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Smarandache Weak Subtraction Algebra

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Abstract : The notions of Smarandache weak subtraction algebra, Q-Smarandache ideals, prime Q-Smarandache ideals and weakly prime Q-Smarandache ideals are introduced. Some examples are given and several properties are investigated. Characterizations of ideals are provided. Different relations between Q-Smarandache ideals, prime Q-Smarandache ideals and weakly prime Q-Smarandache ideals are discussed.

Keywords : weak subtraction algebra; Smarandache weak subtraction algebra; Q-Smarandache ideal; prime Q-Smarandache ideal; weakly prime Q-Smarandache ideal.
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1 Introduction

A Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B of A which is embedded with a strong structure S. In [1], Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by Padilla [2].

Schein [3] considered systems of the form $(\phi, \circ, \diagdown)$, where ϕ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\phi, \circ)$ is a subtraction algebra in the sense of [4]). Jun et al. [5] introduced the notion of ideals in subtraction algebras, and discussed some characterizations of ideals. Lee et al. [6]

introduced the notion of weak subtraction algebras and they investigated relations between a subtraction algebra and a weak subtraction algebra.

It will be very interesting to study the Smarandache structure in this algebraic structures. In [7], Jun discussed the Smarandache structure in BCI-algebras. He introduced the notion of Smarandache (positive implicative, commutative, implicative) BCI-algebras, Smarandache subalgebras and Smarandache ideals, and investigated some related properties. Saeid et al. [8] defined the Smarandache structure in BL-algebras. Smarandache hyper BCC-algebra have been invented by Ahadpanah and Saeid [9], and they deal with Smarandache hyper BCC-ideal structures in Smarandache BCC-algebra. Recently, in [10] Saeid and Rezaei discussed the Smarandache n-Structure on CI-Algebras. In this paper, we introduce the notion of Smarandache weak subtraction algebra. We also introduce the concepts of Q-Smarandache ideals, prime Q-Smarandache ideals and weakly prime Q-Smarandache ideals, and investigate related properties. It's interesting to study the Smarandache Structure in weak subtraction algebras.

2 Preliminaries

By a subtraction algebra we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for all $x, y, z \in X$.

(a1) $x - (y - x) = x$.

(a2) $x - (x - y) = y - (y - x)$.

(a3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x-y)-z$. The subtraction determines an order relation on X: $a \leq b \Longleftrightarrow a-b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [4].

In a subtraction algebra, the following are true (see [5, 11]):

b1) $(x - y) - y = x - y$.

b2) $x - 0 = x$ and $0 - x = 0$.

b3) $(x - y) - x = 0$.

b4) $x - (x - y) \leq y$.

b5) $(x - y) - (y - x) = x - y$.

b6) $x - (x - (x - y)) = x - y$.

b7) $((x - y) - (z - y)) - (x - z) = 0$.

b8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

b9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

Definition 2.1 ([5])**.** A nonempty subset E of a subtraction algebra X is called an *ideal* of X if it satisfies

$$0 \in E \text{ and } (\forall x \in X)(\forall y \in E)\ (x - y \in E \Rightarrow x \in E).$$

Definition 2.2 ([6])**.** By a weak subtraction algebra, we mean a triplet $(W, -, 0)$, where W is a nonempty set, "$-$" is a binary operation on W and $0 \in W$ is a nullary operation, called zero element, such that

(c1) $(\forall x \in W)\ (x - 0 = x,\ x - x = 0)$,

(c2) $(\forall x, y, z \in W)\ ((x - y) - z = (x - z) - y)$,

(c3) $(\forall x, y, z \in W)\ ((x - y) - z = (x - z) - (y - z))$.

Theorem 2.3. *Every subtraction algebra is a weak subtraction algebra.*

The converse of the above theorem is not true in general (see [6, Example 3.7]).

Proposition 2.4 ([6])**.** *For a weak subtraction algebra* $(W, -, 0)$, *we have*

(d1) $(\forall x \in W)\ (0 - x = 0)$,

(d2) $(\forall x, y \in W)\ ((x - y) - x = 0)$,

(d3) $(\forall x, y, z \in W)\ (x - y = 0 \Rightarrow (x - z) - (y - z) = 0)$.

Definition 2.5 ([12])**.** A nonempty subset J of a weak subtraction algebra X is called an *ideal* of X if it satisfies

$$0 \in J \text{ and } (\forall x \in X)(\forall y \in J)\ (x - y \in J \Rightarrow x \in J).$$

3 Smarandache Weak Subtraction Algebra

Definition 3.1. A Smarandache weak subtraction algebra is defined to be a weak subtraction algebra X in which there exists a proper subset A of X such that

(1) $0 \in A$ and $|A| > 2$,

(2) A is a subtraction algebra under the operation of X.

Example 3.2. *Let* $X = \{0, a, b, c\}$. *The following Cayley table shows the weak subtraction algebra structure on* X.

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

If we consider $A = \{0, a, b\}$, then we can see that A is a subtraction algebra which is properly contained in X. Therefore X is a Smarandache weak subtraction algebra.

The following example shows that not every weak subtraction algebra is Smarandache weak subtraction algebra.

Example 3.3. *Let $X = \{0, a, b, c\}$ be a set with the following Cayley table:*

$-$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	c	0

Then X is a weak subtraction algebra. If we consider $Q = \{0, a, b\}$, then we can see that Q is not a subtraction algebra since $a - (a - b) \neq b - (b - a)$. Therefore X is not a Smarandache weak subtraction algebra.

In what follows, let X and Q denote a Smarandache weak subtraction algebra and a subtraction algebra which properly contained in X, respectively.

Definition 3.4. A nonempty subset I of X is called a *Smarandache ideal* of X related to Q (or briefly, Q-Smarandache ideal of X) if it satisfies

(f1) $0 \in I$,

(f2) $(\forall x \in Q)\,(\forall y \in I)\ (x - y \in I \Rightarrow x \in I)\,.$

Example 3.5. *In Example 3.2, let $I_1 = \{0, a, c\}$. It is easily to check that I_1 is a Q-Smarandache ideal of X.*

Proposition 3.6. *An Q-Smarandache ideal I of X has the following property:*

$$(\forall x \in Q)\,(\forall y \in I)\ (x \le y \Rightarrow x \in I)\,.$$

Proof. Straightforward. □

Proposition 3.7. *Any ideal of X is a Q-Smarandache ideal of X.*

The following example shows that the converse of Proposition 3.7 is not always true.

Example 3.8. *In Example 3.2, $I_1 = \{0, a, c\}$ is a Q-Smarandache ideal of X, which is not an ideal of X, since $b - c = 0 \in I_1$ and $c \in I_1$, but $b \notin I_1$.*

Proposition 3.9. *If $\{I_\lambda : \lambda \in \Delta\}$ is an indexed set of Q-Smarandache ideals of X, where $\Delta \neq \emptyset$, then $I = \cap\{I_\lambda : \lambda \in \Delta\}$ is a Q-Smarandache ideal of X.*

Proposition 3.10. *If Q satisfies $Q - X \subset Q$, then every Q-Smarandache ideal I of X satisfies the following implication:*

$$(\forall x, y \in I)\,(\forall z \in Q)\ ((z - y) \le x \Rightarrow z \in I)\,.$$

Proof. Assume that $Q - X \subset Q$ and let I be a Q-Smarandache ideal of X. Suppose that $(z - y) \leq x$ for all $x, y \in I$ and $z \in Q$. Then $(z - y) - x = 0$ for all $x, y \in I$ and $z \in Q$. So $(z - y) \in Q$ by assumption, and $(z - y) - x \in I$, and so $(z - y) \in I$ by $(f2)$. Since $y \in I$ and $z \in Q$, it follows from $(f2)$ that $z \in I$. $\square$

Theorem 3.11. *For any $t \in X$, the set $H_t := \{x \in X \mid x - t = 0\}$ is an Q-Smarandache ideal of X.*

Proof. By $(d1)$, $0 \in H_t$ for all $t \in X$. Let $z \in Q$ and $y \in H_t$ be such that $z - y \in H_t$. Then we have

$$\begin{aligned} z - t &= (z - t) - 0 \\ &= (z - t) - (y - t) && \text{(since } y \in H_t) \\ &= (z - y) - t && \text{(by } (c3)) \\ &= 0 && \text{(since } z - y \in H_t). \end{aligned}$$

Hence $z \in H_t$. Therefore, H_t is an Q-Smarandache ideal of X. $\square$

Theorem 3.12. *Let Q_1 and Q_2 be two subtraction algebras which are properly contained in X and $Q_1 \subseteq Q_2$. Then every Q_2-Smarandache ideal of X is a Q_1-Smarandache ideal of X.*

The converse of Theorem 3.12 may not be true in general as seen in the following example.

Example 3.13. *Let $X = \{0, a, b, c, d\}$ be the set with the following cayley table.*

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Then X is a Smarandache weak subtraction algebra. Note that $Q_1 = \{0, a, b\}$ and $Q_2 = \{0, a, b, c\}$ are two subtraction algebras which are properly contained in X. It is easily to check that $J = \{0, a, b, d\}$ is a Q_1-Smarandache ideal of X, but it is not a Q_2-Smarandache ideal of X since $c - b = a \in J$ and $b \in J$ but $c \notin J$.

Theorem 3.14. *Let I be a subset of X such that*

(i) $0 \in I$,

(ii) $x \in Q$ *and* $y - x = 0$ *imply* $y \in I$,

(iii) *for* $x, y \in I$, *there exists* $z \in Q$ *such that* $x - z = 0$ *and* $y - z = 0$.

Then I is a Q-Smarandache ideal of X.

Proof. Let $x \in Q$ and $y \in I$ be such that $x - y \in I$. Then by (iii), there exists $z \in Q$ such that $y - z = 0$ and $(x - y) - z = 0$. It follows from $(c1)$ and $(c3)$ that $x - z = (x - z) - 0 = (x - z) - (y - z) = ((x - y) - z = 0$. Since $z \in Q$, it follows from (ii) that $x \in I$. Therefore, I is a Q-Smarandache ideal of X. □

The following example shows that the converse of Theorem 3.14 is not valid.

Example 3.15. *In Example 3.13, $J = \{0, a, b, d\}$ is a Q_1-Smarandache ideal of X. If $x = a$ and $y = b$, then there is no $z \in Q_1$ such that $x - z = 0$ and $y - z = 0$, so condition (iii) of Theorem 3.14 is not satisfied.*

Let X be a weak subtraction algebra, $x, y \in X$ and $Q \subset X$ be a subtraction algebra we denote

$$G(x, y) = \{z \in Q \mid z - x \leq y\}.$$

Theorem 3.16. *Let I be a non-empty subset of a weak subtraction algebra X. If $G(x, y) \subseteq I$ for every $x, y \in I$, then I is a Q-Smarandache ideal of X.*

Proof. If $G(x, y) \subseteq I$ for every $x, y \in I$ we have $0 \in I$ since $0 \in G(x, y)$. Let $a \in Q$ and $b \in I$ be such that $a - b \in I$. Then $G(a - b, b) \subseteq I$. Since $a - (a - b) \leq b$, we have $a \in G(a - b, b) \subseteq I$, and so $a \in I$. Hence I is a Q-Smarandache ideal of X. □

Definition 3.17. A nonempty subset I of X is called a *prime Smarandache ideal of X related to Q* (or briefly, prime Q-Smarandache ideal of X) if it satisfies the condition $(f1)$ and $(f3)$ $(\forall x, y \in Q)\,(\forall z \in I)\ ((x - (x - y)) - z \in I \Rightarrow x \in I)$.

Example 3.18. *Let $X = \{0, a, b, c, d\}$ be a set with the following Cayley table:*

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	c	c	0	0
d	d	d	c	b	0

X is a weak subtraction algebra. $Q = \{0, a, b\}$ is a subtraction algebra which is properly contained in X. Therefore, X is a Smarandache weak subtraction algebra. It is clear that $I = \{0, a, c\}$ is a prime Q-Smarandache ideal of X.

Remark 3.19. *If Q satisfies the condition:*

$$(\forall x, y \in Q)\ (x = x - (x - y)),$$

then every Q-Smarandache ideal of X is a prime Q-Smarandache ideal of X.

Theorem 3.20. *Every prime Q-Smarandache ideal of X is a Q-Smarandache ideal of X.*

Proof. Let I be a prime Q-Smarandache ideal of X and let $x \in Q$ and $z \in I$ be such that $x-z \in I$. Taking $x = y$ in $(f3)$, we get $(x-(x-y))-z = (x-0)-z = x-z \in I$. By $(f3)$, it follows that $x \in I$. Hence I is a Q-Smarandache ideal of X. $\square$

In the following example we show that the converse of Theorem 3.20 is not true.

Example 3.21. *In Example 3.13, consider $X = \{0, a, b, c, d\}$ and $Q_2 = \{0, a, b, c\}$. Then X is a Smarandache weak subtraction algebra. It is easy to verify that a subset $K = \{0, b\}$ is a Q_2-Smarandache ideal of X which is not a prime Q-Smarandache ideal of X, since for $x = c$, $y = b$ and $z = b$ in $(f3)$ we have $(c-(c-b))-b = 0 \in K$, but $c \notin K$.*

Proposition 3.22. *Every weak subtraction algebra X satisfies the following equality:*

$$(\forall x, y, z \in X)\ (((x-(x-y))-z)-(y-z) = 0).$$

Proof. For any $x, y, z \in X$, we have

$$\begin{aligned} 0 &= 0 - z \\ &= ((x-y)-(x-y))-z \\ &= ((x-(x-y))-y)-z \\ &= ((x-(x-y))-z)-(y-z). \end{aligned}$$

$\square$

Definition 3.23. A nonempty subset I of X is called a *Smarandache weakly prime ideal of X related to Q* (or briefly, weakly prime Q-Smarandache ideal of X) if it satisfies the condition $(f1)$ and

(f4) $(\forall x, y, z \in Q)\ ((x-y)-z \in I,\ y-z \in I \Rightarrow x-z \in I)$.

Example 3.24. *Let $X = \{0, a, b, c\}$ be the Smarandache weak subtraction algebra with $Q = \{0, a, b\}$ in Example 3.2. It is easy to verify that $I = \{0, c\}$ is a weakly prime Q-Smarandache ideal of X.*

Theorem 3.25. *Every weakly prime Q-Smarandache ideal of X which is contained in Q is a Q-Smarandache ideal of X.*

Proof. Let I be a weakly prime Q-Smarandache ideal of X, $x \in Q$ and $y \in I$ be such that $x-y \in I$. Then $(x-y)-0 = x-y \in I$ and $y-0 = y \in I$. Since $x \in Q$ and $y \in I \subset Q$, it follows from $(f4)$ and $(c1)$ that $x \in I$. Hence I is a Q-Smarandache ideal of X. $\square$

Theorem 3.26. *Every prime Q-Smarandache ideal of X is a weakly prime Q-Smarandache ideal of X.*

Proof. Let J be a prime Q-Smarandache ideal of X. Assume that $(x-y)-z \in J$ and $y-z \in J$ for all $x, y, z \in Q$. From Theorem 3.20, we conclude that J is a Q-Smarandache ideal of X. Since J is a Q-Smarandache ideal of X, it follows from Proposition 3.22 and $(f1)$ that $(x-(x-y))-z \in J$. Since $(x-(x-y))-z = (x-z)-((x-y)-z) \in J$, we have $x-z \in J$. Therefore J is a weakly prime Q-Smarandache ideal of X. $\square$

In the following example we show that the converse of Theorem 3.26 is not true.

Example 3.27. *Let $X = \{0, a, b, c\}$ be the weak subtraction algebra described in Example 3.2. Consider $Q = \{0, a, b\}$. $I = \{0, c\}$ is a weakly prime Q-Smarandache ideal of X (see Example 3.24) but I is not a prime Q-Smarandache ideal of X, since for $x = a$, $y = b$ and $z = c$ in $(f3)$ we have $(a-(a-b))-c = 0 \in I$, but $a \notin I$.*

Proposition 3.28. *If I is a weakly prime Q-Smarandache ideal of X, then*

$$(\forall x, y \in Q)\ ((x-y)-y \in I \Rightarrow x-y \in I).$$

Proof. Assume $(x-y)-y \in I$ for all $x, y \in Q$. If we let $z = y$ in $(f4)$, then we have $x-y \in I$. $\square$

Proposition 3.29. *If I is a weakly prime Q-Smarandache ideal of X which is contained in Q, then*

$$(\forall x, y \in Q)\,(\forall z \in I)\ (((x-y)-y)-z \in I \Rightarrow x-y \in I).$$

Proof. Assume $((x-y)-y)-z \in I$ for all $x, y \in Q$ and $z \in I$. By Theorem 3.25, I is a Q-Smarandache ideal of X. Since I is a Q-Smarandache ideal of X, it follows from $(f2)$ that $(x-y)-y \in I$, and thus by Proposition 3.28, $x-y \in I$. $\square$

Proposition 3.30. *If I is a Q-Smarandache ideal of X such that*

$$(\forall x, y, z \in Q)\ ((x-y)-z \in I \Rightarrow (x-z)-(y-z) \in I),$$

then I is a weakly prime Q-Smarandache ideal of X.

Proof. Assume that $(x-y)-z \in I$ and $y-z \in I$ for all $x, y, z \in Q$. Then $(x-z)-(y-z) \in I$, which implies from $(f2)$ that $x-z \in I$. Hence I is a weakly prime Q-Smarandache ideal of X. $\square$

4 Conclusion

We have introduced the notion of Smarandache weak subtraction algebras and investigated some of their properties. Work is ongoing. Some important issues for future work are:

(1) To develop strategies for obtaining more valuable results.

(2) To apply these definitions and results for studying related notions in other Smarandache structures.

(3) To describe the fuzzy structure of Smarandache weak subtraction algebras and its applications.

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