  ${}_2m_0^{I(F)}(M, p)$  follows from the following relation:  $\{(n, k) \in N \times N : \left[ M \left( \frac{\bar{d}(X_{nk}, \bar{0})}{\rho} \right) \right]^{p_{nk}} \geq \epsilon\} \supseteq \{(n, k) \in N \times N : \left[ M \left( \frac{\bar{d}(Y_{nk}, \bar{0})}{\rho} \right) \right]^{p_{nk}} \geq \epsilon\}$ . Also by Lemma 2.1, it follows that the space is monotone. □

**Corollary 3.8.** *The class of sequences  ${}_2m^{I(F)}(M, p)$  is neither monotone nor solid.*

*Proof.* The result follows from the following example.

**Example 3.9.** *Let  $I_2(\rho) \subset 2^{N \times N}$  denote the class of all subsets of  $N \times N$  of zero natural density. Let  $I_2 = I_2(\rho)$  and  $A \in I_2$ ,  $p_{nk} = 1$  for all  $n, k \in N$ ,  $M(x) = x^2$ . Consider the sequence  $(X_{nk})$  defined by: For  $(n, k) \notin A$*

$$X_{nk}(t) = \begin{cases} 1 + (n+k)(t-1) & \text{for } 1 - \frac{1}{n+k} \leq t \leq 1 \\ 1 - (n+k)(t-1) & \text{for } 1 < t \leq 1 + \frac{1}{n+k} \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise  $X_{nk} = \bar{1}$ . Then  $(X_{nk}) \in {}_2m^{I(F)}(M, p)$ . Let  $K = \{2i : i \in N\}$ . We define the sequence  $(Y_{nk})$  as follows :

$$Y_{nk} = \begin{cases} X_{nk} & \text{if } (n+k) \in K \\ \bar{0} & \text{otherwise.} \end{cases}$$

Then  $(Y_{nk})$  belongs to the canonical pre-image of  $K$  step space of  ${}_2m^{I(F)}(M, p)$ . But  $(Y_{nk}) \notin {}_2m^{I(F)}(M, p)$ . Hence the space  ${}_2m^{I(F)}(M, p)$  is not monotone. So by Lemma 2.1,  ${}_2m^{I(F)}(M, p)$  is not solid.

□

**Theorem 3.10.** The class of sequences  ${}_2m^{I(F)}(M, p)$  and  ${}_2m_0^{I(F)}(M, p)$  are not symmetric in general.

*Proof.* The result follows from the following example.

**Example 3.11.** Let  $I_2(\rho) \subset 2^{N \times N}$  denote the class of all subsets of  $N \times N$  of zero natural density. Let  $I_2 = I_2(\rho)$ ,  $M(x) = x^2$  and

$$p_{nk} = \begin{cases} 1 & \text{for } n \text{ even and all } k \in N \\ 2 & \text{otherwise.} \end{cases}$$

Consider the sequence  $(X_{nk})$  defined by: For  $n = i^2, i \in N$  and for all  $k \in N$

$$X_{nk}(t) = \begin{cases} 1 + \frac{t}{2\sqrt{n}-1} & \text{for } 1 - 2\sqrt{n} \leq t \leq 0 \\ 1 - \frac{t}{2\sqrt{n}-1} & \text{for } 0 < t \leq 2\sqrt{n} - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise  $X_{nk} = \bar{0}$ . Then  $(X_{nk}) \in Z(M, p)$  for  $Z = {}_2m^{I(F)}, {}_2m_0^{I(F)}$ . Consider the rearrangement  $(Y_{nk})$  of  $(X_{nk})$  defined by: For  $k$  odd and for all  $n \in N$ ,

$$Y_{nk}(t) = \begin{cases} 1 + \frac{t}{2n-1} & \text{for } 1 - 2n \leq t \leq 0 \\ 1 - \frac{t}{2n-1} & \text{for } 0 < t \leq 2n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise  $Y_{nk} = \bar{0}$ . Then  $(Y_{nk}) \notin Z(M, p)$  for  $Z = {}_2m^{I(F)}, {}_2m_0^{I(F)}$ . Hence the class of sequences  ${}_2m^{I(F)}(M, p)$  and  ${}_2m_0^{I(F)}(M, p)$  are not symmetric.

□

**Theorem 3.12.** The class of sequences  ${}_2m^{I(F)}(M, p)$  and  ${}_2m_0^{I(F)}(M, p)$  are not convergence free.

*Proof.* The result follows from the following example.

**Example 3.13.** Let  $I_2(\rho) \subset 2^{N \times N}$  denote the class of all subsets of  $N \times N$  of zero natural density. Let  $I_2 = I_2(\rho)$  and  $A \in I_2$ ,  $p_{nk} = \frac{1}{3}$  for all  $n, k \in N$ ,  $M(x) = x$ . Consider the sequence  $(X_{nk})$  defined by: For  $(n, k) \notin A$

$$X_{nk}(t) = \begin{cases} 1 + 2(n+k)t & \text{for } -\frac{1}{2(n+k)} \leq t \leq 0 \\ 1 - 2(n+k)t & \text{for } 0 < t \leq \frac{1}{2(n+k)} \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise  $X_{nk} = \bar{0}$ . Then  $(X_{nk}) \in Z(M, p)$  for  $Z = {}_2m^{I(F)}, {}_2m_0^{I(F)}$ . Consider the sequence  $(Y_{nk})$  defined by: For  $(n, k) \notin A$

$$Y_{nk}(t) = \begin{cases} 1 + \frac{2t}{n+k} & \text{for } -\frac{n+k}{2} \leq t \leq 0 \\ 1 - \frac{2t}{n+k} & \text{for } 0 < t \leq \frac{n+k}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise  $Y_{nk} = \bar{0}$ . Then  $(Y_{nk}) \notin Z(M, p)$  for  $Z = {}_2m^{I(F)}, {}_2m_0^{I(F)}$ . Hence the class of sequences  ${}_2m^{I(F)}(M, p)$  and  ${}_2m_0^{I(F)}(M, p)$  are not convergence free. □

**Theorem 3.14.** The class of sequences  ${}_2m^{I(F)}(M, p)$  and  ${}_2m_0^{I(F)}(M, p)$  are sequence algebras.

*Proof.* We consider the space  ${}_2m_0^{I(F)}(M, p)$ . Let  $(X_{nk}), (Y_{nk}) \in {}_2m_0^{I(F)}(M, p)$  and  $0 < \epsilon < 1$ . Then the result follows from the following inclusion relation:  $\{(n, k) \in N \times N : M\left(\frac{\bar{d}(X_{nk} \otimes Y_{nk}, \bar{0})}{\rho}\right)^{p_{nk}} < \epsilon\} \supset \{(n, k) \in N \times N : M\left(\frac{\bar{d}(X_{nk}, \bar{0})}{\rho}\right)^{p_{nk}} < \epsilon\} \cap \{(n, k) \in N \times N : M\left(\frac{\bar{d}(Y_{nk}, \bar{0})}{\rho}\right)^{p_{nk}} < \epsilon\}$ . Similarly we can prove the result for other case. □

## 4 Conclusion

In this article, we introduced some classes of fuzzy real valued double sequences defined by Orlicz function. We have proved the completeness of the introduced sequence spaces and studied some other properties like solidness, symmetricity sequence algebra etc. We have also proved some inclusion results.

**Acknowledgement :** The authors thank the referees for the careful reading of the manuscript and for their valuable comments and suggestions.

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(Received 7 September 2011)

(Accepted 12 April 2012)